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INTEGRABILITY, RATIONAL SOLITONS AND SYMMETRIES FOR NONLINEAR SYSTEMS IN BIOLOGY AND MATERIALS PHYSICS

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Abstract

Nonlinear systems emerge as an active research topic of growing interest during the last decades due to their versatility when it comes to describing physical phenomena. Such scenarios are typically modelled by nonlinear differential equations, whose mathematical structure has proved to be incredibly rich, but highly nontrivial to treat. In particular, a narrow but surprisingly special group of this kind stands out: the so-called integrable systems.

The present doctoral thesis focuses on the study of some of the extraordinary properties observed for integrable systems. The ultimate purpose of this dissertation lies in providing a unified theoretical framework that allows us to approach nonlinear differential equations that may potentially be considered as integrable. In particular, their integrability characterization is addressed by means of Painlevé analysis, in conjunction with the subsequent quest of the associated spectral problems and the identification of analytical solutions of solitonic nature. The singular manifold method together with auto-Bäckund and Darboux transformations play a critical role in this setting. In addition, a complementary methodology based on Lie symmetries and similarity reductions is proposed so as to analyze integrable systems by studying the symmetry properties of their associated spectral problems.

Taking the ubiquitous nonlinear Schrödinger equation as the starting point, we have investigated several integrable generalizations of this equation that possess copious applications in distinct scientific fields, such as Mathematical Physics, Materials Sciences and Biology.

Keywords

Integrable systems, nonlinear differential equations, Painlevé Property, Singular manifold method, Lax pairs, Darboux transformations, soliton solutions, rational solitons, lump solutions, Lie symmetries, Lie algebras, similarity reductions.

Resumen

Los sistemas no lineales constituyen un tema de investigación de creciente interés en las últimas décadas dada su versatilidad en la descripción de fenómenos físicos en diversos campos de estudio. Generalmente, dichos fenómenos vienen modelizados por ecuaciones diferenciales no lineales, cuya estructura matemática ha demostrado ser sumamente rica, aunque de gran complejidad respecto a su análisis. Dentro del conjunto de los sistemas no lineales, cabe destacar un reducido grupo, pero a la vez selecto, que se distingue por las propiedades extraordinarias que presenta: los denominados sistemas integrables.

La presente tesis doctoral se centra en el estudio de algunas de las propiedades más relevantes observadas para los sistemas integrables. En esta tesis se pretende proporcionar un marco teórico unificado que permita abordar ecuaciones diferenciales no lineales que potencialmente puedan ser consideradas como integrables. En particular, el análisis de integralidad de dichas ecuaciones se realiza a través de técnicas basadas en la Propiedad de Painlevé, en combinación con la subsiguiente búsqueda de los problemas espectrales asociados y la identificación de soluciones analíticas de naturaleza solitónica. El método de la variedad singular junto con las transformaciones de auto-Bäcklund y de Darboux jugarán un papel fundamental en este estudio. Además, también se lleva a cabo un análisis complementario basado en las simetrías de Lie y reducciones de similaridad, que nos permitirán estudiar desde esta nueva perspectiva los problemas espectrales asociados.

Partiendo de la archiconocida ecuación de Schrödinger no lineal, se han investigado diferentes generalizaciones integrables con numerosas aplicaciones en diversos campos científicos, como la Física Matemática, Física de Materiales o Biología.

Palabras clave

Sistemas integrables, ecuaciones diferenciales no lineales, Propiedad de Painlevé, método de la variedad singular, par de Lax, transformaciones de Darboux, soluciones solitónicas, solitones racionales, soluciones de tipo *lump*, simetrías de Lie, álgebras de Lie, reducciones de similaridad.

Alice started to her feet, for it flashed across her mind that she had never before seen a rabbit with either a waistcoat-pocket, or a watch to take out of it, and burning with curiosity, she ran across the field after it, and fortunately was just in time to see it pop down a large rabbit-hole under the hedge.

In another moment down went Alice after it, never once considering how in the world she was to get out again.

<div align="right">

Lewis Carroll
Alice's Adventures in Wonderland
Chapter 1 – Down to the Rabbit-Hole

</div>

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Abbreviations

AKNS (system)	Ablowitz-Kaup-Newell-Segur (system)
ARS (algorithm)	Ablowitz-Ramani-Segur (algorithm)
BKP (equation)	Bogoyanlevski-Kadomtsev-Petviashvili (equation)
CLL (equation)	Chen-Lee-Liu (equation)
DNLS (equation)	Derivative nonlinear Schrödinger (equation)
DS (equation)	Davey–Stewartson (equation)
GI (equation)	Gerdjikov-Ivanov (equation)
IST	Inverse Scattering Transform
KdV (equation)	Korteweg-de Vries (equation)
KN (system)	Kaup-Newell (system)
KP (equation)	Kadomtsev–Petviashvili (equation)
mKdV (equation)	Modified Korteweg-de Vries (equation)
NLS (equation)	Nonlinear Schrödinger (equation)
NNV (equation)	Nizhnik-Novikov-Veselov (equation)
ODE(s)	Ordinary differential equation(s)
PDE(s)	Partial differential equation(s)
PP	Painlevé Property
SMM	Singular manifold method
SOC	Spin-orbit coupling
WKI (equation)	Wadati-Konno-Ichikawa (equation)
WTC (algorithm)	Weiss-Tabor-Carnevale (algorithm)

Chapter 1

Introduction

Nonlinear systems emerge as an active research topic of growing interest during the last decades, due to their versatility when it comes to describing physical phenomena [57, 289, 381]. Such scenarios are typically modelled by nonlinear differential equations, whose mathematical structure has proved to be incredibly rich and fascinating, but highly nontrivial to treat. In particular, a narrow but surprisingly special group of this kind stands out regarding their truly remarkable properties: the so-called *integrable systems*.

Nevertheless, a comprehensive definition of the term *integrable* has turned out to be elusive. Therefore, there is not a unified notion of integrability, leading to different interpretations for what the concept of integrable system may be [3, 39, 218, 262, 323].

1.   Integrable systems

An intuitive but naive notion of *integrability* evokes, in some sense, the concepts of *exact solvability* or *regularity* for the solutions of a given system. Conversely, the term *nonintegrable* is generally associated to an *irregular* or *chaotic behaviour* for such system.

The realm of integrable systems flourished in the context of Classical Mechanics, as an attempt to find exact solutions to Newton's equations [45], frequently described by ordinary differential equations (ODEs). The first notions of integrability are intimately related to the study of dynamical systems in such scenarios, as a synonym of *solvability by quadratures*. Hamiltonian systems play a crucial role in this framework, where the term integrability refers to integrability in the Liouville sense [276]. Loosely speaking, this approach asserts that the identification of a sufficient number of analytic independent first integrals in involution enables the reduction of the system to a form that can be explicitly integrated by quadratures. This notion of

integrability is framed within the Liouville-Arnold Theorem [39, 276] and the action-angle variables formalism [190, 262].

1.1. Criteria of integrability for nonlinear systems

The next step lies in the consideration of systems with infinite degrees of freedom, described by partial differential equations (PDEs). The addition of further dimensions gives rise to a higher level of complexity in terms of the mathematical characterization of the model, providing in turn a riveting and captivating dynamics with astonishing properties. For example, there exist nonlinear PDEs that depict a regular behaviour in their independent variables for any initial condition. These observations enable to induce an analogy of the concept of integrability for this type of systems, which are worth studying.

Generally, nonlinear systems termed as integrable are proved to present many common remarkable properties in their mathematical description. One of the principal indications that certain nonlinear PDEs might belong to this selected group lies with the so-called *solitary waves*, solutions that represent the propagation of waves with a permanent profile, among other peculiar properties. The experimental discovery of such structures is credited to the Scottish engineer John Scott Russell in 1834 [366], when he noticed a particularly rare travelling wave along a narrow canal, which he described as

> I was observing the motion of a boat which was rapidly drawn along a narrow channel by a pair of horses, when the boat suddenly stopped - not so the mass of water in the channel which it had put in motion; it accumulated round the prow of the vessel in a state of violent agitation, then suddenly leaving it behind, rolled forward with great velocity, assuming the form of a large solitary elevation, a rounded, smooth and well-defined heap of water, which continued its course along the channel apparently without change of form or diminution of speed. I followed it on horseback, and overtook it still rolling on at a rate of some eight or nine miles an hour, preserving its original figure some thirty feet long and a foot to a foot and a half in height. Its height gradually diminished, and after a chase of one or two miles I lost it in the windings of the channel. Such, in the month of August 1834, was my first chance interview with that singular and beautiful phenomenon which I have called the Wave of Translation [...] (Russell, 1844, pp. 319–320)

The full mathematical explanation of such phenomenon was later given in 1895 by Korteweg and de Vries [249], who derived the following nonlinear dispersive equation as a model describing the wave propagation on shallow fluid surfaces

$$u_t + u_{xxx} + 6uu_x = 0, \tag{1.1}$$

which constitutes the celebrated Korteweg-de Vries (KdV) equation.

This subject remained forgotten until the second half of the XX century, when the mathematical characterization of nonlinear differential equations reemerged with renewed vigour. It was not until 1955, with the studies on nonlinear chains of coupled oscillators conducted by Fermi, Ulam and Pasta [164], that nonlinear systems became a topic of interest for the scientific community. The previous results on the Fermi-Ulam-Pasta system motivated Zabusky and Kruskal to revisit this model in 1965 [426]. In the continuous limit of such system, they rediscover the KdV equation (1.1), obtaining numerical solutions that presented an analogous behaviour to the nonlinear waves observed by Russell [366]. One of the most captivating features of these solitary waves is found by analyzing their scattering properties, finding a nontrivial behaviour more similar to particles than waves. Solitary waves of this kind are localized structures that propagate with constant and related amplitude and velocity. They may strongly interact among each other with the property that they emerge unchanged from the collision apart from a phase shift in their propagation directions. Then, Zabusky and Kruskal introduced, by consistency with this particle-like behaviour, the term *soliton* to name this type of nonlinear waves [426]. In particular, these authors solved the initial value problem for the KdV equation (1.1) and found the explicit expression for the soliton solution of this equation

$$u(x,t) = 2k^2 \operatorname{sech}^2\Big[k\left(x - 4k^2t\right) + \varphi_0\Big], \tag{1.2}$$

where $\{k, \varphi_0\}$ are arbitrary constants and it is easy to observe that the amplitude $a = 2k^2$ and the propagation velocity $v = 4k^2$ are related via the wavenumber k.

The discovery of such structures gave rise to the beginning of a new area of study in the field of Mathematical Physics, concerning nonlinear differential equations with soliton solutions. Since then, a plethora of nonlinear PDEs have been reported to possess soliton-like solutions, numerous mathematical techniques have been developed in order to understand them and countless applications have arisen in different fields [3, 13, 130, 247, 260, 355].

Within the broad spectrum of mathematical tools employed for the analysis of integrable systems, it is worth highlighting two of them, which may be regarded as two critical pillars of the field itself:

- **The Inverse Scattering Transform** (IST), first developed by Gardner, Green, Kruskal and Miura in 1967 [183], as an alternative method to solve the KdV equation. This method can be understood as a nonlinear generalization of the Fourier transform. The main idea lies in reconstructing the time evolution of the potential $u(x,t)$ from analyzing the time evolution of the associated scat-

tering data for the known potential $u(x, 0)$. The *inverse scattering* procedure requires to solve the Gel'fand–Levitan–Marchenko integral equation [184].

- **Lax formalism**, first introduced by Lax in 1968 [263] in the context of analysis of nonlinear evolution differential equations in $1 + 1$ dimensions. It is possible to express certain nonlinear $(1 + 1)$-PDEs as the compatibility condition of a Lax pair, *i.e.* a pair of linear differential operators such that the corresponding eigenvalue problem, encoded in the spectral parameter λ, is independent of the evolution variable.

Both approaches are considered two solid and complementary proofs for the integrability of a given nonlinear differential equation. They have also been applied to a large number of nonlinear equations with soliton solutions [3, 13, 130, 247, 263, 264].

The rapid development and generalization of the IST method together with the Lax formalism aroused a myriad of new approaches to address this kind of nonlinear models. Hence, several criteria of integrability for such systems may be constructed based on their associated exceptional properties, among which it is worth highlighting:

1. Existence of soliton solutions [3, 130]

2. Solvability by the IST method [1, 3, 13, 247, 428]

3. Existence of the associated spectral problem (Lax pair) [1, 3, 263]

4. Darboux transformations [296]

5. Transformations among families of integrable equations: Bäcklund and Miura transformations [33, 260, 361, 362], hodograph transformations [97, 158], reciprocal transformations [217, 243, 358, 359], etc.

