Advances on a Combinatorial Optimization Approach for Political Districting in Mexico

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Abstract. In this paper we present advances on a combinatorial optimization model and algorithm for political districting in Mexico. We illustrate the challenges arisen when encoding the conforming of districts to administrative boundaries into an objective function. Our approach consists of two steps: First a partitioning process is performed on the set of indivisible territorial units, then Threshold Accepting —a variant of the Simulated Annealing heuristic— is employed in each class of the partition. The preliminary results yielded by a computer implementation are promising.

1 Introduction

Typically, scientific approaches to political districting across the world involve the solution of a combinatorial optimization problem whose aim is to minimize an objective function subject to a set of constraints (good references are [8, 6]). This problem consists in properly partitioning a geographical area (country, state, province, and the like) into a set of connected zones, called districts, by lumping together contiguous, indivisible territorial units. Thus defined, political districting belongs to the class of problems commonly known as territorial design [4].

Desired properties of districts often include a reasonable population balance, compact form, contiguity, and conformity to existing administrative boundaries.

Since the political districting problem is classified as *NP*-hard in the realm of theoretical computation, a variety of heuristic solution approaches have been proposed (for recent ones see [1, 7]).

We present here recent advances of an optimization model and a heuristic procedure in two steps that we propose to provide support along the political districting process in Mexico. Section 2 briefly introduces the particularities of the Mexican case. In Section 3 the combinatorial optimization model is described, which is followed by Section 4, where we propose a methodology in two steps that fully automatizes the generation of a district map. Finally, Section 5 summarizes the preliminary results we have obtained with our approach, and Section 6 presents our conclusions.

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2 Political districting in Mexico

Mexico is politically composed by 32 federal entities (*entities*, for short). Each entity is divided into *municipalities*, which in turn are divided into territorial units called *sections*, each having around 2 000 inhabitants. The number of sections per entity varies from 371 in tiny Colima, to 6 430 in Estado de México.

The National Electoral Institute (INE, by its acronym in Spanish) is in charge of producing both *federal* and *local* political districtings in Mexico. Presently, INE employs combinatorial optimization concepts and tools to help in the construction of a preliminary district map. Once this map is produced, other political actors (national political parties and minorities) have the opportunity to make observations and suggestions to eventually arrive, together with INE and guided by human judgment and common sense, to the definitive districts shape. The current districting process performed by INE, including the documents defining the project and its operational rules, can be consulted in [3].

From a technical point of view, the only difference between the federal and local processes is the number of districts in each entity; entities always have more local districts than federal ones.

In both the federal and local processes, districting consists on partitioning the sections' set of each entity so as each element in the partition forms a district, and corresponds to a connected geographic zone that satisfies a set of criteria. Among these criteria we describe below those which can be modeled as functions to optimize. For clarity sake, in the sequel the discussion is circumscribed to the districting in one single entity, and n stands for the desired number of districts.

- **Population Balance**. This criterion establishes that the districts' population must be as best balanced as possible; namely, if the entity population is P, then for $i = 1, \ldots, n$, the population $p(D_i)$ of district D_i shall be as close as possible to the entity *population average* $\bar{p} = \frac{P}{n}$.
  It is practically impossible to obtain a district map with perfect population balance. This is to say that the *population deviation* $(p(D_i)/\bar{p}) - 1$ is in general different from 0. Hence, for $i = 1, \ldots, n$, population deviations are allowed as long as $|p(D_i)/\bar{p} - 1| \leq 0.15$.

- **Municipal Integrity**. This criterion refers to conformity to administrative boundaries. Namely, it specifies to avoid as much as possible breaking up municipalities to form districts, establishing as a constraint a maximum of three municipal fractions in any district.

- **Geometric Compactness**. This criterion is reluctant to any kind of gerrymandering. It stipulates that the districts should be as 'compact' as possible; that is to say, 'weird' and 'octopus' district shapes are unwanted, being preferred those close to regular polygons.
  To meet his somewhat subjective criterion is a true challenge since the building blocks are the sections which most of time are not compact themselves.

3 A combinatorial optimization model

To construct a preliminary district map for each entity, incorporating the three criteria described in Section 2, we deal here with the objective function:

$$(\min)\ Z = \alpha_1 C_1 + \alpha_2 C_2 + \alpha_3 C_3, \tag{1}$$

where C_1, C_2, and C_3 are computed as shown below, and correspond, respectively, to population balance, municipal integrity, and geometric compactness. The weighting factors $\alpha_1, \alpha_2, \alpha_3$, set the criteria priority. In what follows, if X is a geographical zone with population $p(X)$, then its *population index* is denoted $I(X) = p(X)/\bar{p}$, where $\bar{p} = P/n$ is the population average.

