Aggregating T-Equivalence Relations

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Abstract. This contribution deals with the problem of aggregating T-equivalence relations, in the sense that we are looking for functions that preserve reflexivity, symmetry, and transitivity with respect to a given t-norm T. We obtain a complete description of those functions in terms of that we call T-triangular triplets. Any extra condition on the t-norm is assumed.

Keywords: t-norm, additive generator, T-equivalence relation, preserving properties, triangular triplet.

1 Introduction

An important issue in fuzzy sets theory is given by the concept of fuzzy equivalence relation, that measures the degree to which two points of an universe are indistinguishable, generalizing the concept of classical equivalence relation.
Fuzzy equivalence relations were introduced in [10] under the name *similarity relations* (with respect to the minimum). The generalization to t-norms was considered in [8]. Other names have been used for this concept in the literature (sometimes in connection with a specific t-norm), such as *likeness relation, indistinguishability relation, fuzzy equality, proximity relation*, etc. We shall use in the sequel the term T-equivalence relation which, in our opinion, reflects in the best way the mathematical motivation in the axioms we recall in Section 2. The term T-indistinguishability operator is also widely used in the literature [3, 6, 8, 9].

In many situations, there can be more than one T-indistinguishabilities defined on a universe and, in these cases, we may need to aggregate them. The most commmon way to aggregate a collection of T-equivalence relations is calculating their minimum, which also is a T-equivalence relation. However, somtimes this way of aggregating fuzzy relations leads to undesirable results since the Minimum only takes the smaller value for every couple into account and disregards the information of the other values. Similar drawback occurs when the Minimum is replaced by the t-norm T, specially when it is non-strict Archimedean. Thus, more general procedures to aggregate T-indistinguishability are needed.

Several authors have dealt the problem of the aggregation of some classes of fuzzy relations. With the same spirit as in [9, 6], we revisit this topic in order to give, whatever the t-norm T we use, a characterization of those functions that combine a collection of T-equivalence relations in a single one.

2 Preliminaries

Despite the fact that triangular norms (t-norms, for short) were first introduced in the context of statistical metric spaces [5], they have become an important tool in many other fields: fuzzy sets, decision making, statistics, theories of non-additive measures, etc. Comprehensive monographs on t-norms are [1, 4]. We use the set of axioms provided by Schweizer and Sklar [7]. Thus, our requirements on a t–norm $T : [0,1] \times [0,1] \to [0,1]$ for all a, b, c, d in $[0,1]$ are:

(i) $T(a,b) = T(b,a)$,
(ii) $T(T(a,b),c) = T(a,T(b,c))$,
(iii) $T(a,b) \leq T(c,d)$ whenever $a \leq c$ and $b \leq d$,
(iv) $T(a,1) = a$.

The following are the four basic t–norms T_M, T_L, T_P and T_D:

- $T_M(a,b) = \min(a,b)$ (minimum)
- $T_L(a,b) = \max(a+b-1,0)$ (Łukasiewicz t–norm)
- $T_P(a,b) = ab$ (product)
- $T_D(a,b) = \begin{cases} \min(a,b), & \text{if } a = 1 \text{ or } b = 1 \\ 0, & \text{otherwise} \end{cases}$ (drastic t–norm).

The associativity allows us to extend a t–norm in a unique manner to an n-ary operation in the usual way by induction, defining for each n-tuple $(a_1,\ldots,a_n)$, $n \geq 3$, $T(a_1,\ldots,a_n) = T(T((a_1,\ldots,a_{n-1}),a_n)$. Also by convention we formulate $T(a) = a$ for all a in $[0,1]$.