6. (Multi-)Hamiltonian formalism and hereditary recursion operators [73, 169, 383]

7. Infinite number of conservation laws and infinite-dimensional Lie algebras [13, 130, 398]

8. Formulation in terms of Hirota's bilinear formalism [204, 211–213]

9. The nonlinear system is said to possess the Painlevé Property (up to a change of variables) [9–12, 332, 417]

A separate issue regarding the integrability characterization is the inspection of the analytical and geometric properties of such differential equations. This approach is inspired by the works of the famous mathematicians Kovalevskaya, Fuchs or Painlevé,

just to mention a few, on the study of singularities in the complex plane for differential equations [175, 250, 251, 330, 331]. The major breakthrough regarding this topic is due to Painlevé and collaborators [176, 181, 332], whose investigations yielded an analytical criterion of integrability for ODEs encoded in the so-called *Painlevé Property* (PP). His conjecture mainly states that the solutions of integrable ODEs exhibit no movable branch points, in other other words, solutions must be single-valued in the initial conditions, but in an arbitrary number of isolated movable poles. This approach provides a straightforward methodology, which may be algorithmically implemented, to confirm whether or not a given ODE has the PP [9–12]. Besides, it can be easily generalized to the study of PDEs [417]. The aforementioned algorithms, based on the Painlevé Property, are often called *Painlevé tests.* The key advantage of such tests is that they allow us to check the integrability of a given nonlinear differential equation *a priori*, without the need to solve it.

There is still no rigorous proof of the relation among the integrability criteria based on the PP and the unique features arising from soliton theory. Nevertheless, there is ample evidence of the validity of both approaches when it comes to identifying the integrability of nonlinear systems. Besides, there exist several procedures, such as the singular manifold method (SMM) [410], that may shed some light about the conjunction of these two conceptions. This particular methodology constitutes a central axis of this doctoral thesis, as it will illustrated hereinafter.

1.2. Lie symmetries

Symmetry analysis emerges as one of the most fruitful techniques to study (nonlinear) differential equations and derive exact analytical solutions for such equations. This procedure was first developed by the Norwegian mathematician S. Lie in the second half of the XIX century [273–275]. In his seminal work, S. Lie introduced the notion of continuous groups, now known as (local) Lie groups, to unify and extend several specialized methods to simplify problems in partial differential equations and geometry. The basis of the theory of Lie symmetries lies in the invariance of differential equations under one-parameter groups of transformations of their variables. Then, in this context, Lie groups naturally arise as local groups of transformations acting on manifolds, such that any point is mapped into another point of the same manifold. Besides, local properties of such transformation are sufficient to fully characterize the transformation globally, by the analysis of the associated Lie algebra. This method has been extensively investigated along the last decades and classical references about this subject may be found in [52, 323, 378].

A standard method to find solutions of differential equations can be implemented by using Lie's method. This procedure has numerous applications in the context of

(nonlinear) differential equations, among which it is worth highlighting:

- *Integration of ordinary differential equations*: symmetry groups of ODEs can be used to reduce the order of the equation, and for first-order ODEs, the reduction process leads to the complete integration (by quadratures) of the equation.

- *Reduction of partial differential equations*: symmetry groups of PDEs allow to reduce the total number of independent variables by one. If the symmetry group is large enough, this process enables us to achieve a reduced problem expressed in terms of ODEs, which can be integrated by the usual procedures.

- *Obtain new solutions from old solutions*: symmetry groups establish a map between solutions for differential equations. Thus, the application of the symmetry group to a known solution of a given differential equation will yield a family of new solutions.

- *Classification of equations*: symmetry groups can be used to classify differential equations into equivalence classes [52].

- *Conservation laws*: there exists an intimate connection between symmetries and conservation laws, due to Noether's Theorem [318, 394, 395].

One of the underlying reasons for the tremendous success of Lie's symmetry method, apart from its numerous applications, relies on the fact that it provides a consolidated framework to study any sort of problems involving differential equations. Lie symmetry analysis does not depend upon whether or not the system is integrable.

Nevertheless, of avid interest will be the application of Lie symmetries to nonlinear integrable systems, specially when it comes to analyzing the associated spectral problems. In particular, the conjunction of the Lie symmetry formalism together with their distinguished properties provides the perfect complement to give a unified approach to these systems.

2. Structure of the thesis and further remarks

The present doctoral thesis focuses on the study of some of these extraordinary properties observed for integrable systems. The ultimate purpose of this dissertation lies in providing a unified theoretical framework that allows us to approach nonlinear differential equations that may potentially be considered as integrable systems. In particular, the integrability characterization for such systems is addressed

by means of the quest of the associated spectral problems, in conjunction with the identification of analytical solutions of solitonic nature. Auto-Bäckund and Darboux transformations play a critical role in this approach. In addition, a complementary methodology based on Lie symmetries and similarity reductions is proposed so as to analyze integrable systems by studying the symmetry properties of their associated spectral problems.

Among the celebrated classical nonlinear PDEs spread out in the realm of integrable systems, we find of particular interest the so-called *nonlinear Schrödinger* (NLS) equation [3, 13, 130], which can be written as

$$iu_t + u_{xx} + \kappa|u|^2u = 0, \tag{1.3}$$

where κ is a real parameter and $u(x,t)$ is a complex field.

This equation is found to be integrable based on several of the criteria established above and it also displays a rich spectrum of soliton solutions [13, 431, 432]. NLS equation first arose in the context of Fluid Dynamics [202,380,427], and since then, it has proved to be present in countless scenarios describing the most diverse nonlinear phenomena [221,222,240,353,387]. More specifically, we will be truly interested in its numerous applications in Condensed Matter Physics, with great relevance in Solid-State Physics [2, 133, 335, 382, 388] or Bose-Einstein condensates [128, 194, 225, 344]. This nonlinear system also stands out in the field of Biological Physics, with the modelization of energy transport phenomena in α-helical molecules by means of Davydov's theory [114,122,123]. Its overwhelming versatility makes this equation an ideal starting point on which construct integrable generalizations with innumerable applications in diverse related disciplines. Hence, NLS equation (1.3) will be considered as the central pillar of the nonlinear integrable systems to be investigated in the course of the present doctoral thesis.

The contents of this thesis are organized as follows. In Chapter 2 we provide a general overview about some of the remarkable properties for nonlinear integrable systems. In particular, we characterize a criterion of integrability for such systems based on the so-called Painlevé Property. Besides, we define an algorithmic procedure which lays the foundations of the methodology of analysis to be applied in the ensuing Chapters. Chapters 3 and 4 are devoted precisely to the practical applications of such machinery to integrable systems in $1 + 1$ dimensions and $2 + 1$ dimensions, respectively. On the other hand, Chapter 5 is aimed at the theoretical description from a geometric point of view of Lie symmetries and their applications to differential equations. Similarity reductions and the computation of group invariant solutions are also duly emphasized. Chapter 6 therefore addresses the applications of Lie's formalism to diverse integrable models in several dimensions. Appendices A and B

include additional and supplementary calculations for Chapters 5 and 6, respectively.

It should be duly stressed that the research conducted through this doctoral work is purely analytic, all the expressions and solutions arising from the diverse analyses are exact. Due to the high computational complexity, certain symbolic calculus packages have been employed. MAPLE constitutes our principal source in this matter, to deal with both intermediate calculations and plots for the derived solutions. MATHEMATICA has been used as support to obtain certain graphical representations, whilst the computer algebra system REDUCE has helped with the calculations involving Lie symmetries.

Publications

The central results of this thesis have led to the following scientific publications, sorted chronologically in the list hereafter:

1. Lumps and rogue waves of generalized Nizhnik–Novikov–Veselov equation,
   P. Albares, P. G. Estévez, R. Radha and R. Saranya,
   Nonlinear Dynamics **90**, 2305–2315 (2017)
   https://doi.org/10.1007/s11071-017-3804-7

2. Classical Lie symmetries and reductions for a generalized NLS equation in $2+1$ dimensions,
   P. Albares, J. M. Conde and P. G. Estévez,
   Journal of Nonlinear Mathematical Physics **24**, Suppl. 1, 48–60 (2017)
   DOI: 10.1080/14029251.2017.1418053

3. Solitons in a nonlinear model of spin transport in helical molecules,
   P. Albares, E. Díaz, J. M. Cerveró, F. Domínguez-Adame, E. Diez and P. G. Estévez
   Physical Review E **97**, 022210 (2018)
   https://doi.org/10.1103/PhysRevE.97.022210

4. Spin dynamics in helical molecules with non-linear interactions,
   E. Díaz, P. Albares, P. G. Estévez, J. M. Cerveró, C. Gaul, E. Diez and F. Domínguez-Adame
   New Journal of Physics **20**, 043055 (2018)
   https://doi.org/10.1088/1367-2630/aabb91

5. Spectral problem for a two-component nonlinear Schrödinger equation in 2+1 dimensions: Singular manifold method and Lie point symmetries,
   P. Albares, J. M. Conde and P. G. Estévez

Applied Mathematics and Computation **355**, 585–594 (2019)
https://doi.org/10.1016/j.amc.2019.03.013

6. Derivative non-linear Schrödinger equation: Singular manifold method and Lie symmetries,
   P. Albares, P. G. Estévez and J. D. Lejarreta
   Applied Mathematics and Computation **400**, 126089 (2021)
   https://doi.org/10.1016/j.amc.2021.126089

where complementary results to the latter publication are given in

7. Integrability and rational soliton solutions for gauge invariant derivative nonlinear Schrödinger equations,
   P. Albares
   In A. Espuny Díaz, E. López-Navarro, A. Navarro-Quiles (Eds.), *TEMat monográficos*, Volume 2, Proceedings of the 3rd BYMAT Conference, Asociación Nacional de Estudiantes de Matemáticas, pp. 35–38 (2021)
   https://temat.es/monograficos/article/view/vol2-p35

Additional related work to the present research, which has not been enclosed in this dissertation, can be found in

8. Reciprocal transformations and their role in the integrability and classification of PDEs
   P. Albares, C. Sardón and P. G. Estévez
   In N. Euler, M. C. Nucci (Ed.), *Nonlinear Systems and Their Remarkable Mathematical Structures*, Volume 2, Chapman and Hall & CRC Press, pp. 1–28 (2019)

9. Miura-Reciprocal Transformation and Symmetries for the Spectral Problems of KdV and mKdV,
   P. Albares and P. G. Estévez
   Mathematics **9**(9), 916 (2021)
   https://doi.org/10.3390/math9090926

Chapter 2

Painlevé Property and the singular manifold method for differential equations

This Chapter attempts to give a thorough outline about some of the remarkable properties stated in Chapter 1 for nonlinear integrable PDEs. More specifically, the particular techniques to be employed in the future research of this thesis should be duly emphasized. The structure of this Chapter is given as follows.

The first Section provides a general setting concerning the fundamental aspects related to the Painlevé Property for differential equations. This purely analytic condition allows to construct an extremely effective algorithmic criterion to evaluate and successfully identify the integrability for such equations. Sections 2 and 3 are devoted to the second and third keystones in the integrability analysis for differential equations: the singular manifold method and Lax pairs. The relation between them is deeply examined, together with the so-called binary Darboux transformations, introduced in Section 4. The conjunction of these three elements results in an optimal methodology to approach the study of nonlinear systems modelled by integrable partial differential equations. This procedure may yield some of the extraordinary properties of this kind of systems, among which it is worth highlighting the obtention of the intriguing soliton solutions. Section 5 is aimed at the detailed illustration of the whole procedure described above for a toy example: the nonlinear Schrödinger (NLS) equation equation in $1 + 1$ dimensions.

For further applications of this method to different integrable systems, defined in several dimensions, we refer the reader to Chapters 3 and 4.

1.   Integrability and the Painlevé Property

This Section begins with a brief overview on the Painlevé Property for differential equations, developed from the pioneering work of the French mathematician P. Painlevé [331,332], which is based on the singularity analysis in the complex plane for such differential equations [207,228]. These ideas have enabled the construction of algorithmic methods, known as Painlevé tests, which may act as a criterion (or at least as an identificator) of integrability, for both ODEs [9–12] and PDEs [417]. The latter constitutes a true cornerstone of this dissertation, since it will be the fundamental ingredient to identify the integrability of the nonlinear systems treated hereafter. Further details on this topic may be found in the classical references [105, 350, 383, 429].

1.1.   Singularities in the complex plane

The beginning of the study of singularities in the complex plane for differential equations dates to the XIX century, with the pioneering contribution of the French mathematician A. L. Cauchy [74]. His main idea lies in the obtention of global solutions for ordinary differential equations by analytical continuation to larger domains starting from local solutions. This procedure requires knowledge of the singularities of the solutions for the differential equation and their loci on the complex plane. In this sense, singularities for ODEs of this kind can be classified as follows [4, 350]

- *Fixed singularities*: these singularities are determined by the coefficients of the differential equation and their location on the complex plane does not depend on the initial conditions.

- *Movable singularities*: the location of these singularities does depend on the initial conditions of the differential equation.