- The population balance component is $C_1 = \frac{1}{n}\sum_{i=1}^{n}\left(\frac{1 - I(D_i)}{0.15}\right)^2$, where the denominator 0.15 corresponds to the maximum population deviation allowed in any district. Note that $C_1 = 0$ if and only if the population of every district is identical to $\bar{p}$. We establish $\alpha_1 = 1.0$ as weighting factor.
- The municipal integrity component is computed as $C_2 = \frac{1}{n}\sum_{i=1}^{n}\left(\frac{\gamma_i}{3}\right)^2$, where γ_i is the number of municipal fractions in district i. The denominator 3 is the maximum number of municipal fractions allowed in any district. Note that $C_2 = 0$ if and only if the number of municipal fractions in every district is zero. We set $\alpha_2 = 9.0$ to reflect the priority currently given by INE to the municipal integrity criterion.
  Assume the municipalities indexed by k. Then $p(M_k)$ is the population of municipality M_k, and $p(M_k \cap D_i)$ is the population of municipality M_k that belongs to district D_i. Thus, $\gamma_i = \sum_{k=1}^{m} F_{ik}$, for $i = 1, \ldots, n$, where
  $$F_{ik} = \begin{cases} 1 & \text{if } 0 < p(D_i \cap M_k) < p(M_k), \\ 0 & \text{otherwise.} \end{cases}$$
- Finally, we come to the geometric compactness component, computed as $C_3 = \frac{1}{n}\sum_{i=1}^{n}\left(\frac{1}{4}\frac{Q_i}{\sqrt{A_i}} - 1\right)$, where Q_i and A_i are the perimeter and the area of district i, respectively. Note that $C_3 = 0$ if and only if every district is a perfect square. As the geometric compactness criterion has the lowest priority we set $\alpha_3 = 0.4$.

Therefore, from (1) the combinatorial optimization problem $\mathcal{D}$ is:

$$(\min)\ Z = \frac{1}{n}\sum_{i=1}^{n}\left(\frac{1 - I(D_i)}{0.15}\right)^2 + \frac{9}{n}\sum_{i=1}^{n}\left(\frac{\gamma_i}{3}\right)^2 + \frac{0.4}{n}\sum_{i=1}^{n}\left(\frac{0.25Q_i}{\sqrt{A_i}} - 1\right), \tag{2}$$

subject to the following constraints for each district: (a) the population deviation must be lower or equal to 0.15, (b) the number of municipal fractions cannot exceed three, and (c) the district must be formed by contiguous, entire sections.

A district map that satisfies all constraints is called *feasible*.

4 Heuristic procedure

As it has been said, the combinatorial optimization problem as described in Section 3 is intractable. Further, the local search nature of the Simulated Annealing heuristic or any of its variants is such that its behavior is rather erratic when aimed to solve problem $\mathcal{D}$: Small changes on the current solution may cause undesirable large changes in the Z value.

To overcome this inconvenient we resort to an approach that closely follows the one presently used by INE: It proceeds in two steps guided by the divide-and-conquer principle. In the first step a non-biased, fully automatized strategic partition of the municipalities' set in the entity is determined. In the second step a variant of the Simulated Annealing heuristic (SA, see [5]) known as Threshold Accepting (TA), is independently applied in each class of the partition using the objective function Z without the municipal integrity component, namely,

$$(\min)\ Z' = \alpha_1 C_1 + \alpha_3 C_3. \tag{3}$$

With TA —as with any other local search heuristic— small changes in the current solution result in relatively small changes of Z' value. Further, TA has the advantage over classical SA of not needing the computation of exponentials. Its details can be consulted in [2].

In summary, we approach the optimal solution to $\mathcal{D}$ by means of a two-step process as indicated above. These steps are now sketched.

