Relating multi-adjoint algebras to general residuated structures

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Abstract. This paper summarizes the relationships established among multi-adjoint algebras and other residuated structures. Different relational diagrams provide an overview about the comparative study.
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1 Introduction

The study of algebraic structures is fundamental in many domains in mathematics and information sciences, since they are the base from which the computations in these domains are given. The consideration of general structures offer the possibility of remove artificial constrains introduced in practical examples. Moreover, a good balance between structures with optimal properties for the computations and general structures is mandatory.

Adjoint triples were introduced as a flexible generalization of t-norms and their residuated implications. These operators provide more flexibility and applicability in important frameworks such as logic programming [16, 17], formal concept analysis [4, 14, 15], rough set theory [7], fuzzy relation equations [9, 10].

This paper considers and extends the results given in [5, 6], in order to present different relations among multi-adjoint algebras and other general algebraic structures. These relations are given on four diagrams. The first diagram shows the relationships without restrictions on the operators and structures. In the other three diagrams the conditions required in each particular framework is taken into account. Specifically, we consider multi-adjoint concept lattices, multi-adjoint logic programming and multi-adjoint fuzzy rough sets.

2 A comparative study of multi-adjoint algebras

Different logical approaches have carried out the survey of the algebraic operators related to logical connectives, as well as a detailed analysis of the possible relations between those connectives. Following this idea, the operators associated with multi-adjoint algebras were compared with some of the most general algebraic operators. In what follows, we will recall the general structures considered in the comparative study, which continue the ones given in [5, 6].

2.1 Multi-adjoint algebras

This section presents the definitions of the different multi-adjoint algebras. An adjoint triple $(\&, \swarrow, \nwarrow)$ with respect to three posets $(P_1, \leq_1)$, $(P_2, \leq_2)$, $(P_3, \leq_3)$ is composed by three mappings $\&\colon P_1 \times P_2 \to P_3$, $\swarrow\colon P_3 \times P_2 \to P_1$, $\nwarrow\colon P_3 \times P_1 \to P_2$ verifying, for all $x \in P_1$, $y \in P_2$ and $z \in P_3$, the *adjoint property*:

$$x \leq_1 z \swarrow y \quad \text{iff} \quad x \,\&\, y \leq_3 z \quad \text{iff} \quad y \leq_2 z \nwarrow x$$

The algebraic structure $(P_1, P_2, P_3, \leq_1, \leq_2, \leq_3, \&_1, \swarrow^1, \nwarrow_1, \dots, \&_n, \swarrow^n, \nwarrow_n)$ is called *biresiduated multi-adjoint algebra* which is composed by a family of adjoint triples $(\&_i, \swarrow^i, \nwarrow_i)$ w.r.t P_1, P_2, P_3, with $i \in \{1, \dots, n\}$.

In certain frameworks, the requirements of adjoint triples are weakened since only pairs are needed to make computations [11]. For example, we can consider the *right adjoint pair* $(\&, \swarrow)$ formed by two operators $\&\colon P_1 \times P_2 \to P_3$ and $\swarrow\colon P_3 \times P_2 \to P_1$, satisfying the condition $x \leq_1 z \swarrow y$ iff $x \,\&\, y \leq_3 z$, for all $x \in P_1$, $y \in P_2$, $z \in P_3$. The tuple $(P_1, P_2, P_3, \leq_1, \leq_2, \leq_3, \&_1, \swarrow^1, \dots, \&_n, \swarrow^n)$, where $(\&_i, \swarrow^i)$ is a family of right adjoint pairs w.r.t P_1, P_2, P_3, with $i \in \{1, \dots, n\}$, is called *right multi-adjoint algebra*.

Analogously, we can define the *left multi-adjoint algebra* and the *antitone multi-adjoint algebra* which are build from a family of *left adjoint pairs* and *weak Galois pairs*, respectively. More information about multi-adjoint algebras and several properties of their corresponding operators such as the monotonicity, the uniqueness of the adjoint implications, as well as different useful inequalities of adjoint triples and pairs were exposed in [3, 6].

2.2 Other residuated algebras

Adjointness algebras and sup-preserving aggregation structures are general algebras whose operators verify the adjoint property. This fact justifies the comparison between multi-adjoint algebras and these residuated structures.