A general property of linear ordinary differential equations is that all the singularities of their solutions are fixed (cf. Theorem 3.7.3 in [4]). Let us consider the following generic nth-order homogeneous ODE in the complex domain D

$$\sum_{k=0}^{n} p_k(z)\frac{d^k w}{dz^k} = 0, \tag{2.1}$$

where one can assume $p_n(z) = 1$ without loss of generality. The point $z = z_0 \in D$ is said to be a *regular point* of the ODE if the coefficients $p_k(z)$, $k = 0, \ldots, n$, are analytic in a neighbourhood of z_0, $|z - z_0| < R$. Then, there exists a unique solution

of (2.1) expressed as the Taylor series

$$w(z) = \sum_{k=0}^{\infty} \frac{c_k}{k!}(z - z_0)^k, \tag{2.2}$$

with constant coefficients c_k, which converges in a neighbourhood of z_0, $|z - z_0| < R$. Hence, the possible singularities of the solution of a linear equation are exclusively located at the singularities of the coefficients $p_k(z)$, $k = 0, \ldots, n$. And, therefore, linear ordinary differential equations do not have movable singular points. All their singularities are fixed by definition.

As opposed to their linear counterparts, nonlinear ODEs may exhibit both fixed or movable singularities of diverse kinds.

Example 2.1. Let us consider the examples given in [4, 383]. Firstly, let us analyze the linear differential equation of first-order

$$\frac{dw}{dz} + \frac{w}{z^2} = 0. \tag{2.3}$$

This ODE has a fixed singular point in $z = 0$, since $p_0(z) = \frac{1}{z^2}$ is singular at this point. The general solution of the equation above is given by

$$w(z) = a_0 e^{\frac{1}{z}}, \tag{2.4}$$

where a_0 is the constant of integration. This solution actually possesses an essential singularity at $z = 0$.

Let us now consider the following initial value problem, given by the nonlinear differential equation

$$\frac{dw}{dz} + w^p = 0, \qquad w(0) = w_0, \qquad p \geq 2, \tag{2.5}$$

whose solution is

$$w(z) = [(p-1)(z - z_0)]^{\frac{1}{1-p}}, \tag{2.6}$$

where z_0 is a constant of integration related to the initial condition w_0, as $z_0 = -\frac{w_0^{1-p}}{p-1}$. Depending on the value of p, different types of singularities arise. The function $w(z)$ has a single pole at $z = z_0$ if $p = 2$, whereas $z = z_0$ behaves as a branch point for $p \geq 3$. In both situations the point $z = z_0$ constitutes a movable singularity, since it is directly related to the initial condition w_0.

The former applications of singularity analysis to differential equations were initiated by Fuchs [175] and Briot and Bouquet [60] with the inspection of Riccati-type

equations. Major contributions were made by the Russian mathematician S. Kovalevskaya [250, 251], who studied the motion of a rotating rigid body and explored the connection between its integrability and the presence of ordinary poles in the associated solution.

Based on these primeval ideas, contemporary mathematicians such as Painlevé, Picard, Gambier, Fuchs, etc. [176, 181, 331, 332, 339] focused their attention on the classification problem of ordinary differential equations according to the types of singularities of their solutions. It was Painlevé and collaborators who provided the crucial breakthrough in this topic, as it will be illustrated in the next Section.

1.2. Painlevé Property

In the late XIX and the early XX centuries, Painlevé, Gambier *et al.* [176, 181, 332] addressed the study and classification of the integrability of second-order ordinary differential equations of the form

$$\frac{d^2 w}{dx^2} = F\left(z, w, \frac{dw}{dz}\right),\tag{2.7}$$

where F is a rational function in $\frac{dw}{dz}$, algebraic in w and locally analytic in z. Painlevé found that there were fifty canonical equations of the form (2.7) with the property that their critical points (branch points and essential singularities) are fixed singularities. Forty-four of these equations may be integrated in terms of elementary functions, such as Riccati equations, elliptic functions, linear equations, etc. The remaining six equations do not have algebraic integrals and cannot be integrated by quadratures. These six equations are known as the *Painlevé equations* ($P_I - P_{VI}$), whose explicit expressions are given by

$$P_I: \qquad \frac{d^2 w}{dz^2} = 6w^2 + z,$$

$$P_{II}: \qquad \frac{d^2 w}{dz^2} = 2w^3 + zw + \alpha,$$

$$P_{III}: \qquad \frac{d^2 w}{dz^2} = \frac{1}{w}\left(\frac{dw}{dz}\right)^2 - \frac{1}{z}\frac{dw}{dz} + \frac{\alpha w^2 + \beta}{z} + \gamma w^3 + \frac{\delta}{w},$$

$$P_{IV}: \qquad \frac{d^2 w}{dz^2} = \frac{1}{2w}\left(\frac{dw}{dz}\right)^2 + \frac{3}{2}w^3 + 4zw^2 + 2(z^2 - \alpha)w + \frac{\beta}{w},\tag{2.8}$$

$$P_V: \qquad \frac{d^2 w}{dz^2} = \left(\frac{1}{2w} + \frac{1}{w-1}\right)\left(\frac{dw}{dz}\right)^2 - \frac{1}{z}\frac{dw}{dz} + \frac{(w-1)^2}{z^2}\left(\alpha w + \frac{\beta}{w}\right) + \frac{\gamma w}{z}$$

$$\mathrm{P_{VI}}: \qquad \frac{d^2w}{dz^2} = \frac{1}{2}\left(\frac{1}{w} + \frac{1}{w-1} + \frac{1}{w-z}\right)\left(\frac{dw}{dz}\right)^2 - \left(\frac{1}{z} + \frac{1}{z-1} + \frac{1}{w-z}\right)\frac{dw}{dz}$$
$$+ \frac{\delta w(w+1)}{w-1},$$
$$+ \frac{w(w-1)(w-z)}{z^2(z-1)^2}\left(\alpha + \frac{\beta z}{w^2} + \frac{\gamma(z-1)}{(w-1)^2} + \frac{\delta z(z-1)}{(w-z^2)}\right),$$

where $\alpha, \beta, \gamma, \delta$ are arbitrary constants. These equations require the introduction of new transcendental functions to be solved, giving rise to the so-called *Painlevé transcendents* [95, 120, 207, 228].

The analytical feature underlying the Painlevé classification for second-order ODEs allows us to introduce the concept of *Painlevé Property* (PP). An ODE is said to possess the Painlevé Property if all the movable singularities of its solutions are ordinary poles, *i.e.* its solutions exhibit no movable branch points. This condition implies that the solutions are single-valued everywhere, in particular in the initial conditions, but in an arbitrary number of isolated movable poles. The core contribution of Painlevé rests on the fact that he established the foundations of a theory exclusively based on the analytical properties of a differential equation (singularity analysis) that addresses valuable information about its integrability *a priori*, without the need to solve it.

Chazy [79] and Bureau [63] explored third-order ODEs as an attempt to extend Painlevé's ideas, and some works have been developed in this fashion for higher-order ODEs [63, 64, 79, 110, 111, 192] and the references therein. Nevertheless, there is not a complete classification for higher-order ODEs with the Painlevé Property yet.

1.2.1. Painlevé Property for ODEs. ARS algorithm

The work of Painlevé faded into oblivion until the second half of the XX century. During the decades of 1950 and 1960, great advances were made in the context of nonlinear Mathematical Physics and integrable systems. In 1965, Zabusky and Kruskal [426] introduced the concept of *soliton* as a travelling wave solution with remarkable properties for the ubiquitous KdV equation [249]. In 1967, Gardner, Green, Kruskal and Miura [183] first developed the now celebrated IST method, successfully solving the KdV equation by this procedure. This technique [13] has also been applied to a large number of integrable equations with soliton solutions [130].

The conspicuous success of the IST method led Ablowitz, Ramani and Segur (ARS) [9–12] to develop an algorithm to determine whether a nonlinear ODE (or system of ODEs) possesses the Painlevé Property. Strictly speaking, this algorithm provides

the necessary conditions for the absence of movable branch points, or equivalently, multivalued solutions in the initial conditions. This procedure does not contemplate other types of singularities that are not branch points, such as essential singularities. In the same fashion as the work of performed by Kovalevskaya, the ARS algorithm focuses on the analysis of the local properties of the aforementioned ODE.

Let us consider an nth-order ordinary differential equation of the form (this procedure can be analogously extended for systems of ODEs)

$$F\left(z, w, \frac{dw}{dz}, \ldots, \frac{d^n w}{dz^n}\right) = 0, \tag{2.9}$$

where F is analytic in z and rational in the remaining arguments.

The behaviour of its solutions is determined by a leading-order analysis around the singularities. A solution of (2.9) is single-valued in a neighbourhood of a movable singularity $z = z_0$ if it admits a local Laurent series expansion of the form

$$w(z) = \sum_{j=0}^{\infty} a_j (z - z_0)^{j-\alpha}, \tag{2.10}$$

where z_0 denotes the arbitrary location of the singularity in the complex plane, the coefficients a_j, for all j, are constants to be determined and α is a positive integer. If all the solutions of equation (2.9) are of the form (2.10), it is said that this ODE has the Painlevé Property, and then, it is conjectured integrable.

The ARS algorithm consists on three steps, which are illustrated as follows:

1. *Dominant behaviour*

   The leading-order analysis corresponds to the case $j = 0$ in the expansion (2.10). We may determine the values of α and the first coefficient a_0 by balance of the dominant terms. α should result in a positive integer, or else, z_0 would be an algebraic branch point. In some cases, a change of variables suffices to transform the equation into another equation without movable branch points. Otherwise, other advanced techniques should be considered in order to overcome this inconvenience [350, 383].

   The values of α and a_0 are not necessarily unique, which means that the ODE may have several branches of expansion. If so, we must then analyze all possible dominant behaviours separately. This aspect of the algorithm turns out to be crucial, since the absence of any of the cases in the analysis can directly lead to misleading results.

2. *Recursion relations and resonances*

We should now substitute the complete series expansion (2.10) into the ODE (2.9), which leads to recursion relations for the coefficient a_j of the form

$$(j - \beta_1) \cdot \cdots \cdot (j - \beta_n)\, a_j = F_j(z, a_0, \ldots, a_{j-1}), \qquad (2.11)$$

for some nonnegative integers β_l, $l = 1, \ldots, n$ and where $F_j, \forall j$ are some analytic functions in the independent variable z and the coefficients a_l, $l = 0, \ldots, j - 1$. Then, equation (2.11) allows us to obtain, in principle, the value of any coefficient a_j for a given $j > 0$, with the prior knowledge of a_0. Nevertheless, if $j = \beta_i$, $i = 1, ..., n$, the left-hand side of (2.11) vanishes and the associated a_{β_i} is arbitrary. These values of j, which retrieve arbitrary coefficients, are called *resonances* and the corresponding equation (2.11) is known as *the resonance condition*.

The general solution of an nth-order ODE is expressed in terms of n arbitrary constants. Since z_0 is already an arbitrary constant, the series expansion (2.10) must therefore depend on $n - 1$ arbitrary coefficients. This fact implies the existence of $n - 1$ resonances of integer j in the present analysis.

3. *Resonance conditions*

The last step concerns about the validation of the resonance conditions. For each $j = \beta_i$, $i = 1, \ldots, n$ in (2.11), the resonance condition $F_{\beta_i} = 0$ must be identically satisfied, so that the ODE has the Painlevé Property (and then, it is considered integrable). We should obtain the following:

- A resonance in $j = -1$, associated to the arbitrariness of z_0.

- $n-1$ resonances for $j \geq 0$, associated to the arbitrariness of the coefficients a_j in the power $(z - z_0)^{j-\alpha}$ for each $j = \beta_i$. If there exists a resonance in $j = 0$, it corresponds to the leading-order term being arbitrary.

Example 2.2. In order to illustrate the technique described above, let us consider the following nonlinear ordinary differential equation

$$\frac{d^2 w}{dz^2} = 2w^3, \qquad (2.12)$$

which is solvable in terms of Jacobi elliptic functions (cf. equation VII of [228]). We consider the following Laurent series expansion for solutions of (2.12)

$$w(z) = \sum_{j=0}^{\infty} a_j (z - z_0)^{j-\alpha}, \qquad (2.13)$$

where z_0 is arbitrary and the coefficients a_j, $\forall j$ and α have to be determined. The ARS algorithm to verify if equation (2.12) has the PP proceeds as follows:

1. The dominant behaviour of (2.13) is $w(z) \sim a_0(z - z_0)^{-\alpha}$, whose substitution into (2.12) results in

$$\alpha(\alpha + 1)a_0(z - z_0)^{-\alpha-2} = 2a_0^3(z - z_0)^{-3\alpha}. \tag{2.14}$$

Equating the exponents in the powers of $(z - z_0)$ provides

$$\alpha + 2 = 3\alpha \quad \Rightarrow \quad \alpha = 1,$$

whilst the balance in the coefficients yields

$$2a_0(a_0^2 - 1) = 0 \quad \Rightarrow \quad a_0 = \pm 1.$$

Since a_0 is not unique, we will have two branches of expansion in (2.13), and each of them should be analyzed separately. This fact implies that steps 2 and 3 from the ARS algorithm must be independently implemented and verified for both cases $a_0 = 1$ and $a_0 = -1$.