4.1 First step: The partitioning process

For a given set of adjacent municipalities U, let $\Phi(U)$ be the set of integers $\lambda \geq 1$ such that $0.85\lambda \leq I(U) \leq 1.15\lambda$. In words, for each $\lambda \in \Phi(U)$ it is possible to form λ feasible districts in U. For example, if population index $I(U) = 6.3$ then $\Phi(U) = \{6, 7\}$, because in the geographic zone U it is theoretically possible to form either 6 districts with index $\frac{6.3}{6} = 1.05$, or 7 districts with index 0.9; all within the ± 0.15 allowed deviation. Note that if, for example, $I(U) = 0.5$ or 1.6 then $\Phi(U)$ is empty.

In this step we process all *kernels* in the entity, namely, those municipalities M_k with $I(M_k) > 1.15$. For each kernel M_k we retain every possible grouping U with its adjacent municipalities whenever

$$\begin{array}{c} [\ \Phi(M_k) \text{ is empty, and } \Phi(U) = \{\lceil I(M_k) \rceil\}\ ] \\ \text{or } [\ \Phi(M_k) \neq \emptyset \text{ and } \max\{\lambda \in \Phi(U)\} \leq \max\{\lambda \in \Phi(M_k)\} + 1\ ]. \end{array} \tag{4}$$

An example of eq. (4) is when, say, $I(M_k) = 1.42$ for some $k \in \{1, \ldots, m\}$; hence $\Psi(M_k) = \emptyset$ and $\lceil I(M_k) \rceil = 2$. Assuming there is a grouping U with kernel M_k such that $I(U) = 2.08$ we get $\Phi(U) = \{2\}$. Two districts perfectly fit in U, each with index 1.04 (ideally). One district is formed by some sections of M_k, while the other district is formed by the rest of U.

The number of possible groupings for each kernel is exponential in the number of municipalities adjacent to it. In every entity in Mexico this figure is lower than 20, and in most it is below 10, so the computation time required to consider all groupings is negligible.

Clearly, in a feasible district map every kernel is broken down. On the other hand, it is expected that any municipality not being a kernel entirely belongs to one single district.

Denote $\mathcal{K}$ the set of all groupings U that can be formed from kernels, and denote $\mathcal{L}$ the set of municipalities M_k such that $0.85 \leq I(M_k) \leq 1.15$.

Note that any set of non-overlapping elements of $\mathcal{K} \cup \mathcal{L}$ induces a partition, say $\{U_1, \ldots, U_w\}$, on the set of municipalities in the entity, where U_j is a set of adjacent municipalities, for $j = 1, \ldots, w$. This partition is feasible if there exist $n_j \in \Phi(U_j)$, for $j = 1, \ldots, w$, satisfying $\sum_{j=1}^{w} n_j = n$. Each U_j is called a *part*.

Let $\mathcal{P}$ be the collection of all feasible partitions in this sense. Of course not all partitions in $\mathcal{P}$ have the same cardinality.

Our procedure heuristically selects in $\mathcal{P}$ a feasible partition P^* that, expectedly by means of the TA heuristic, will lead to district maps with low population deviation. At this point the geometric compactness criterion is not taken into account.

4.2 Second step: Threshold accepting

In this step a TA procedure is used to optimize the objective function (3) in each part of the partition P^* produced in the first step.

Initially, every kernel is conceptually subdivided in the sections composing it, so as any two sections of the kernel could belong to distinct districts; whereas the remaining municipalities are considered as indivisible elements. However, when no feasible solution can be obtained under these initial conditions, municipalities are iteratively broken down in sections, as necessary, until a feasible solution is found for each part.

Finally, the district map obtained for the entity from the union of the final solutions is evaluated with the objective function (2).

An advantage of our methodology is that, as TA runs independently in each part, it allows easy parallelism speeding up the process.

5 Results

We made computational experiments for several Mexican entities, so as to assess the performance of the two-steps procedure described in Section 4. To this aim, we compared its results with those obtained through the simpler strategy of using TA without the partitioning process (Section 4.1). In all cases either our methodology outperformed a direct use of TA, or the latter did not lead to feasible solutions, thus confirming its expected erratic behavior.

When using the partitioning process and then executing the TA heuristic in parallel on each part, there is always an execution time improvement compared to simply executing TA on the whole entity. This improvement varies from 15%-20% less execution time on small entities, to 60%-70% less execution time on more complex entities.

As an example of our sayings we present computer results for Aguascalientes, a relatively small entity with $n = 18$, and population average $\bar{p} = 1\,184\,996/18 = 65\,833$ (rounded). In this entity our methodology outperformed a direct use of TA.

For clarity sake, in all tables below the population deviations are shown multiplied by 100.