Adjointness algebras [18]. Let $(L, \leq_L)$, $(P, \leq_P)$ be two posets with a top element $\top_P$ in $(P, \leq_P)$. An *adjointness algebra* is a tuple $(L, \leq_L, P, \leq_P, \top_P, A, K, H)$, satisfying the following four conditions:

1. The operation $A\colon P \times L \to L$ is antitone in the left argument and isotone in the right argument, and the equality $A(\top_P, \gamma) = \gamma$ holds, for all $\gamma \in L$. We call A an *implication on* (L, P).
2. The operation $K\colon P \times L \to L$ is isotone in each argument and the equality $K(\top_P, \beta) = \beta$ holds, for all $\beta \in L$. We call K a *conjunction on* (L, P).
3. The operation $H\colon L \times L \to P$ is antitone in the left argument and isotone in the right argument, and it satisfies, for all $\beta, \gamma \in L$, that

   $$H(\beta, \gamma) = \top_P \quad \text{iff} \quad \beta \leq_L \gamma$$

   We call H a *forcing-implication on* L.

4. The three operations A, K and H, are mutually related by the following condition, for all $\alpha \in P$ and $\beta, \gamma \in L$:

$$\beta \leq_L A(\alpha, \gamma) \quad \text{iff} \quad K(\alpha, \beta) \leq_L \gamma \quad \text{iff} \quad \alpha \leq_P H(\beta, \gamma)$$

which is called the *adjointness condition*.

We call the ordered triple (A, K, H) an *implication triple* on (L, P).

As a consequence of the properties proven in [3], sufficient conditions were stated in order to obtain an implication triple from an adjoint triple. Moreover, it is easy to see that each implication triple is an adjoint triple.

Sup-preserving aggregation operators [1]. A *sup-preserving aggregation structure* is a quadruple $(\mathbf{L}_1, \mathbf{L}_2, \mathbf{L}_3, \Box)$, where $\mathbf{L}_i = (L_i, \preceq_i)$, with $i \in \{1, 2, 3\}$, are complete lattices and $\Box\colon L_1 \times L_2 \to L_3$ is a function which commutes with suprema in both arguments, for all $a, a_j \in L_1$ $(j \in J)$, $b, b_{j'} \in L_2$ $(j' \in J')$, that is:

$$\left(\bigvee_{j \in J} a_j\right) \Box\, b = \bigvee_{j \in J} (a_j \Box\, b) \qquad a \Box \left(\bigvee_{j' \in J'} b_{j'}\right) = \bigvee_{j' \in J'} (a \Box\, b_{j'})$$

A biresiduated multi-adjoint algebra can be defined from a sup-preserving aggregation structure. However, the counterpart is not true since only posets are needed in the definition of an adjoint triple. Hence, multi-adjoint algebras provide a more flexible framework than a sup-preserving aggregation structure and, moreover, than every particular algebra as quantales [2].

2.3 Implicative algebras

Two of the most important implicative algebras are extended-order algebras and implicative structures. A detailed study with several properties and remarks showing the relationships among multi-adjoint algebras and the structures previously mentioned was presented in [6].

Extended-order algebras [12]. Let P be a non-empty set, $\rightarrow\colon P \times P \to P$ a binary operation and $\top$ a fixed element of P. The triple $(P, \rightarrow, \top)$ is a *weak extended-order algebra*, in short *w-eo algebra*, if for all $a, b, c \in P$ the following conditions are satisfied:

(o_1) $a \rightarrow \top = \top$ (upper bound condition)
(o_2) $a \rightarrow a = \top$ (reflexivity condition)
(o_3) $a \rightarrow b = \top$ and $b \rightarrow a = \top$ then $a = b$ (antisymmetry condition)
(o_4) $a \rightarrow b = \top$ and $b \rightarrow c = \top$ then $a \rightarrow c = \top$ (weak transitivity condition)

In [8], when the carrier of a w-eo algebra is a complete lattice $(L, \preceq)$, the following structures are defined:

(a) $(L, \rightarrow, \top)$ is called *right-distributive w-ceo algebra* if the following condition is satisfied, for any $a \in L$, $B \subseteq L$: $a \rightarrow \bigwedge_{b \in B} b = \bigwedge_{b \in B} (a \rightarrow b)$
(b) $(L, \rightarrow, \top)$ is called *symmetrical w-ceo algebra* if there exists a binary operation $\rightsquigarrow: L \times L \rightarrow L$ such that $(L, \rightsquigarrow, \top)$ is a complete w-eo algebra, $\rightarrow$ and $\rightsquigarrow$ induce the same ordering $\leq$, and the equivalence $y \leq x \rightsquigarrow b$ if and only if $x \leq y \rightarrow b$ holds, for all $b, x, y \in L$.