2. The general series (2.13) becomes

$$w(z) = \sum_{j=0}^{\infty} a_j(z - z_0)^{j-1}, \tag{2.15}$$

with $a_0 = \pm 1$. Substitution of (2.15) in (2.12) gives rise to the following recursion relations for the coefficients

$$(j+1)(j-4)a_j = 2\sum_{n=1}^{j-1} a_{j-n}\left(3a_0 a_n + \sum_{m=1}^{n-1} a_{n-m}a_m\right), \tag{2.16}$$

which provides the expression of a_j in terms of the coefficients of lower order, and where we have set $a_0^2 = 1$.

From the left-hand side of (2.16) it is easy to see that the ARS algorithm retrieves two resonances. The first resonance in $j = -1$ validates the arbitrariness of z_0, and the second resonance in $j = 4$ implies that a_4 should be a free parameter. Since equation (2.12) is a second-order ODE, we have obtained two constants of integration, z_0 and a_4.

3. We should now verify if the resonance condition for $j = 4$ is identically satisfied.

By solving expression (2.16) recursively for the coefficients a_j, one gets

$$
\begin{aligned}
j = 1: && a_1 &= 0, \\
j = 2: && a_2 &= 0, \\
j = 3: && a_3 &= 0, \\
j = 4: && 6\left(2a_0 a_1 a_3 + a_0 a_2^2 + a_1^2 a_2\right) &= 0.
\end{aligned}
$$

Since the coefficients a_1, a_2, a_3 vanish, the resonance condition for $j = 4$ identically holds for any value of a_0. Hence, the coefficient a_4 turns out to be arbitrary for both values of $a_0 = \pm 1$.

The three steps of the ARS algorithm have been applied to equation (2.12), successfully verifying that the solutions of this equation can be expressed as (2.13). Thus, we can conclude that the only movable singularities of (2.12) are poles, which means that this equation has the PP and it is therefore integrable in the Painlevé sense.

1.2.2. Painlevé Property for PDEs. WTC algorithm

In view of the tremendous success of singularity analysis as a vehicle to identify integrable ODEs, it is natural to wonder if it is possible to build a similar approach for partial differential equations.

ARS conjecture

The first answer to this question was addressed by Ablowitz, Ramani and Segur [9–11] and Hastings and McLeod [203] by means of the so-called *ARS conjecture* (also referred as the *Painlevé conjecture*). This conjecture states that every nonlinear ODE obtained by a similarity reduction of a nonlinear PDE solvable by the IST method has the Painlevé Property (though perhaps only after a transformation of variables). Only weaker versions of this conjecture have been proved [10, 11, 297]. ARS conjecture can be interpreted as follows: it provides a necessary condition for the integrability of a PDE. Conversely, if a PDE is reduced to an ODE that does not possess the PP, the aforementioned PDE is not completely integrable.

There exists an algorithmic method to compute the similarity reductions of a given PDE arising from its Lie symmetry analysis (cf. Section 4 from Chapter 5 of this manuscript). Nonetheless, the application of ARS conjecture to test the integrability of a given PDE is not a simple procedure. This technique requires the obtention of *all* possible similarity reductions, which is already a titanic task. Besides, there also exist similarity reductions that cannot be obtained by the classical group techniques.

Thereupon, we should check whether the resulting reduced ODEs have the PP or not, up to a change of variables.

WTC algorithm

Weiss, Tabor and Carnevale (WTC) proposed a generalization of Painlevé's ideas by extending the notion of the PP to partial differential equations [417].

Singularities of analytical functions of several complex variables are no longer isolated [328]. Let $w : (z_1, \ldots, z_N) \in D \mapsto w(z_1, \ldots, z_N) \in \mathbb{C}$ be a meromorphic function depending on N complex independent variables $z = (z_1, \ldots, z_N)$ defined in a subdomain $D \in \mathbb{C}^N$. Then, the singularities of w lie on analytic manifolds of dimension $(2N-2)$, referred as *singular manifolds*, which are determined by conditions of the form

$$\phi(z_1, \ldots, z_N) = 0, \tag{2.17}$$

where ϕ is analytic in the neighbourhood of the manifold defined as (2.17). When this manifold depends on the initial conditions, it is called a *movable singularity manifold*.

The existence of this kind of manifolds provides a natural extension of the concept of *Painlevé Property for PDEs*. A PDE is said to possess the Painlevé Property if all its solutions are single-valued in a neighbourhood of the movable singularity manifolds. Weiss, Tabor and Carnevale then generalized the ARS algorithm for PDEs, giving rise to the so-called WTC method. This procedure allows us to algorithmically test the integrability of a given PDE, as it will be illustrated hereafter.

The solutions of the PDE under consideration should be locally expressed as a generalized Laurent expansion of the form

$$w(z_1, \ldots, z_N) = \sum_{j=0}^{\infty} w_j(z_1, \ldots, z_N) \left[\phi(z_1, \ldots, z_N)\right]^{j-\alpha}, \tag{2.18}$$

where $\phi(z)$, $w_j(z)$, $\forall j$ are analytic functions in a neighbourhood of the singular manifold (2.17) and α must be an integer. Comparing the Laurent expansions for the WTC algorithm (2.18) and the ARS algorithm (2.10), we may conclude that the former one includes variable coefficients $w_j(z)$, $\forall j$ and the associated expansion is performed in a neighbourhood of the singular manifold $\phi(z) = 0$.

Steps 1-3 from the ARS algorithm for ODEs should be slightly modified in order to establish the new prescription for testing the integrability of PDEs:

1. *Dominant behaviour*

The leading-order analysis, corresponding to $j = 0$ in (2.18), also retrieves the values of α and the first coefficient $w_0(z)$. Once again, if either α or $w_0(z)$ turn out to be non-unique, several branches of expansion arise, which should be analyzed independently.

2. *Recursion relations and resonances*

The substitution of (2.18) into the PDE, supposed to be of nth-order, now provides a more general expression for the recursion relations for the coefficients $w_j(z)$. These relations may incorporate terms on the derivatives up to order n of the coefficients themselves or the singular manifold, as

$$(j - \beta_1) \cdot \cdots \cdot (j - \beta_n) w_j = F_j \left(z, w_J, \phi, \frac{\partial w_J}{\partial z_i}, \frac{\partial \phi}{\partial z_i}, \frac{\partial^2 w_J}{\partial z_i \partial z_k}, \frac{\partial^2 \phi}{\partial z_i \partial z_k}, \cdots \right),$$
(2.19)

for all $J \leq j - 1$ and all combinations $i, k = 1, \ldots, N$.

The resonances are found in the values $j = \beta_l$, $l = 1, \ldots, n$ and, consequently, the associated coefficients $w_{\beta_l}(z)$ should be arbitrary.

An nth-order PDE should possess a resonance in $j = -1$, representing the arbitrariness of the singular manifold (2.17) and $n - 1$ resonances for $j \geq 0$.

3. *Resonance conditions*

If every resonance condition $F_{\beta_l} = 0$, $l = 1, \ldots, n$ is identically satisfied, we may therefore say that the corresponding PDE has the Painlevé Property.

Sometimes the resonance conditions are satisfied for certain values of the parameters of the equation. In this case, the equation is said to be integrable in the Painlevé sense only for those values of the parameters, and we should be dealing with *conditional integrability* [71, 412].

A detailed example on the application of the WTC algorithm to a system of nonlinear PDEs is given in Section 5 of the present Chapter of this manuscript.

1.3.   Further remarks

Algorithmic methods based on the Painlevé Property, such as the ARS algorithm or the WTC algorithm, are often referred as Painlevé tests. There exists a vast amount of literature concerning this topic and its applications to differential equations [105, 107, 314, 350, 377, 383, 429].

There is no rigorous proof about the connection between the Painlevé Property and the complete integrability of a differential equation. Nonetheless, Painlevé tests appear to emerge as a potent criterion to identify integrable differential equations.

Furthermore, Painlevé analysis may be used to derive other important information of this kind of equations, such as Bäcklund transformations, Lax pairs or special solutions [83, 186, 314, 383, 414]. These latter properties constitute one of the keystones of the present research. All these topics of paramount interest will be carefully investigated in the ensuing Sections.

On the other hand, it should be taken into account that Painlevé tests present some limitations. In the following, we exclusively mention two of them, further remarks concerning this topic may be consulted in [254, 350] and the references therein.

The core observation regarding this matter lies in the fact that the Painlevé Property for a given differential equation is not preserved under transformations of the variables of the equation. This means that a certain equation that does not possess the Painlevé Property in a given set of variables may be considered integrable in the Painlevé sense after a change of variables. Finding the suitable transformation of variables in those cases may become a titanic task. In this context, transformations among families of integrable equations, such as Bäcklund and Miura transformations [361, 362], hodograph transformations [97, 158] or reciprocal transformations [30, 217, 243, 358, 359], emerge as a useful tool when it comes to identifying their integrability. These techniques have been successfully applied in literature, highlighting the work carried out by Estévez and collaborators [27, 30, 139, 140, 158, 162], two of which are contributions of the author of this thesis.

The Painlevé Property, as defined previously, might turn out to be an overly restrictive imposition to characterize the integrability of a differential equation, since it exclusively considers ordinary poles for the singularities of the solutions. There exist several variants of the Painlevé test in this context. First of all, it is worth mentioning the so-called weak Painlevé test [350], related to the *weak Painlevé Property*, which allows the presence of movable *rational* branch points (positive rational leading indices in the series expansions). Secondly, we shall remark other approaches, such as the poly-Painlevé test, developed by Kruskal [254], or the perturbative Painlevé test [106, 170], which deals with negative resonances.

2.   The singular manifold method

J. Weiss first introduced [410] and developed the first applications [410–414] of the so-called *singular manifold method* (SMM). The SMM constitutes an algorithmic procedure, based on the WTC method, which allows to derive crucial properties of these (integrable) systems, such as Bäcklund transformations or Lax pairs. If a given PDE has the PP, its solutions can be therefore expressed as a Laurent series of the form (2.18). The SMM focuses on truncated solutions of this series expansion in

order to obtain particular solutions of the PDE. Expression (2.18) may be truncated at constant level (up to terms in the power ϕ^0) as

$$w(z) = \sum_{j=0}^{\alpha} w_j \phi^{j-\alpha} = w_0 \phi^{-\alpha} + w_1 \phi^{1-\alpha} + \cdots + w_\alpha, \qquad (2.20)$$

where the dependence in the independent variables $z = (z_1, \ldots, z_N)$ has been omitted.

If expression (2.20) is imposed to be a solution for the PDE under study, this yields an overdetermined system of equations for ϕ, the coefficients $w_j, 0 \leq j \leq \alpha$ and their derivatives[1]. Hence, $\phi(z)$ is no longer an arbitrary function since it must fulfill this system arising from the truncation condition. If w, as defined in (2.20), is a solution of the PDE, then it is evident that w_α is also a solution. Consequently, relation (2.20) defines an auto-Bäcklund transformation between two solutions w and w_α of the PDE under consideration [410, 411, 414–416]. Or, in other words, the truncation of the Painlevé series allows us to construct new solutions for the PDE, via this auto-Bäcklund transformation, starting from a known initial solution w_α, called the *seed solution*.

The easiest way to proceed in order to deal with this system of equations for the singular manifold requires to set as null every coefficient in the different powers of ϕ. If these conditions hold, then it is possible to establish algebraic relations among the seed solution, its derivatives and the singular manifold, which are called the *singular manifold equations*. Occasionally, this imposition may turn excessively limiting, and this procedure needs to be slightly modified [153].

The SMM may present a major flaw when it is applied to PDEs with several branches of expansion. Several branches of expansion means that the truncation (2.20) needs to be performed for every value of α and the leading-order coefficients, whilst the procedure itself [410] restricts us to one of the branches. Nevertheless, the latter restriction to just one of the possible branches may result in a lose of information about the PDE under consideration [149]. There exist special techniques to overcome this inconvenience. The first of them requires the splitting of the fields of the PDE in a certain mode [141, 150, 157], whilst the second one involves the use of the so-called *double singular manifold method* [109, 148, 149, 191, 309]. In both cases, the application of this modified SMM provides an additional Miura transformation that relates the original PDE to another one with a single branch of expansion [28, 141, 149, 191].

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[1] Properly speaking, this overdetermined system may be expressed as a polynomial in powers of ϕ with variable coefficients depending on the fields and their derivatives.

Homographic invariance

As a property inherited from the PP for ODEs, the Painlevé Property for PDEs is invariant under the action of the Möbius group (also referred as the projective group or homographic transformations) [103, 308, 410, 412], of the form

$$\phi \quad \rightarrow \quad \frac{a\phi + b}{c\phi + d}, \qquad ad - bc \neq 0. \tag{2.21}$$

Hence, it is expected that the invariants of this group play an important role in the Painlevé analysis for the PDE under study. In fact, the relations to be satisfied by ϕ after the truncation ansatz required by the SMM can be expressed more conveniently in terms of these invariants. There exists indeed an alternative reformulation of the SMM, developed by Conte and collaborators [102–104, 340] based on the homographic invariants.