The partitioning process produced the partition displayed in Figure 1; its corresponding numerical information is summarized in Table 1. Note that the deviation in each district of any part is idealized: It is tacitly assumed that the second step (Section 4.2) evenly distributes the population of each part in the required number of districts. For example, part 01 has population 794 853 and is composed by 12 districts; the quotient of these figures is 66 237 (rounded). Thus, ideally, every district of this part has 66 237 inhabitants, hence $66\,237/65\,833 - 1 = 0.0061481$ as expected population deviation in each of the 12 districts.

Once the partition was determined the second step yielded the district map shown in Figure 2; Table 2 summarizes its corresponding numerical information. Note that each district is within the allowed deviation of $\pm 15\%$, and has at most three municipal fractions.

For comparison, Figure 3 displays the district map generated by means of the TA heuristic without the partitioning process. The corresponding numerical information is shown in Table 3.

6 Conclusions

Clearly, the three criteria we dealt with are in conflict with each other. Particularly when we consider the trade-off between the population balance and the municipal integrity criteria: District maps with low (high) population deviation usually have many (few) municipal fractions. In general, this is an observed characteristic of district maps obtained by the direct use of TA, with no partitioning whatsoever. Even worse, although TA produced district maps with low population deviation, they were unfeasible, namely, presenting more than three municipal fractions in many districts, and resulting in a staggeringly high number of total municipal fractions.

Notwithstanding the mentioned conflict, our approach was able of somewhat smoothing the effect of one criteria over the other: We got feasible district maps with low population deviation and few municipal fractions.

Furthermore, the proposed methodology yields significant gains in the total executing time required to produce a district map by processing each part of a partition in parallel.

Through computational experiments we verified that either the two-step approach outperformed the direct use of TA, or the latter did not lead to feasible solutions, thus confirming its expected erratic behavior.

A final remark: The automatized partitioning of the set of municipalities is done without human intervention, with no bias whatsoever, adding objectivity and impartiality to the political districting process.

We look forward to extend the applicability of our methodology, as well as to investigate improvements on the algorithms performance.

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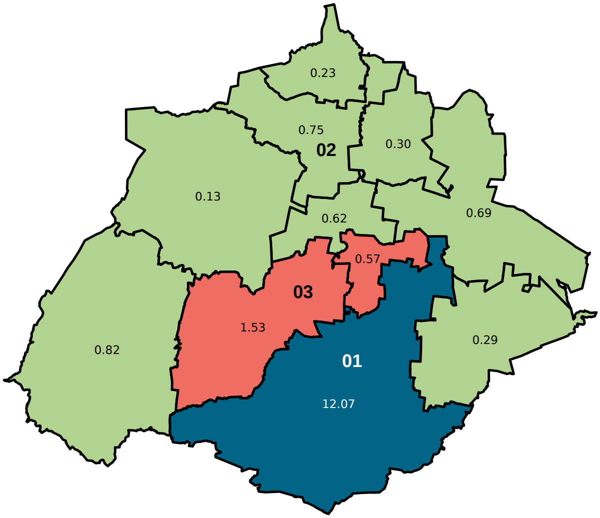

Fig. 1: Aguascalientes. The 3-partition generated by the first step of our methodology. Each municipality has its index printed inside.

Part	Districts	Municipalities	Population	Part pop.	Deviation
01	12	1	66 237	794 853	0.61481
02	4	8	63 020	252 083	-4.27179
03	2	2	69 030	138 060	4.85623
Total	18	11		1 184 996	1.89874

Table 1: Numerical information for the partition in Figure 1.

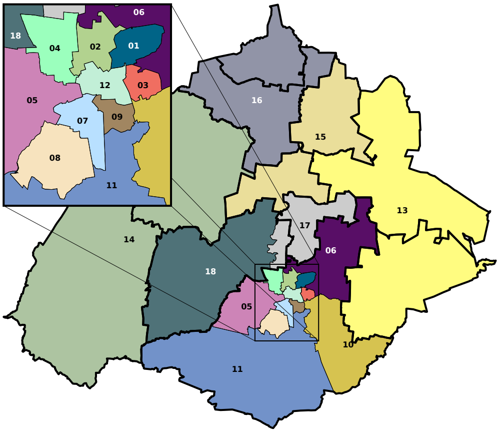

Fig. 2: Aguascalientes. District map generated by minimizing Z' on each part displayed on Figure 1: Thick lines are municipal boundaries; thin lines are district boundaries.