The natural ordering relation defined in [12] plays an important role in different results which relate these previous notions to implication triples and adjoint triples. More information can be found in [5, 6].

Implication structures [20]. An independent axiomatization of extended-order algebras are provided by implication structures. An *implication structure* is a triple $(P, e, \rightarrow)$, where $(P, \leq)$ is a poset, e is an element of P interpreted as true and $\rightarrow: P \times P \rightarrow P$ is a binary function, called the *implication*, satisfying the following properties, for all $a, b, c \in P$:

(I1) If $b \leq c$ then $a \rightarrow b \leq a \rightarrow c$;
(I2) If $b \leq a$ then $a \rightarrow c \leq b \rightarrow c$;
(I3) $a \rightarrow b \leq (c \rightarrow a) \rightarrow (c \rightarrow b)$;
(I4) $e \leq a \rightarrow b$ if and only if $a \leq b$;
(I5) $e \rightarrow a \leq a$.

Observe that, an implication structure is not an w-eo algebra since the forcing implication property is not equivalent to Property $(I4)$. Whereas, symmetric implication structures are a particular case of antitone multi-adjoint algebras. For more details, see [6].

2.4 Conjunctive algebras

Conjunctive algebras are fundamental structures in different frameworks of soft computing. Three of the most general and used conjunctors in that structures are t-norms, uninorms and u-norms. Although these conjunctors not need to have a residuated operator, this is needed in useful frameworks, such as in formal concept analysis, in fuzzy relation equations, in fuzzy rough sets or in fuzzy logic programming.

Uninorms [19]. These operators are another interesting generalization of t-norms and t-conorms whose neutral element is not necessarily equal to 0 or 1. If we interpret the adjoint property in terms of a multiple-valued inference, it is easy to see that adjoint implications of a semi-uninorm or uninorm may fail to exist and so, they might not necessarily be part of an adjoint triple. In [6], it was shown that these operators are particular cases of an adjoint conjunctor, when they are $\vee$-distributive.

U-norms [13]. U-norms arose as a generalization of the arithmetic mean, the continuous Archimedean t-norm and the operator "fuzzy and" with $\gamma < 1$. An example to illustrate that an adjoint triple cannot be obtained from an arbitrary u-norm was shown in [6]. However, continuous u-norms are a particular case of a conjunctor in an adjoint triple.

3 Relational diagrams

This section presents different diagrams which offer an overview about the possible relations among multi-adjoint algebras and the algebraic structures shown in the previous section. To begin with, we will include a diagram showing the relationships without restrictions (Fig. 1). After that, we will present three diagrams taking into account the conditions required in environments where multi-adjoint concept lattices (Fig. 2), multi-adjoint fuzzy rough sets (Fig. 3) and multi-adjoint logic programming (Fig. 4) are defined.

For a proper understanding of the information supplied by these diagrams, the following considerations must be assumed. Algebraic structures joined by arrows with a single tip indicate that the structure located at the initial node is a particular case of the algebraic structure placed in the final node. We will interpret that two algebraic structures are equivalent if they are connected by arrows with double tips. Moreover, the flow of the arrows in the diagrams is, more or less, bottom-up. It is important to mention that the diagrams are transitive, that is, arrows that are deducted by the transitivity of the relations represented are not drawn. Finally, it is worth to note that the differences among the first diagram and the given ones to particular frameworks are emphasized by arrows with different colors. More details about a general understanding of these diagrams can be seen in [6], which cannot be included here due to lack of space.

4 Conclusions and future work

We have summarized and completed the comparisons between the multi-adjoint algebras and other general algebraic structures given in [5, 6]. By means of illustrative diagrams, we have shown that the use of these algebraic structures, in environments needed of residuated implications such as formal concept analysis, fuzzy logic programming and fuzzy rough sets, provides particular cases of biresiduated multi-adjoint algebras.

On this research topic, we have some prospects for future work. We will study more properties and applications of multi-adjoint algebras, which will also be enriched with a disjunction or summation operator.