For PDEs defined in $1+1$ dimensions, with coordinates (x, t), the first of the homographic invariants is

$$s \equiv (S\phi)(x) = \{\phi, x\} = \frac{\partial}{\partial x}\left(\frac{\phi_{xx}}{\phi_x}\right) - \frac{1}{2}\left(\frac{\phi_{xx}}{\phi_x}\right)^2 = \frac{\phi_{xxx}}{\phi_x} - \frac{3}{2}\left(\frac{\phi_{xx}}{\phi_x}\right)^2, \tag{2.22}$$

which corresponds to the so-called *Schwarzian derivative* [171, 172, 207]. The second invariant [103, 308] is given by

$$r = \frac{\phi_t}{\phi_x}. \tag{2.23}$$

If we introduce the quantity

$$v = \frac{\phi_{xx}}{\phi_x}, \tag{2.24}$$

the compatibility condition $\phi_{xxt} = \phi_{txx}$ then provides the following identities, which are valid for every $(1+1)$-PDE

$$\begin{aligned} v_t &= (r_x + vr)_x, \\ s_t &= r_{xxx} + 2sr_x + rs_x. \end{aligned} \tag{2.25}$$

Finally, the equations for the singular manifold ϕ arising from the SMM, expressed in terms of the variables $\{v, r, s\}$, together with expressions (2.25), constitute an alternative (and more approachable) setting to describe the *singular manifold equations*.

It should be worth highlighting that this procedure may be easily generalized to higher dimensions [141, 144, 148, 340, 412], as it will be illustrated in Chapter 4.

3. Lax pairs

The SMM has proved to be an extraordinarily fruitful procedure to derive Lax pairs for PDEs by means of the linearization of the singular manifold equations [102, 103, 141, 149, 410–412, 414]. The key aspect of the Lax pair associated to a (nonlinear) PDE lies in the fact that it is consider a proof of integrability for such differential equation and it may constitute a useful tool for its resolution [3,13]. In general, there is not any standard formalism to obtain a Lax pair for a given nonlinear PDE, and the quest for Lax pairs may become an arduous task. In this sense, although it is not a trivial procedure, the SMM provides a straightforward and algorithmic technique to deduce Lax pairs for differential equations. The fundamentals about Lax pairs will be exhibited hereunder, following the references [3, 13, 263, 264]. Throughout the present dissertation, the notion of Lax pair for nonlinear differential equations will play a primary role, since it will be considered as a synonym of the integrability of such equations. Lax pairs also constitute a key cornerstone in the obtention of soliton-like solutions of these equations, as expounded in the ensuing Sections.

3.1. Lax formalism

The concept of Lax pair emerges from the modern studies of integrable systems in the 1960s. Its formulation was first introduced by the mathematician P. Lax [263] in the context of the analysis of nonlinear evolution differential equations in $1+1$ dimensions

$$u_t = K(x, t, u, u_x, \dots), \tag{2.26}$$

where $u = u(x, t)$ is defined in some function space $\mathscr{B}$. The concept of Lax pair is referred to a set of differential operators whose compatibility condition provides the starting evolution problem (2.26).

A *Lax pair*, also known as (the associated linear) *spectral problem*, consists in a pair of linear operators $L(t)$, $A(t)$, acting on a Hilbert space $\mathcal{H}$, which satisfy the so-called *Lax equation*

$$L_t + [L, A] = 0, \tag{2.27}$$

where t is the evolution variable, which usually alludes to the evolution in time, $L_t \equiv \frac{dL}{dt}$ and $[L, A] = LA - AL$ is the usual commutator between two differential operators. Moreover, $L(t)$ is a self-adjoint operator, whilst $A(t)$ is skew-symmetric for all t.

Lax equation (2.27), also known as the *Lax representation of* (2.26), can be regarded

as the compatibility condition of the linear problem

$$L\psi = \lambda\psi, \qquad\qquad (2.28a)$$

$$\psi_t = A\psi, \qquad\qquad (2.28b)$$

where the operator L characterizes the associated eigenvalue problem and A accounts for the operator governing the evolution in time. $\psi = \psi(x,t) \in \mathcal{H}$ is the *spectral function* or *eigenfunction* and λ is the so-called *spectral parameter* or *eigenvalue*. The operator L is said to be *isospectral* if its spectrum is independent of the evolution variable t, *i.e.* $\lambda_t = 0$. Thus, equation (2.27) provides an isospectral evolution equation for $L(t)$, since the isospectrality condition holds [3]. Then, the eigenvalues of L constitute a set of integrals (conserved quantities in time) for the evolution equation (2.26). Moreover, $\det L$ and $\operatorname{tr} L^k$, $k \in \mathbb{N}$ are spectral invariants as well.

Lax showed in [263, 264] that the isospectrality of $L(t)$ follows from the fact that the operators $L(t)$, considered as a one-parameter family of self-adjoint operators in $\mathcal{H}$, are *unitarily equivalent* to each other. In particular $L(t)$ is similar to $L(0)$, *i.e.* there exists a one-parameter family of unitary operators $U(t)$ such that

$$L(0) = U^\star(t)L(t)U(t), \qquad \forall t \in \mathbb{R}, \qquad\qquad (2.29)$$

where $U^\star(t)$ denotes the adjoint operator of $U(t)$ and $L(0)$ is independent of t. The unitary operator $U(t)$ in (2.29) is solution of the initial value problem

$$U_t \equiv \frac{dU}{dt} = A(t)U(t), \qquad U(0) = \mathcal{I}, \qquad\qquad (2.30)$$

for any $A^\star(t) = -A(t)$ and where $\mathcal{I}$ is the identity operator in $\mathcal{H}$. Then, by differentiating (2.29), it is straightforward to see that the operators $L(t)$, $A(t)$ satisfy (2.27). The spectra of $L(t)$ and $L(0)$ coincide [357] and hence the eigenvalues of L are conserved in time. Finally, the eigenvalue problem (2.28) reads

$$L(0)\psi = \lambda\psi \quad \Rightarrow \quad L(t)\tilde{\psi} = \lambda\tilde{\psi} \quad \text{with} \quad \tilde{\psi} = U(t)\psi \in \mathcal{H}. \qquad\qquad (2.31)$$

Lax equation (2.27) is invariant under similarity transformations

$$L \to \tilde{L} = gLg^{-1}, \qquad A \to \tilde{A} = gAg^{-1} + g_t g^{-1}, \qquad\qquad (2.32)$$

where g is an invertible linear operator on $\mathcal{H}$ and $g_t \equiv \frac{dg}{dt}$. Then, the associated spectral problem propagates as

$$\tilde{L}\tilde{\psi} = \lambda\tilde{\psi}, \qquad \tilde{\psi}_t = \tilde{M}\tilde{\psi}, \qquad \text{with} \quad \tilde{\psi} = g\psi \in \mathcal{H}. \qquad\qquad (2.33)$$

There exists an infinite number of Lax pairs which are gauge equivalent through (2.32), and hence, one may conclude that a Lax pair is nowise unique.

Thus, we state that integrable nonlinear differential equations may be written, in an completely equivalent form, as the compatibility condition of a spectral problem of the form (2.28). As already mentioned above, finding Lax pairs for a given nonlinear PDE is not a trivial quest. In that sense, it will be shown that the singular manifold method emerges as a powerful procedure to obtain Lax pairs. It is worthwhile to remark that one of the advantages of Lax formulation is that the associated linear equations are easier to treat and solve than the starting nonlinear PDE [138,141,159]. Lax formulation is also linked to a plethora of techniques related to the resolution of nonlinear differential equations, such as the IST method [3,183].

There also exists a relaxed condition regarding the Lax formalism, called *weak Lax pair* [3,13]. Boiti *et al.* [53–56] considered this approach in the inspection of integrable evolution equations in $2 + 1$ dimensions, and successive applications have appeared in literature regarding this matter [206,255,279,285,347]. The Lax pair is conceived as a pair of differential operators whose compatibility condition retrieves the desired nonlinear evolution equation. Conversely, in a weak Lax pair this condition only needs to be satisfied in the subspace of solutions of the spectral problem.

3.2.   Zero-curvature representation

The Lax formalism, given by the pair of linear operators L, A satisfying (2.27) and (2.28), admits a geometric reformulation in terms of the so-called *zero-curvature representation* [13, 361, 434]. We may rewrite the eigenvalue problem (2.28) as the auxiliary linear system

$$\Psi_x = U\Psi, \qquad \Psi_t = V\Psi, \tag{2.34}$$

where U, V are a pair of linear operators on $\mathcal{H}$, depending on the spectral parameter λ, and $\Psi \in \mathcal{H}$ is a matrix eigenvector whose size depend on the order of L. Hence, the compatibility condition of (2.34) is translated into the relation

$$U_t - V_x + [U, V] = 0. \tag{2.35}$$

Both representations are proved to be equivalent [3, 6] and the preference for one or the other formalism depends on each case under consideration. As in the Lax formalism, the zero-curvature representation (2.35) is invariant under the similarity transformation

$$U \to \tilde{U} = GUG^{-1} + G_xG^{-1}, \qquad V \to \tilde{V} = GVG^{-1} + G_tG^{-1}, \tag{2.36}$$

where G is an invertible matrix operator and $G_x = \frac{\partial G}{\partial x}$, $G_t = \frac{\partial G}{\partial t}$, such that (2.34) transforms as

$$\tilde{\Psi}_x = \tilde{U}\tilde{\Psi}, \qquad \tilde{\Psi}_t = \tilde{V}\tilde{\Psi}, \qquad \text{with} \qquad \tilde{\Psi} = G\Psi. \tag{2.37}$$

Example 2.3. The revisited Korteweg-de Vries (KdV) equation [249]

$$u_t + u_{xxx} - 6uu_x = 0, \tag{2.38}$$

can be written as the compatibility condition of the linear system [263]

$$\begin{aligned} \psi_{xx} &= (u - \lambda)\psi, \\ \psi_t &= (2u + 4\lambda)\psi_x - u_x\psi, \end{aligned} \tag{2.39}$$

where λ is the spectral parameter and ψ acts as the eigenfunction. The system of equations (2.39) corresponds to the following Lax representation

$$L = -\partial_{xx} + u, \qquad A = -4\partial_{xxx} + 6u\partial_x + 3u_x. \tag{2.40}$$

It is easy to check that if the operators defined in (2.40) satisfy the Lax equation (2.27), u therefore satisfies (2.38). Hence, the linear system (2.39) is said to be a (scalar) Lax pair for (2.38).

The zero-curvature representation for the above scalar spectral problem (2.39) can be easily obtained as

$$\Psi_x = \begin{pmatrix} 0 & 1 \\ u - \lambda & 0 \end{pmatrix} \Psi, \qquad \Psi_t = \begin{pmatrix} -u_x & 2u + 4\lambda \\ 2u^2 + 2u\lambda - 4\lambda^2 - u_{xx} & u_x \end{pmatrix} \Psi, \tag{2.41}$$

by considering $\Psi = (\psi, \psi_x)^{\mathsf{T}}$ as the eingenvector, where the symbol T indicates the transpose of the corresponding vector. It is immediate to see that condition (2.35) retrieves (2.38), and then, (2.41) constitutes a Lax pair for the KdV equation.

4.   Darboux transformations and soliton solutions

Transformations among (integrable) nonlinear partial differential equations constitute a valuable and advantageous technique to deal with such equations, specially in the study of nonlinear evolution equations [220]. In this Section we review two of the most important transformations among integrable differential equations, the so-called Bäcklund transformations and Darboux transformations [361]. We will also illustrate their connection with the SMM and the Lax formalism described in the previous Sections, and their potential to derive solutions of major significance in nonlinear differential equations arising from soliton theory.

4.1. Bäcklund transformations

Bäcklund transformations were introduced by the Swedish mathematician A. V. Bäcklund in the late XIX century [67] as generalized surface transformations in differential geometry. They have proved to be of special relevance in the field of Mathematical Physics with numerous applications [129, 359, 361, 362]. In the context of this dissertation, Bäcklund transformations constitute explicit relations between the solutions of two nonlinear differential equations (also extendable to hierarchies of PDEs). Hence, Bäcklund transformations provide a powerful method for generating solutions of nonlinear PDEs [362]. From the classical point of view, it is worthwhile to mention the Clairin's method to generate Bäcklund transformations [88, 259], while a modern approach can be found in [33, 342, 343] or by means of the Wahlquist-Estabrook procedure [401, 402].