A t-norm T is called Archimedean if for each $a, b \in (0,1)^2$ there is $n \geq 1$ such that $T(\overbrace{a,\ldots,a}^{n}) < b$. One special property of a continuous Archimedean t-norm is that it is strictly increasing, except for the subset of $[0,1]^2$ where its value is 0. A remarkable fact is that any continuous Archimedean t-norm T can be expressed with the help of an additive generator [1]: $T(a,b) = g^{(-1)}(g(a)+g(b))$, where $g^{(-1)}$ is the pseudo-inverse[2] of g. Note that T_L and T_P are continuous Archimedean t-norms with additive generators $g(a) = 1-a$ and $g(a) = -\log a$ respectively.

Given a set X and a t-norm T, we say that a fuzzy relation $E : X \times X \longrightarrow [0,1]$ is a T-equivalence (or a T-indistinguishability) if for all x, y, z in X the following conditions hold:

[1] An additive generator is a continuous and strictly decreasing function $g : [0,1] \longrightarrow [0,\infty]$ such that $g(1) = 0$.
[2] $g^{(-1)}(t) = \sup \{c \in [0,1]; g(c) > t\}$, $\sup \emptyset = 0$

(i) $E(x,x) = 1$ (reflexivity)
(ii) $E(x,y) = E(y,x)$ (symmetry)
(iii) $E(x,y) \geq T(E(x,z), E(z,y))$ (T-transitivity)

As it is known, $E(x,y)$ is interpreted as the degree of indistinguishablity (or similarity) between x and y. The axioms of reflexivity, symmetry and T-transitivity fuzzify the ones of a crisp equivalence relation.

Given a left continuous t-norm T, we can introduce the function on $[0,1]^2$ defined by $\overrightarrow{T}(a,b)=$ sup $\{c \in [0,1]; T(a,c) \leq b\}$ that we call the residuation of T. It is easy to see that $\overrightarrow{T}$ is a T-preorder[3] on [0,1]. The biresiduation of T is the function on $[0,1]^2$ defined by $\overleftrightarrow{T}(a,b) = T(\overrightarrow{T}(a,b), \overrightarrow{T}(b,a))=$ min $(\overrightarrow{T}(a,b), \overrightarrow{T}(b,a))$. It is an important example of T-equivalence[4] on [0,1] that usually is called the natural T-equivalence associated to T, denoted by E_T. If T is a continuous Archimedean t-norm with additive generator g, then $E_T(x,y) = g^{(-1)}(\mid g(x) - g(y) \mid)$ for all $x, y \in [0,1]$.

Complete information on indistinguishability operators can be found in the recent monograph [6].

3 *T*-triangular triplets

Definition 1. *We say that a triplet* $(a,b,c) \in [0,\infty]^3$ *is triangular if* $a \leq b+c, b \leq a+c$ *and* $c \leq a+b$.
Being $\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c} \in [0,\infty]^m$, $m \geq 1$, *we say that* $(\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c})$ *is a (m-dimensional) triangular triplet if* (a_i, b_i, c_i) *is triangular for all* $i = 1, \ldots, m$, *where* $\boldsymbol{a} = (a_1, \ldots, a_m), \boldsymbol{b} = (b_1, \ldots, b_m), \boldsymbol{c} = (c_1, \ldots, c_m)$.

Note that if ($\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$) is triangular then so is any reordering of $\mathbf{a}, \mathbf{b}, \mathbf{c}$.

Proposition 1. *A triplet* $(a,b,c) \in [0,\infty]^3$ *is triangular if and only if it is of one of the following forms:*

(i) (∞, ∞, c) , $c \in [0,\infty]$

(ii) $c = \sqrt{a^2 + b^2 + \lambda ab}$, $\quad 0 \leq a, b < \infty$, $-2 \leq \lambda \leq 2$

Definition 2. *Let* T *be a t-norm. We say that* $(a,b,c) \in [0,1]^3$ *is* T*-triangular if* $a \geq T(b,c)$, $b \geq T(a,c)$, $c \geq T(a,b)$.
Being $\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c} \in [0,1]^m$, $m \geq 1$, *we say that* $(\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c})$ *is a (m-dimensional)* T*-triangular triplet if* (a_i, b_i, c_i) *is* T*-triangular for all* $i = 1, \ldots, m$, *where* $\boldsymbol{a} = (a_1, \ldots, a_m), \boldsymbol{b} = (b_1, \ldots, b_m), \boldsymbol{c} = (c_1, \ldots, c_m)$.