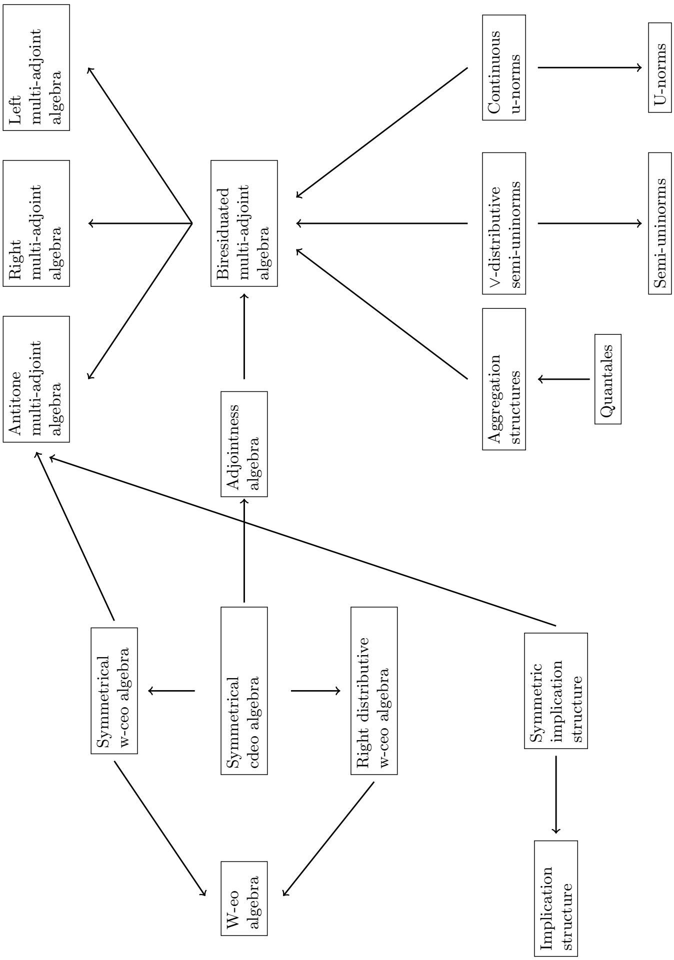

Fig. 1. General comparative diagram

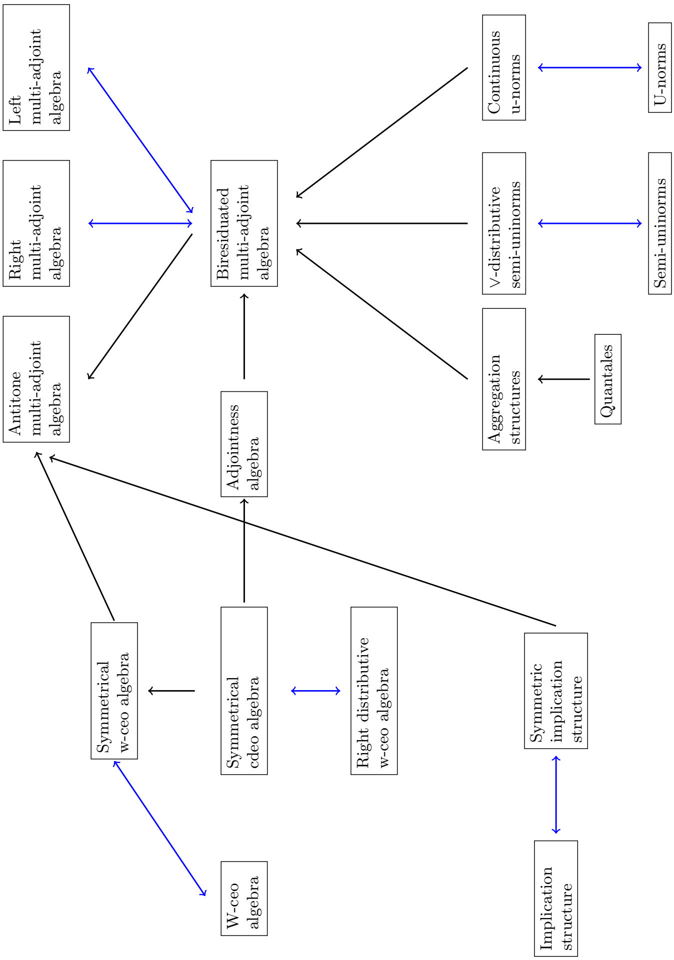

Fig. 2. Comparative diagram in the multi-adjoint concept lattices framework

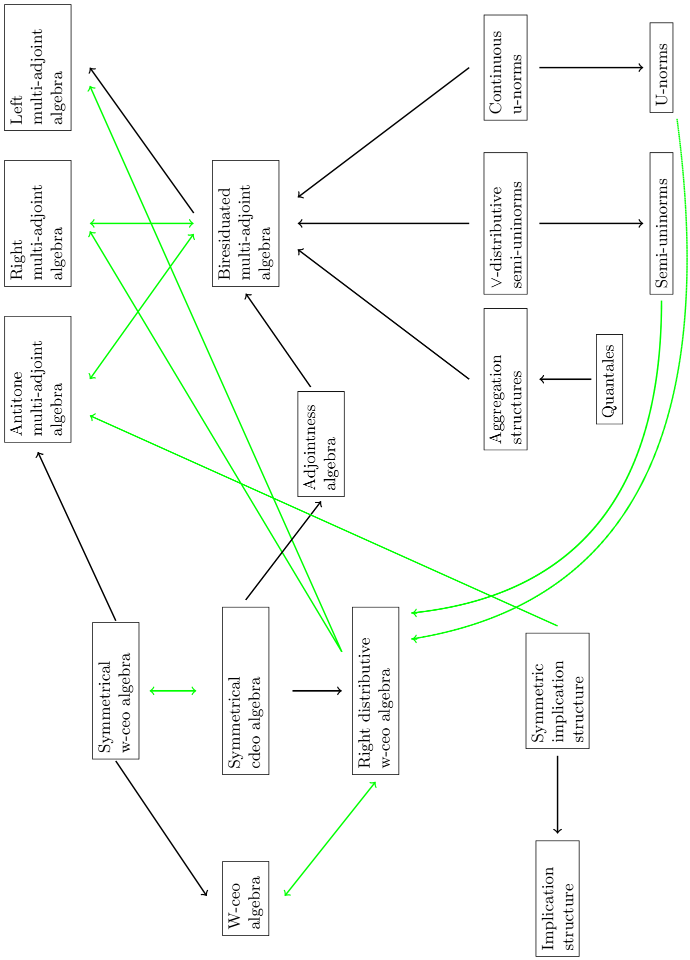