Example 2.4. A well-known example of a Bäcklund transformation is the celebrated Miura transformation [303]

$$u = v^2 \pm v_x, \tag{2.42}$$

that establishes a relation between two solutions of the KdV equation [249]

$$u_t + u_{xxx} - 6uu_x = 0, \tag{2.43}$$

and the solution v for the modified Korteweg-de Vries (mKdV) equation [425]

$$v_t + v_{xxx} - 6v^2 v_x = 0. \tag{2.44}$$

A Bäcklund transformation which relates solutions of the same equation is called an *auto-Bäcklund transformation*. Many examples of auto-Bäcklund transformations for renowed integrable equations may be found in [259, 361, 362] and the references therein. Then, auto-Bäcklund transformations may yield an iterative procedure to construct solutions once a particular solution is known. As already said, the truncated series expansion arising from the Painlevé analysis (2.20) constitutes an auto-Bäcklund transformation between w and w_α.

4.2. Darboux transformations

Another important class of transformations with their origin in the XIX century corresponds to the so-called Darboux transformations, introduced by the French mathematician G. Darboux [115, 116], based on the previous work of M. Moutard [305, 306], in the study of the Sturm-Liouville problem for differential equations of physical relevance. More advances concerning the Sturm-Liouville problem were performed

by Crum in [112] and several classical generalizations and early applications were developed in [43, 44, 46, 269, 293–295, 398]. Further remarks can be found in the monographs [70, 195, 296, 361].

Given a nonlinear PDE, the basis of Darboux transformations lies in the invariance of its associated spectral problem under a simultaneous and combined mapping between both the fields and the corresponding eigenfunctions. It may be formulated as a covariance principle for the corresponding operators [296]. This type of transformations have proved to be extremely useful to construct exact solutions for integrable nonlinear PDEs. In particular, this technique provides an iterative algebraic procedure that allows to construct multi-soliton solutions by the sequential application of Darboux transformations together with solutions of the associated Lax pair and the evolution equation. This approach has been widely applied in recent years for a plethora of multi-dimensional integrable systems [86, 265, 266, 296, 361].

Example 2.5. Let us consider two different eigenfunctions ψ, χ for the spectral problem (2.39) for the KdV equation (2.38), defined in terms of the field u and the spectral parameter λ. Then, this Lax pair admits the following Darboux transformation [296]

$$
\begin{aligned}
\hat{u} &= u - 2\left(\frac{\psi_x}{\psi}\right)_x, \\
\hat{\psi} &= \chi_x - \chi\frac{\psi_x}{\psi},
\end{aligned}
\tag{2.45}
$$

such that $\hat{\psi}$, $\hat{u}$ solves the linear problem (2.39) with spectral parameter λ. Hence, it is easy to check that the linear problem (2.39) is invariant under the simultaneous transformation $u \to \hat{u}$, $\psi \to \hat{\psi}$.

4.3.   Binary Darboux transformations

The approach developed in the present research offers a slightly different perspective. The solid methodology to be used in the following is based on the previous work of Estévez and collaborators [141, 148, 152, 153, 157, 161], among other references. The core idea resides in the conjunction of the SMM and Darboux transformations to obtain exact solutions for nonlinear integrable equations whose associated spectral problem is known. The aforementioned procedure is outlined hereinafter.

Let us consider a pth-order nonlinear differential equation in $1 + 1$ dimensions, depending on the independent variables (x, t) and a single field $u = u(x, t)$[2]. This

---

[2]This procedure can be easily generalized to both higher dimensions or number of fields, as it will be illustrated in the examples from Chapters 3 and 4.

nonlinear PDE is supposed to pass the Painlevé test[3], and then, be considered integrable in the Painlevé sense. We may apply the SMM in order to derive a Lax pair for it, which means that the solution of the PDE u admits a truncated series expansion at constant level in a neighbourhood of the singular manifold $\phi = 0$ of the form

$$u(x,t) = \sum_{j=0}^{\alpha} a_j \phi^{j-\alpha} = a_\alpha + \frac{a_{\alpha-1}}{\phi} + \cdots + \frac{a_0}{\phi^\alpha}, \tag{2.46}$$

where $a_j = a_j(x,t)$, $0 \leq j \leq \alpha$. The coefficient $a_0(x,t)$ and the leading index α are determined by the balance of the dominant terms. The remaining coefficients $a_j(x,t), j = 1, \ldots, \alpha$ are either determined or arbitrary by virtue of the Painlevé analysis.

The application of the SMM over the nonlinear PDE under study straightforwardly leads to the singular manifold equations. The linearization of such equations allows us to introduce the eigenfunctions and a free parameter (that will act as the spectral parameter) such that the arising linear system possesses a Lax pair structure. The corresponding spectral problem may be generally expressed as a system of 2μ differential equations, depending on μ eigenfunctions ψ^l, $l = 1, \ldots, \mu$ and the spectral parameter λ, of the form

$$S_r\left(x,t,u,u_{(k)},\psi^l,\frac{\partial \psi^l}{\partial x},\ldots,\frac{\partial^m \psi^l}{\partial x^m},\lambda\right) = 0,$$
$$T_r\left(x,t,u,u_{(k')},\psi^l,\frac{\partial \psi^l}{\partial x},\ldots,\frac{\partial^{m-1}\psi^l}{\partial x^{m-1}},\frac{\partial \psi^l}{\partial t},\lambda\right) = 0, \qquad r = 1,\ldots,\mu, \tag{2.47}$$

where $u_{(k)}$ denotes the set of all the derivatives of u up to order k (cf. Section 2 from Chapter 5), with $k, k' < p$ and either $k + 1 = p$ or $m + k' = p$. The system of equations $S_r = 0$, $r = 1, \ldots, \mu$ is often referred as the *spatial part of the Lax pair*, since it only involves spatial derivatives of the eigenfunctions, up to a highest order m. The system of equations $T_r = 0$, $r = 1, \ldots, \mu$ represents the *temporal part of the Lax pair*, due to the presence of the temporal derivatives $\frac{\partial \psi^l}{\partial t}$. In principle, it is not possible to determine a priori the number of eigenfunctions μ required to construct the Lax pair, nor its order m, since this information depends on the particular process of linearization of the corresponding singular manifold equations in each case.

The SMM also retrieves the relation between the eigenfunctions ψ^l, $l = 1, \ldots, \mu$ and the singular manifold ϕ, which may depend on the spectral parameter and the field,

---

[3]Without loss of generality, we will assume that the series expansion has a unique branch of expansion. If it is not the case, there exist several techniques [141, 149] that allow us to properly treat this scenario.

as

$$P\left(x, t, u, \psi^l, \frac{\partial \psi^l}{\partial x}, \ldots, \frac{\partial^m \psi^l}{\partial x^m}, \lambda, \phi, \phi_x, \phi_t\right) = 0, \qquad (2.48)$$

which might be occasionally expressed as an exact derivative.

We shall consider now a Darboux transformation for the Lax pair (2.47), *i.e.* a simultaneous transformation for the fields and the eigenfunctions such that the spectral problem remains invariant. The sheer novelty of this formulation with respect to the classical Darboux approach rests on the fact that these transformations have to be extended to the singular manifold, such that expression (2.48) is also preserved. This implies that iterated solutions for the field u, the eigenfunctions ψ^l, $l = 1, \ldots, \mu$ and the singular manifold ϕ must be constructed. The SMM therefore plays a crucial role in this process, since it will precisely prescribe those transformations.

The truncated series expansion (2.46) defines an auto-Bäcklund transformation for the field of the form

$$u^{[1]} = u^{[0]} + \sum_{j=0}^{\alpha-1} \frac{a_j}{\phi^{\alpha-j}}, \qquad (2.49)$$

where $u^{[0]}$ denotes the seed solution (a_α in (2.46), which clearly satisfies the starting nonlinear PDE) and $u^{[1]}$ represents the iterated solution (u in (2.46)), which also satisfies the same differential equation.

Let us consider now two different sets of eigenfunctions $\{\psi_1^l, \psi_2^{l'}\}$, $l, l' = 1, \ldots, \mu$, satisfying the linear problem (2.47) for the seed solution $u^{[0]}$ with two different eigenvalues $\{\lambda_1, \lambda_2\}$, respectively, as

$$S_r\left(x, t, u^{[0]}, \psi_1^l, \lambda_1\right) = 0, \qquad T_r\left(x, t, u^{[0]}, \psi_1^l, \lambda_1\right) = 0, \qquad (2.50a)$$

$$S_r\left(x, t, u^{[0]}, \psi_2^{l'}, \lambda_2\right) = 0, \qquad T_r\left(x, t, u^{[0]}, \psi_2^{l'}, \lambda_2\right) = 0, \qquad (2.50b)$$

for $r, l, l' = 1, \ldots, \mu$, and where we have omitted the dependence in the derivatives of the fields and the eigenfunctions for simplicity.

The consideration of two sets of eigenfunctions with their respective spectral parameters needfully defines two singular manifolds $\{\phi_1, \phi_2\}$ through (2.48) as

$$P\left(x, t, u^{[0]}, \psi_1^l, \lambda_1, \phi_1\right) = 0, \qquad (2.51a)$$

$$P\left(x, t, u^{[0]}, \psi_2^{l'}, \lambda_2, \phi_2\right) = 0. \qquad (2.51b)$$

Following the ideas developed in [83, 248], the spectral problem (2.47) should be now regarded as a coupled system of nonlinear differential equations depending on the field u and the eigenfunctions ψ^l, $l = 1, \ldots, \mu$. Thus, the SMM can be now applied to the Lax pair itself. This procedure requires the following simultaneous Painlevé expansions, both constructed in a neighbourhood of the singular manifold $\phi_1 = 0$, which can be truncated at constant level as

$$u^{[1]} = u^{[0]} + \sum_{j=0}^{\alpha-1} \frac{a_j}{\phi_1^{\alpha-j}}, \qquad \psi_{1,2}^l = \psi_2^l + \sum_{j=0}^{\beta_l-1} \frac{b_j}{\phi_1^{\beta_l-j}}, \tag{2.52}$$

where α, β_l, $l = 1, \ldots, \mu$ are the leading-order indices, and the coefficients $a_j(x,t)$, $b_{j'}(x,t)$, $j = 0, \ldots, \alpha - 1$, $j' = 0, \ldots, \beta_l - 1$ have to be determined by imposing that the iterated solutions $(u^{[1]}, \psi_{1,2}^l)$, $l = 1, \ldots, \mu$ satisfy the spectral problem

$$S_r\left(x, t, u^{[1]}, \psi_{1,2}^l, \lambda_2\right) = 0,$$
$$\tag{2.53}$$
$$T_r\left(x, t, u^{[1]}, \psi_{1,2}^l, \lambda_2\right) = 0,$$

with associated spectral parameter λ_2. The truncated expansion for $u^{[1]}$ should coincide with the one provided for the nonlinear PDE (2.49). The coefficients $b_j(x,t)$, $j = 1, \ldots, \beta_l - 1, \forall l$ may depend on the seed solution $u^{[0]}$, both sets of eigenfunctions $\{\psi_1^l, \psi_2^{l'}\}$, their derivatives and the spectral parameters $\{\lambda_1, \lambda_2\}$. The first iterated eigenfunction will be denoted as $\psi_{1,2}^l$, $\forall l$, alluding to the fact that it is constructed in terms of two seed eigenfunctions $\{\psi_1^{l'}, \psi_2^{l''}\}, \forall l', l''$, and the last subindex $\psi_{\_,2}^l$ indicates the associated spectral parameter λ_2 for this eigenfuction.

Besides, transformations (2.52) induce an analogous truncated series expansion for the singular manifold as

$$\phi_{1,2} = \phi_2 + \sum_{j=0}^{\gamma-1} \frac{c_j}{\phi_1^{\gamma-j}}, \tag{2.54}$$

with $\gamma \in \mathbb{N}$ and coefficients $c_j(x,t), j = 0, \ldots, \gamma - 1$ to be determined such that

$$P\left(x, t, u^{[1]}, \psi_{1,2}^l, \lambda_2, \phi_{1,2}\right) = 0, \qquad l = 1, \ldots, \mu, \tag{2.55}$$

is identically satisfied. The first iteration of the singular manifold is again denoted as $\phi_{1,2}$, following the same notation convention as for the iterated eigenfunction.

The main difference of this approach with respect to the classical Darboux transformations is that the series (2.52) use the singular manifold ϕ_1 as the expansion variable rather than the eigenfunction ψ_1 of the former (2.47) spectral problem (cf.

Example 2.5). Nevertheless, expansions (2.52) may be regarded as (binary) Darboux transformations in the sense that these transformations leave invariant the Lax pair (2.47), since (2.53) is satisfied by construction. Besides, relation (2.48) is also preserved.

In summation, two sets of eigenfunctions $\{\psi_1^l, \psi_2^{l'}\}$, $l, l' = 1, \ldots, \mu$ for the Lax pair associated to the seed solution $u^{[0]}$ suffice to construct a new set of eigenfunctions $\psi_{1,2}^l, l = 1 \ldots, \mu$ that solve the spectral problem for the iterated solution $u^{[1]}$. Analogously, the additional introduction of two singular manifolds $\{\phi_1, \phi_2\}$ allows us to define a new singular manifold $\phi_{1,2}$ associated to these solutions $u^{[1]}, \psi_{1,2}^l, \forall l$. Hence, the SMM yields the suitable truncated expansions, interpreted as (binary) Darboux transformations, which map the Lax pair into itself, providing and iterative method to construct new solutions of the spectral problem and the initial nonlinear PDE under consideration.