Proposition 2. *Let* T *be a left continuous t-norm. A triplet* $(a,b,c) \in [0,1]^3$ *is* T*-triangular if and only if* $T(a,b) \leq c \leq \overleftrightarrow{T}(a,b)$.

[3] Reflexive and T-transitive.
[4] Reflexive, symmetric and T-transitive.

Proof: Let us suppose first that (a, b, c) is T-triangular. From $T(a, c) \leq b$ and $T(b, c) \leq a$ we deduce $c \leq \overrightarrow{T}(a, b)$ and $c \leq \overrightarrow{T}(b, a)$, hence $c \leq \min\ (\overrightarrow{T}(a, b), \overrightarrow{T}(b, a)) = \overleftrightarrow{T}(a, b)$. Then $T(a, b) \leq c \leq \overleftrightarrow{T}(a, b)$. Reciprocally, assuming $T(a, b) \leq c \leq \overleftrightarrow{T}(a, b)$ we have to prove that (a, b, c) is T-triangular. From $c \leq \overrightarrow{T}(a, b)$ and, applying left continuity and monotonicity of T, we obtain $T(a, c) \leq b$. Similarly, from $c \leq \overrightarrow{T}(b, a)$ we obtain $T(b, c) \leq a$. Thus, the triplet (a, b, c) is T-triangular.

Remark 1 - *A triplet is T_M-triangular if and only if there exists a reordering (a, b, c) such that $a = b$ and $c \geq a$.*
- *A triplet is $T_L -$ triangular if and only if there exists a reordering (a, b, c) such that $max(a + b - 1, 0) \leq c \leq 1- \mid a - b \mid$.*
- *A triplet is $T_P -$ triangular if and only if it is $(0, 0, 0)$ or there exists a reordering (a, b, c) with $a, b, c > 0$, such that $ab \leq c \leq min(\frac{a}{b}, \frac{b}{a})$.*

Remark 2 *Denoting by $\Delta(T)$ the set of T-triangular triplets, observe that $T_1 \leq T_2$ implies $\Delta(T_1) \supset \Delta(T_2)$. Thus for any t-norm T we have $[0, 1]^3 \supset \Delta(T_D) \supset \Delta(T) \supset \Delta(T_M) \supset \{(a, a, a); a \in [0, 1]\}$.*

4 Aggregating *T*-equivalence relations

Definition 3. *We say that a function $F : [0, 1]^m \longrightarrow [0, 1], m \geq 1$, aggregates T-equivalence relations if for any set X and any collection of T-equivalence relations on X, $(E_1, \ldots, E_m)$, then $F(E_1, \ldots, E_m)$ is also a T-equivalence relation on X, where $F(E_1, \ldots, E_m)$ is the fuzzy binary relation $F(E_1, \ldots, E_m)(x, y) = F(E_1(x, y), \ldots, E_m(x, y))$.*

Remark 3 *Any t-norm T aggregates T-equivalence relations (for any $m \geq 1$).*

In [3] an aggregation method with respect to E_T is introduced. Being g an additive generator of T, then the corresponding aggregation function coincides with the quasi-arithmetic mean generated by g. Next proposition states that this function aggregates T-equivalence relations.

Proposition 3. *Let T be a continuous Archimedean t-norm with g as additive generator. The quasi-arithmetic mean generated by g, $M_g(a_1, \ldots, a_m) = g^{-1}(\frac{g(a_1)+\ldots+g(a_m)}{m})$, aggregates T-equivalence relations.*

The main result in this contribution is collected in the following proposition.