Fig. 3. Comparative diagram in the multi-adjoint rough sets framework

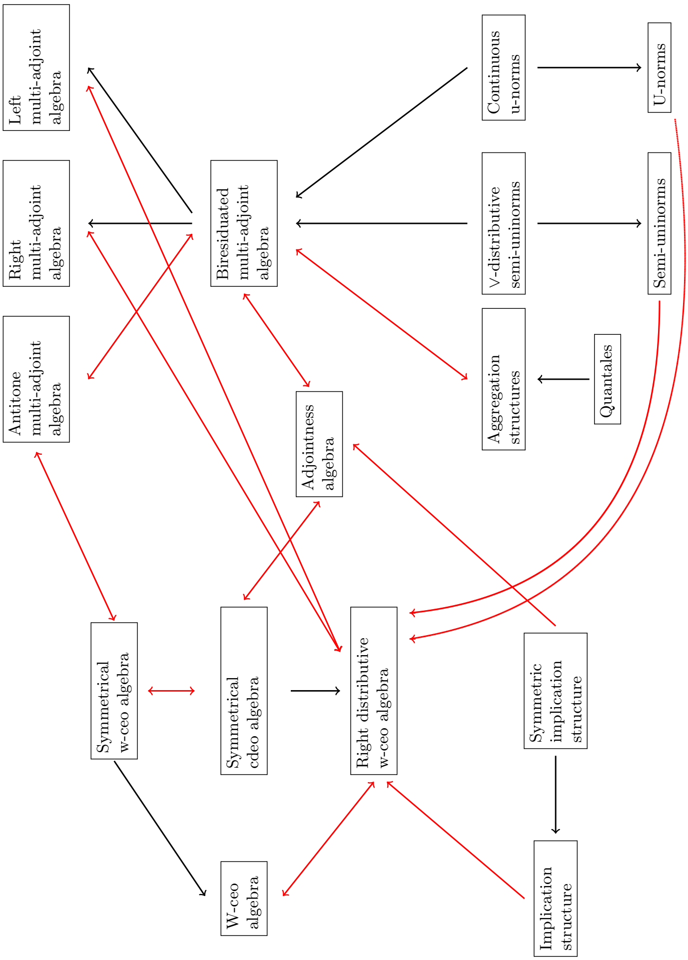

Fig. 4. Comparative diagram in the multi-adjoint logic programming framework

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