This procedure may be recursively applied in order to obtain further iterations, and consequently, novel solutions. The nth iteration introduces the new triad $\{u^{[n]}, \psi_{1,2,\ldots,n,n+1}^l, \phi_{1,2,\ldots,n,n+1}\}$, $l = 1, \ldots, \mu$, such that $\{\psi_{1,2,\ldots,n+1}^l\}$ represents the set of eigenfunctions associated to the field $u^{[n]}$ with eigenvalue λ_{n+1}, and $\phi_{1,2,\ldots,n+1}$ is the associated singular manifold. This implies that the following relations hold

$$S_r\left(x, t, u^{[n]}, \psi_{1,2,\ldots,n+1}^l, \lambda_{n+1}\right) = 0,$$
$$T_r\left(x, t, u^{[n]}, \psi_{1,2,\ldots,n+1}^l, \lambda_{n+1}\right) = 0,$$

$$\tag{2.56a}$$

$$P\left(x, t, u^{[n]}, \psi_{1,2,\ldots,n+1}^l, \lambda_{n+1}, \phi_{1,2,\ldots,n+1}\right) = 0, \tag{2.56b}$$

for every $r, l, = 1, \ldots, \mu$.

The triad $\{u^{[n]}, \psi_{1,2,\ldots,n+1}^l, \phi_{1,2,\ldots,n+1}\}$ can be expressed in terms of the following quantities,

- Seed solution $u^{[0]}$,

- μ different sets of $n + 1$ eigenfunctions $\{\psi_1^{l_1}, \psi_2^{l_2}, \ldots, \psi_{n+1}^{l_{n+1}}\}$, $l_1, \ldots, l_{n+1} = 1, \ldots, \mu$, for this seed solution $u^{[0]}$, each of them associated to $n + 1$ different eigenvalues $\{\lambda_1, \lambda_2, \ldots, \lambda_{n+1}\}$, respectively,

- $n + 1$ singular manifolds $\{\phi_1, \phi_2, \ldots, \phi_{n+1}\}$

that satisfy (2.50) and (2.51), respectively.

This last statement implies that, with a suitable seed solution and the corresponding resolution of the associated spectral problem, we will be able to generate an extensive

class of diverse solutions for the former nonlinear PDE. More specifically, this thesis addresses the quest for solutions with solitonic nature, with special emphasis on the particular case of rational solitons. Applications of this procedure can be found in the references already cited above, as well as in the example described in the following Section and the different models analyzed in Chapters 3 and 4 of the present dissertation.

4.4.   Connection to Hirota's bilinear formalism

There also exist other methods to determine multi-soliton solutions for a given non-linear PDE. Among them, the celebrated *Hirota's bilinear method*, first introduced in [208], should be duly emphasized. This technique provides a systematic procedure to derive, in a relatively straightforward methodology, such kind of special solutions.

Hirota's method requires that the nonlinear PDE is transformed, by means of a (nontrivial) dependent variable transformation (cf. Section 1.7 in [213]), in a bilinear differential equation, often called *Hirota form*, which should be expressed in terms of the Hirota's D-operators [211, 212]

$$D_x^n(f,g) \equiv D_x^n f \cdot g = \left(\frac{\partial}{\partial x} - \frac{\partial}{\partial x'} \right)^n f(x)g(x') \bigg|_{x=x'} = \frac{\partial^n}{\partial x'^n} f(x+x')g(x-x') \bigg|_{x'=0}, \tag{2.57}$$

with $n \in \mathbb{N}$. This operator acts as a bilinear derivative and a list of properties of D-operators can be found in [210, 212, 213, 215]. Generalizations of these operators to higher dimensions have been studied in [193].

Once that the former nonlinear PDE has been expressed in a bilinear form, the exact solutions are obtained by a perturbation algebraic method. The N-soliton solution formula can be expressed as quotients of Wronskian-type determinants in terms of the *Hirota's τ-function* [174, 214], and soliton solutions appear as polynomials of a finite number of exponential functions. In contrast to the IST method, this procedure allows solutions to be obtained directly, so that it might be known as the *direct method*. Countless applications of this method to remarkable nonlinear PDEs in soliton theory can be found in the monograph [213] and the references therein.

Since the integrability of a given PDE has been connected to the existence of multi-soliton solutions, Hirota's method may be regarded as a criterion of integrability [204, 350]. The main problem of Hirota's procedure lies in the process of bilinearization, which is definitively not algorithmic neither obvious to perform. There have been reported visible similarities between the bilinear transformation and the truncated series expansion arising from the SMM [187, 205, 314]. Furthermore, the iterative procedure arising from the combination of the SMM and the implementation of

binary Darboux transformations to derive multi-soliton solutions may be related with Hirota's τ-function, as established and evidenced in [148, 149, 153, 182].

5. Toy example: nonlinear Schrödinger equation in $1 + 1$ dimensions

In order to provide a complete overview of the procedure described above, this Chapter closes with an illustrative and detailed example.

This Section is then devoted to the study of the so-called *nonlinear Schrödinger* (NLS) equation [3, 13, 130], a classical integrable equation of reference in the field of Mathematical Physics and soliton theory. NLS equation has been widely studied in literature in recent years, in terms of integrability characterization and mathematical properties. This nonlinear dispersive equation appears in the description and modelization of countless physical scenarios, with remarkable applications in diverse applied disciplines.

NLS equation first emerges in the context of the evolution of gravity waves in fluid dynamics, with the pioneering research of G. G. Stokes [380] and more recent works [47, 82, 117, 118, 202, 229, 427]. It also plays an important role in the area of nonlinear optics [42, 48, 240, 387], and it is found in the description of hydromagnetic and plasma waves [221, 222, 374, 408, 409], Bose-Einstein condensates [128, 194, 225, 344], propagation of heat pulses in solids [388] and other nonlinear waves in different scenarios [114, 122, 123, 335, 353].

Besides to its physical relevance and applications, NLS equation presents numerous remarkable mathematical properties and analytical solutions of interest. As it is well known, NLS equation constitutes a complete integrable system, it was first solved by Zakharov and Shabat via the IST method [431, 432] and it admits a Lax pair [13]. More properties regarding its integrability can be found in [13, 130, 385, 386, 430, 432, 433]. This system has also been proved to exhibit a plethora of soliton-like solutions: N-soliton solution [16, 313, 431, 432], breather solutions [19, 21, 132, 241, 242, 257, 284, 337], rogue waves [17, 18, 22, 35, 239], etc. There also exist diverse integrable generalizations, such as multi-component generalizations [286], extensions to higher dimensions [69], discretized [7] or quantized versions of NLS [200].

The following Section addresses the study of the so-called *defocusing NLS* equation[4] in $1 + 1$ dimensions, which is written as the following coupled system of differential

---

[4]The general NLS equation is usually written as $iu_t + u_{xx} + \kappa|u|^2 u = 0$, as introduced in Chapter 1, with $\kappa \neq 0$. If $\kappa > 0$, this equation is also denoted as *focusing NLS*, whilst the case $\kappa < 0$ refers to the *defocusing NLS*. Although these two equations appear in distinct physical scenarios and exhibit different kinds of solutions, both of them possess remarkable properties regarding their integrability.

equations

$$iu_t + u_{xx} - 2u^2w = 0,$$
$$-iw_t + w_{xx} - 2w^2u = 0,$$

(2.58)

where $u(x,t)$ is a complex-valued function and $w(x,t)$ stands for its complex conjugate, $\overline{u} = w$.

For practical purposes in the ensuing calculations, let us introduce a new real field $m(x,t)$, which is related to the density of probability of the system as $m_x = -|u|^2 = -uw$. Then, NLS system (2.58) may be expressed in nonlocal form as

$$iu_t + u_{xx} + 2m_xu = 0,$$
$$-iw_t + w_{xx} + 2m_xw = 0,$$
$$m_x + uw = 0.$$

(2.59)

5.1. Painlevé test for NLS

The NLS system (2.59) defines a coupled system of three nonlinear PDEs of second order. We should now apply the WTC algorithm, as described in Subsection 1.2.2 of this Chapter, in order to verify if (2.59) has the Painlevé Property, and then satisfies the necessary conditions to be integrable.

The Painlevé test for (2.59) requires the following generalized Laurent expansions for the fields u, w, m of the form

$$u(x,t) = \sum_{j=0}^{\infty} a_j(x,t)\,\phi(x,t)^{j-\alpha},$$

$$w(x,t) = \sum_{j=0}^{\infty} b_j(x,t)\,\phi(x,t)^{j-\beta},$$

(2.60)

$$m(x,t) = \sum_{j=0}^{\infty} c_j(x,t)\,\phi(x,t)^{j-\gamma},$$

where $a_j(x,t)$, $b_j(x,t)$, $c_j(x,t)$, $\forall j$ are arbitrary coefficients for the series expansion, α, β, γ are the leading indices and $\phi(x,t)$ is the so-called singular manifold.

1. A leading-order analysis for the case $j = 0$ results in

$$\alpha = \beta = \gamma = 1, \qquad a_0\,b_0 = \phi_x^2, \quad c_0 = \phi_x,$$

(2.61)

where we may appreciate that both indices and coefficients are uniquely deter-

mined. The coupled relation $a_0 b_0 = \phi_x^2$ indicates the presence of a resonance in $j = 0$, as it will be proved later.

2. Then, expressions (2.60) read

$$u = \frac{a_0}{\phi} + \sum_{j=1}^{\infty} a_j \, \phi^{j-1}, \quad w = \frac{b_0}{\phi} + \sum_{j=1}^{\infty} b_j \, \phi^{j-1}, \quad m = \frac{\phi_x}{\phi} + \sum_{j=1}^{\infty} c_j \phi^{j-1}, \quad (2.62)$$

where we have dropped the dependence in the independent variables, and $a_0 b_0 = \phi_x^2$.

If we substitute expressions (2.62) in the starting system (2.59), the following recursion relations arise

$$j(j-3)\phi_x^2 a_j + 2(j-1)a_0 \phi_x c_j = -i\frac{\partial a_{j-2}}{\partial t} - \frac{\partial^2 a_{j-2}}{\partial x^2} - (j-2)\Big[i\phi_t + \phi_{xx}\Big]a_{j-1}$$

$$- 2(j-2)\phi_x \frac{\partial a_{j-1}}{\partial x} - 2\sum_{k=1}^{j-1}(j-k-1)\phi_x a_k c_{j-k}$$

$$- 2\sum_{k=0}^{j-1} a_k \frac{\partial c_{j-k-1}}{\partial x}, \qquad\qquad (2.63a)$$

$$j(j-3)\phi_x^2 b_j + 2(j-1)b_0 \phi_x c_j = i\frac{\partial b_{j-2}}{\partial t} - \frac{\partial^2 b_{j-2}}{\partial x^2} + (j-2)\Big[i\phi_t - \phi_{xx}\Big]b_{j-1}$$

$$- 2(j-2)\phi_x \frac{\partial b_{j-1}}{\partial x} - 2\sum_{k=1}^{j-1}(j-k-1)\phi_x b_k c_{j-k}$$

$$- 2\sum_{k=0}^{j-1} b_k \frac{\partial c_{j-k-1}}{\partial x}, \qquad\qquad (2.63b)$$

$$b_0 a_j + a_0 b_j + (j-1)\phi_x c_j = -\frac{\partial c_{j-1}}{\partial x} - \sum_{k=1}^{j-1} a_k b_{j-k}, \qquad\qquad (2.63c)$$

where we have isolated the higher coefficients a_j, b_j, c_j in the left-hand side of (2.63). The resonance analysis is easily generalizable for systems of differential equations, as prescribed in [350], providing the following resonance condition

$$j(j-1)(j-3)(j-4)(j+1) = 0, \qquad\qquad (2.64)$$

which retrieves five resonances. The resonance in $j = -1$ validates the arbitrariness of the singular manifold $\phi = 0$. The remaining four resonances in $j = 0, 1, 3, 4$ imply the presence of four arbitrary parameters in the associated

coefficients for those values of j in the recursion relations (2.63). There is indeed a resonance in $j = 0$, ratified by the condition $a_0 b_0 = \phi_x^2$, which implies that either a_0 or b_0 are arbitrary.

3. The resonance conditions for $j = 0, 1, 3, 4$ have been verified with the aid of MAPLE, successfully checking that all of them hold. In this case, the coefficients a_0, c_1, a_3, c_4 have been reported as arbitrary.

Hence, the NLS system (2.59) passes the Painlevé test, which means that it possesses the Painlevé Property and it is therefore conjectured integrable.