Proposition 4. *A function $F : [0, 1]^m \longrightarrow [0, 1], m \geq 1$, aggregates T-equivalence relations if and only if the following conditions hold:*

(i) $F(\overbrace{1, \ldots, 1}^{m}) = 1$.

(ii) F transforms $m-dimensional$ T-triangular triplets into $1-dimensional$ T-triangular triplets.

Proof: Firstly, let us suppose that F satisfies (i) and (ii)[5] and prove that $F(E_1,\ldots,E_m)$ is a T-equivalence relation for all T-equivalence relations $E_1,\ldots,E_m$. We know that, for each $i=1,\ldots,m$, it is $E_i(x,y)\geq T(E_i(x,z),E_i(z,y))$. Thus, the triplet $(\mathbf{a},\mathbf{b},\mathbf{c})$, where $a_i=E_i(x,y), b_i=E_i(x,z), c_i=E_i(z,y), i=1,\ldots,m$, is T-triangular, and from (ii) we have that $(F(\mathbf{a}),F(\mathbf{b}),F(\mathbf{c}))$ so is, and, consequently, we can write:
$F(E_1,\ldots,E_m)(x,y)=F(E_1(x,y),\ldots,E_m(x,y))\geq$
$T(F(E_1(x,z),\ldots,E_m(x,z)),F(E_1(z,y),\ldots,E_m(z,y)))=$
$T(F(E_1,\ldots,E_m)(x,z)),F(E_1,\ldots,E_m)(z,y))$.
Hence, $F(E_1,\ldots,E_m)$ is T-transitive. Reflexivity and symmetry follow immediately from (i) and symmetry of T.
Reciprocally, let us suppose that F aggregates T-equivalence relations. We have to prove that it satisfies conditions (i) and (ii). First, it is $F(1,\ldots,1)=1$ because $F(1,\ldots,1)=F(E(x,x),\ldots,E(x,x))=F(E,\ldots,E)(x,x)=1$, where E is a T-equivalence relation on a set X and $x\in X$. Now, let us prove that $(F(\mathbf{a}),F(\mathbf{b}),F(\mathbf{c}))$ is T-triangular whenever $(\mathbf{a},\mathbf{b},\mathbf{c})$ also is. There exist a set X , T-equivalence relations on X, $E_1,\ldots,E_m$, and $x,y,z\in X$ such that $E_i(x,y)=a_i,E_i(x,z)=b_i$ and $E_i(z,y)=c_i$ for all $i=1,\ldots,m$[6], then we can write
$F(\mathbf{a})=F(E_1(x,y),\ldots,E_m(x,y))=F(E_1,\ldots,E_m)(x,y)\geq$
$T(F(E_1,\ldots,E_m)(x,z),F(E_1,\ldots,E_m)(y,z))=T(F(\mathbf{b}),F(\mathbf{c}))$. Similarly, we obtain $F(\mathbf{b})\geq T(F(\mathbf{a},F(\mathbf{c}))$ and $F(\mathbf{c})\geq T(F(\mathbf{a},F(\mathbf{b}))$ and we have proven that $(F(\mathbf{a}),F(\mathbf{b}),F(\mathbf{c}))$ is T-triangular.

Next, an immediate consequence is shown.

Proposition 5. *A function $F:[0,1]^m\longrightarrow[0,1],m\geq1$, aggregates T_M-equivalence relations (similarity relations) if and only if it is increasing in each variable and $F(1,\ldots,1)=1$.*

Proof: Obvious, because F transforms $m-dimensional$ T_M-triangular triplets into $1-dimensional$ T_M-triangular triplets if and only if it is increasing in each variable.

When T is a continuous Archimedean t-norm, a characterization of those functions that aggregate T-equivalence relations can be formulated in terms of an additive generator of T as follows.

[5] If $(\mathbf{a},\mathbf{b},\mathbf{c})$ is a T-triangular triplet in $[0,1]^m$ then $(F(\mathbf{a}),F(\mathbf{b}),F(\mathbf{c}))$ is a T-triangular triplet in $[0,1]$.