District	Z	C_1	C_2	C_3	**Population**	**Deviation**	**Fractions**
01	0.11952	0.00296	0.00617	0.15253	66 370	0.81570	1
02	0.28964	0.00012	0.00617	0.58492	65 939	0.16101	1
03	0.15386	0.01928	0.00617	0.19758	67 204	2.08254	1
04	0.21653	0.00013	0.00617	0.40211	65 722	-0.16861	1
05	0.18990	0.00014	0.00617	0.33549	65 951	0.17924	1
06	0.47450	0.00025	0.00617	1.04674	65 678	-0.23544	1
07	0.21674	0.00050	0.00617	0.40170	66 054	0.33570	1
08	0.14525	0.00042	0.00617	0.22319	66 036	0.30836	1
09	0.15925	0.00308	0.00617	0.25154	66 381	0.83241	1
10	0.28466	0.01154	0.00617	0.54391	66 894	1.61165	1
11	0.29191	0.01118	0.00617	0.56295	66 877	1.58583	1
12	0.17864	0.00008	0.00617	0.30753	65 747	-0.13063	1
13	0.41773	0.02246	0.00000	0.98817	64 353	-2.24811	0
14	0.32530	0.10214	0.00000	0.55789	62 677	-4.79395	0
15	0.72411	0.25147	0.00000	1.18158	60 881	-7.52206	0
16	0.26515	0.02829	0.00000	0.59215	64 172	-2.52305	0
17	0.54820	0.13940	0.00617	0.88311	69 520	5.60053	1
18	0.37436	0.07515	0.00617	0.60913	68 540	4.11192	1
Total	5.37525	0.66858	0.08642	9.82223	1 184 996	1.95815	14

Table 2: Numerical results of the district map in Figure 2, using Z as objective function.

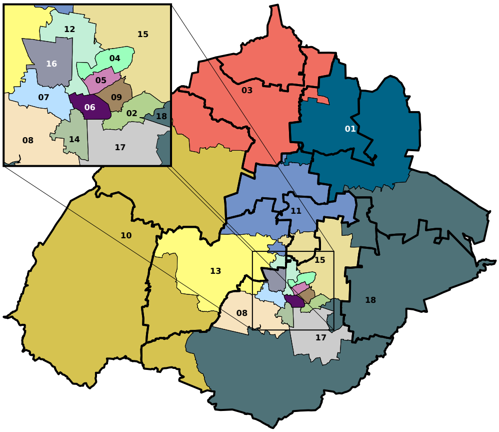

Fig. 3: Aguascalientes. District map generated by minimizing Z on the whole entity, without previous partitioning.

District	Z	C_1	C_2	C_3	Population	Deviation	Fractions
01	0.41243	0.00003	0.01852	0.61435	65 783	-0.07595	3
02	0.25388	0.00302	0.00617	0.48824	65 290	-0.82481	1
03	0.32161	0.01050	0.01235	0.49998	66 845	1.53722	2
04	0.23640	0.00002	0.00617	0.45206	65 793	-0.06076	1
05	0.15140	0.00084	0.00617	0.23750	66 120	0.43595	1
06	0.10885	0.01017	0.00617	0.10779	66 829	1.51292	1
07	0.27410	0.01194	0.00617	0.51651	66 912	1.63900	1
08	0.26536	0.00011	0.00617	0.52424	65 937	0.15798	1
09	0.14743	0.00369	0.00617	0.22046	65 233	-0.91140	1
10	0.47696	0.00517	0.01852	0.76281	65 123	-1.07849	3
11	0.50107	0.01641	0.01852	0.79497	64 568	-1.92153	3
12	0.50746	0.01409	0.01235	0.95565	67 005	1.78026	2
13	0.34795	0.00505	0.01235	0.57947	65 131	-1.06633	2
14	0.23970	0.00701	0.00617	0.44282	66 660	1.25621	1
15	0.42018	0.01282	0.01852	0.60173	64 715	-1.69824	3
16	0.16040	0.00188	0.00617	0.25742	66 261	0.65013	1
17	0.25751	0.00000	0.00617	0.50487	65 844	0.01671	1
18	0.61135	0.00805	0.01235	1.23048	64 947	-1.34583	2
Total	5.69402	0.11081	0.18519	9.79135	1 184 996	0.99832	30

Table 3: Numerical results of the district map in Figure 3.