[6] It is sufficient we consider a 3-element set $X=\{x,y,z\}$ and define $E_i(x,x)=E_i(y,y)=E_i(z,z)=1,E_i(x,y)=E_i(y,x)=a_i,E_i(x,z)=E_i(z,x)=b_i,E_i(z,y)=E_i(y,z)=c_i,i=1,\ldots,m$. Note that each E_i is a T-equivalence relation on X.

Proposition 6. *If T is a continuous Archimedean t-norm with additive generator g, then $F : [0,1]^m \longrightarrow [0,1]$ aggregates T-equivalence relations if and only if the function $G = gFg^{(-1)}$ transforms (ordinary) triangular triplets of $[0,\infty]^m$ (with elements in $[0,g(0)]^m$) into (ordinary) triangle triplets of $[0,\infty]$ (with elements in $[0,g(0)]$) and $G(0,\ldots,0) = 0$.*

Proof: Straightforward. Note that we consider $g^{(-1)}(a_1,\ldots,a_m) = (g^{(-1)}(a_1),\ldots, g^{(-1)}(a_m))$.

Example 1 *A function $F : [0,1]^m \longrightarrow [0,1], m \geq 1$, aggregates T_L-equivalence relations if and only if $G(a_1,\ldots,a_m) = 1-F(\max(1-a_1,0),\ldots,\max(1-a_m,0))$ transforms triangular triplets of $[0,\infty]^m$ (with elements in $[0,1]^m$) into triangle triplets of $[0,\infty]$ (with elements in $[0,1]$) and $G(0,\ldots,0) = 0$.*

Under increasingness, subadditivity[7] is equivalent to the property of transforming triangular triplets into triangle triplets.

Proposition 7. *Consider $G : [0,\infty]^m \longrightarrow [0,\infty]$. Then:*

(i) If G transforms triangular triplets of $[0,\infty]^m$ into triangular triplets of $[0,\infty]$ then it is subadditive.
(ii) If G is increasing and subadditive then it transforms triangular triplets of $[0,\infty]^m$ into triangular triplets of $[0,\infty]$.

Thus, from the two previous propositions, we can enunciate the following result.

Proposition 8. *An increasing function $F : [0,1]^m \longrightarrow [0,1]$, with $F(1,\ldots,1) = 1$, aggregates T-equivalence relations (T is a continuous Archimedean t-norm with additive generator g) if and only if the function $G = gFg^{(-1)}$ is subadditive.*

Consequences of the previous propositions are two known results concerning the role of weighted arithmetic means and ordered weighted arithmetic means (OWA operators) in this approach. More details on these classes of aggregation functions can be found in the recent monograph [2].

Proposition 9. *A weighted quasi-arithmetic mean $F(a_1,\ldots,a_m) = g^{-1}(\Sigma w_i g(a_i))$, where the components of the weighting list $(w_1,\ldots,w_m)$ are non-negative real numbers satisfying $\Sigma w_i = 1$ and g is an additive generator of a given t-norm T, aggregates T-equivalence relations.*

Proposition 10. *An ordered weighted quasi-arithmetic mean $F(a_1,\ldots,a_m) = g^{-1}(\Sigma w_i g(a_{(m-i)}))$, where $a_{(k)}$ denotes the k-largest input in the list $(a_1,\ldots,a_m)$ and g is an additive generator of a given t-norm T, aggregates T-equivalence relations.*

[7] $G(\mathbf{a}+\mathbf{b}) \leq G(\mathbf{a}) + G(\mathbf{b})$.

5 Conclusions

In this contribution we revisit the problem of the aggregation of T-equivalence relations. After introducing the concept of T-triangular triplet, we characterize those functions that transform any collection of T-equivalence relations into a single one. The interest of this characterization is that we do not assume any extra condition on the t-norm T.

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