5.2. Singular manifold method for NLS

The SMM requires that the generalized Laurent series expansion (2.60) shall be truncated at constant level, *i.e.* $j = 1$. The presence of a resonance at leading-order $j = 0$ implies the relation $a_0 b_0 = \phi_x^2$, so that a_0 and b_0 are no longer independent. This fact allows us to introduce the arbitrary function $g_0 = g_0(x, t)$ such that $a_0 = ig_0\phi_x$ and $b_0 = -\frac{i\phi_x}{g_0}$. Then, the truncated expansions (2.60) result in

$$u^{[1]} = u^{[0]} + \frac{ig_0\phi_x}{\phi}, \qquad w^{[1]} = w^{[0]} - \frac{i\phi_x}{g_0\phi}, \qquad m^{[1]} = m^{[0]} + \frac{\phi_x}{\phi}, \tag{2.65}$$

where both the triads $\{u^{[0]}, w^{[0]}, m^{[0]}\}$ (seed solution) and $\{u^{[1]}, w^{[1]}, m^{[1]}\}$ (iterated solution) are solutions of the NLS system (2.59). Expression (2.65) therefore constitutes an auto-Bäcklund for the involved fields.

At this point, it is convenient to introduce the quantities

$$v = \frac{\phi_{xx}}{\phi_x}, \qquad r = \frac{\phi_t}{\phi_x}, \qquad s = v_x - \frac{v^2}{2}, \tag{2.66}$$

where r and the Schwarzian derivative s are the homographic invariants defined in (2.22) and (2.23).

We should now substitute the truncated expansions (2.65) into the original NLS system (2.59), yielding a polynomial result in the singular manifold ϕ. The usual (and simplest) procedure to approach the resulting equations demands that the coefficients associated to the different powers of ϕ vanish. This assumption, after some algebraic

manipulations, provides

$$u^{[0]} = \frac{1}{2}g_0 r - i\frac{\partial g_0}{\partial x} - \frac{i}{2}g_0 v,$$

$$w^{[0]} = \frac{r}{2g_0} - \frac{i}{g_0^2}\frac{\partial g_0}{\partial x} + \frac{iv}{2g_0},$$

(2.67a)

$$ir_x + \left(\frac{1}{g_0}\frac{\partial g_0}{\partial x}\right)_x = 0,$$

(2.67b)

$$\frac{i}{g_0}\frac{\partial g_0}{\partial t} + \frac{3}{g_0^2}\left(\frac{\partial g_0}{\partial x}\right)^2 + \frac{2ir}{g_0}\frac{\partial g_0}{\partial x} - \frac{r^2}{2} + v_x - \frac{v^2}{2} = 0,$$

(2.67c)

where the quantities $\{v, r, s\}$ have been already introduced according to their definitions (2.66).

Equation (2.67b) can be easily integrated as

$$g_0 = e^{i\left(2\lambda x - \int r\,dx\right)},$$

(2.68)

where λ is the constant of integration[5].

Hence, expressions (2.67) may be rewritten as follows:

- The equations for the seed solutions (2.67a) yield

$$u^{[0]} = \frac{g_0}{2}\left(4\lambda - r - iv\right), \qquad w^{[0]} = \frac{1}{2g_0}\left(4\lambda - r + iv\right).$$

(2.69)

- The singular manifold equations finally take the form

$$r_t = \left(\frac{3}{2}r^2 - 8\lambda r - v_x + \frac{v^2}{2}\right)_x,$$

(2.70a)

$$v_t = (r_x + rv)_x,$$

(2.70b)

where the second expression (2.70b) arises from the compatibility condition $\phi_{xxt} = \phi_{xxt}$ in (2.66).

The singular manifold equations can be expressed in terms of the homographic

---

invariants as the following system of nonlinear PDEs

$$r_t = \left(\frac{3}{2} r^2 - 8\lambda r - s \right)_x,$$
$$s_t = r_{xxx} + 2sr_x + rs_x. \tag{2.71}$$

5.3. Spectral problem for NLS

The SMM provides a straightforward procedure to construct the Lax pair for (2.59). The next step concerns the linearization of the singular manifold equations (2.70) in order to derive the associated spectral problem. This process is far from being trivial and finding the suitable change of variables to linearize the system (2.70) is not immediate. Once again, Painlevé analysis yields the successful linearization scheme.

The singular manifold equations (2.70) can be regarded as a coupled nonlinear system of two differential equations in the variables $\{v, r\}$. If we apply the Painlevé test to such system, the fields $\{v, r\}$ should behave as the series expansion

$$v = \sum_{j=0}^{\infty} v_j \, \psi^{j-\delta}, \qquad r = \sum_{j=0}^{\infty} r_j \, \psi^{j-\eta}, \tag{2.72}$$

in a neighbourhood of a new singular manifold $\psi = 0$. A leading-order analysis provides

$$\delta = 1, \qquad \eta = 1, \qquad v_0 = \psi_x, \qquad r_0 = \pm i\psi_x. \tag{2.73}$$

The duality in the sign of r_0 implies that this coefficient is not uniquely determined, and therefore we are dealing with two branches of expansions. It can be trivially checked that the Painlevé test is satisfied for both branches of expansions, with resonances in $j = -1, 2$ (double), 3. These leading terms render the precise transformation that leads to the linearization of (2.70), [141, 148]. Since the Painlevé expansion possesses two branches, we should introduce two singular manifolds $\{\psi, \chi\}$ [141, 148, 149, 191], one for each expansion branch, such that the fields $\{v, r\}$ are written as the truncated series expansion (2.72) at constant level

$$v = \frac{\psi_x}{\psi} + \frac{\chi_x}{\chi}, \qquad r = i \left(\frac{\psi_x}{\psi} - \frac{\chi_x}{\chi} \right) + 2\lambda, \tag{2.74}$$

where we have introduced the additional term 2λ for convenience and simplicity in the forthcoming calculations.

Spatial Lax pair

Substitution of expressions (2.74) into the equations for the seed solutions (2.69) directly provides the spatial part of the spectral problem for (2.59), which reads

$$\psi_x = -i\lambda\psi + iu^{[0]}\chi,$$
$$\chi_x = i\lambda\chi - iw^{[0]}\psi,$$

(2.75)

where $\{\psi,\chi\}$ are the eigenfunctions and λ plays the role of the spectral parameter.

Temporal Lax pair

The singular manifold equations (2.70) provide in this occasion the temporal part of the spectral problem for (2.59). By substituting (2.74) into (2.70) and after performing and integration in the variable x, we get the following system of PDEs

$$i\left(\frac{\psi_t}{\psi} - \frac{\chi_t}{\chi}\right) + \frac{\psi_{xx}}{\psi} + \frac{\chi_{xx}}{\chi} + 2i\lambda\left(\frac{\psi_x}{\psi} - \frac{\chi_x}{\chi}\right) - 4\frac{\psi_x\chi_x}{\psi\chi} - 2\lambda^2 = 0,$$
$$\frac{\psi_t}{\psi} + \frac{\chi_t}{\chi} - i\left(\frac{\psi_{xx}}{\psi} - \frac{\chi_{xx}}{\chi}\right) - 2\lambda\left(\frac{\psi_x}{\psi} + \frac{\chi_x}{\chi}\right) = 0,$$

(2.76)

which can be solved and expressed in the following form with the aid of (2.75),

$$\psi_t = -i(u^{[0]}w^{[0]} + 2\lambda^2)\psi + (2i\lambda u^{[0]} - u_x^{[0]})\chi,$$
$$\chi_t = i(u^{[0]}w^{[0]} + 2\lambda^2)\chi - (2i\lambda w^{[0]} + w_x^{[0]})\psi.$$

(2.77)

It is immediate to verify that the compatibility condition of the linear system (2.75) and (2.77) retrieves the NLS system (2.59), and hence, expressions (2.75) and (2.77) define a two-component spectral problem for the NLS equation.

Operator representation

The spectral problem (2.75) and (2.77) may be expressed in matrix form as the zero-curvature Lax pair

$$\Psi_x = \begin{pmatrix} -i\lambda & iu^{[0]} \\ -iw^{[0]} & i\lambda \end{pmatrix}\Psi, \qquad \Psi_t = \begin{pmatrix} -i(u^{[0]}w^{[0]} + 2\lambda^2) & 2i\lambda u^{[0]} - u_x^{[0]} \\ -2i\lambda w^{[0]} - w_x^{[0]} & i(u^{[0]}w^{[0]} + 2\lambda^2) \end{pmatrix}\Psi,$$

(2.78)

where we have defined the two-component eigenvector $\Psi = (\psi, \chi)^\intercal$, which clearly satisfies the compatibility condition (2.35).

On the other hand, the Lax representation of (2.75) and (2.77) is given in terms of the matrix differential operators

$$L = \begin{pmatrix} i\partial_x & u^{[0]} \\ w^{[0]} & -i\partial_x \end{pmatrix}, \qquad A = \begin{pmatrix} i\left(2\partial_{xx} - u^{[0]}w^{[0]}\right) & 2u^{[0]}\partial_x + u^{[0]}_x \\ 2w^{[0]}\partial_x + w^{[0]}_x & -i\left(2\partial_{xx} - u^{[0]}w^{[0]}\right) \end{pmatrix} \Psi,$$

(2.79)

where it is immediate to check that the Lax equation (2.27) successfully provides (2.59).

Eigenfunctions and singular manifold

The combination of definitions (2.66) and the linearization ansatz (2.74) allows us to write the relation between the singular manifold ϕ and the eigenfunctions ψ and χ, as

$$\phi_x = \psi\chi, \qquad \phi_t = 2\lambda\psi\chi + i\left(\psi_x\chi - \psi\chi_x\right), \tag{2.80}$$

which may be expressed in a more compact form through the exact derivative

$$d\phi = \psi\chi\,dx + \left[2\lambda\psi\chi + i\left(\psi_x\chi - \psi\chi_x\right)\right]dt. \tag{2.81}$$

5.4. Darboux transformations for NLS

Let us proceed with an illustrative example of the Darboux transformation approach hereunder. The SMM allows us to construct an iterative procedure to obtain non-trivial solutions by means of the eigenfunctions of a trivial seed solution.

Let $\{u^{[0]}, w^{[0]}, m^{[0]}\}$ be a seed solution for (2.59). Let us now consider two sets of eigenfunctions $\{\psi_1, \chi_1\}$ and $\{\psi_2, \chi_2\}$ associated to this seed solution with respective eigenvalues $\{\lambda_1, \lambda_2\}$, *i.e.*

$$\begin{aligned}
\psi_{1,x} &= -i\lambda_1\psi_1 + iu^{[0]}\chi_1, \\
\chi_{1,x} &= i\lambda_1\chi_1 - iw^{[0]}\psi_1, \\
\psi_{1,t} &= -i(u^{[0]}w^{[0]} + 2\lambda_1^2)\psi_1 + (2i\lambda_1 u^{[0]} - u^{[0]}_x)\chi_1, \\
\chi_{1,t} &= i(u^{[0]}w^{[0]} + 2\lambda_1^2)\chi_1 - (2i\lambda_1 w^{[0]} + w^{[0]}_x)\psi_1,
\end{aligned}$$

(2.82)

and

$$\psi_{2,x} = -i\lambda_2\psi_2 + iu^{[0]}\chi_2,$$
$$\chi_{2,x} = i\lambda_2\chi_2 - iw^{[0]}\psi_2,$$
$$\psi_{2,t} = -i(u^{[0]}w^{[0]} + 2\lambda_2^2)\psi_2 + (2i\lambda_2 u^{[0]} - u_x^{[0]})\chi_2,$$
$$\chi_{2,t} = i(u^{[0]}w^{[0]} + 2\lambda_2^2)\chi_2 - (2i\lambda_2 w^{[0]} + w_x^{[0]})\psi_2,$$

$$(2.83)$$

where the notation $\psi_{j,x}$, $\psi_{j,t}$ stands for the derivatives of the eigenfunction ψ_j with respect to x and t, respectively. The associated pair of singular manifolds $\{\phi_1, \phi_2\}$ are given by (2.81)

$$d\phi_1 = \psi_1\chi_1\,dx + \left[2\lambda_1\psi_1\chi_1 + i\left(\psi_{1,x}\chi_1 - \psi_1\chi_{1,x}\right)\right]dt, \qquad (2.84)$$

$$d\phi_2 = \psi_2\chi_2\,dx + \left[2\lambda_2\psi_2\chi_2 + i\left(\psi_{2,x}\chi_2 - \psi_2\chi_{2,x}\right)\right]dt. \qquad (2.85)$$

First iteration

The truncated series expansion (2.65) induces the iterated fields $\left\{u^{[1]}, w^{[1]}, m^{[1]}\right\}$, and then it is possible to define the eigenfunctions $\{\psi_{1,2}, \chi_{1,2}\}$ that satisfy the spectral problem for these iterated fields, with spectral parameter λ_2, of the form

$$(\psi_{1,2})_x = -i\lambda_2\psi_{1,2} + iu^{[1]}\chi_{1,2},$$
$$(\chi_{1,2})_x = i\lambda_2\chi_{1,2} - iw^{[1]}\psi_{1,2},$$
$$(\psi_{1,2})_t = -i(u^{[1]}w^{[1]} + 2\lambda_2^2)\psi_{1,2} + (2i\lambda_2 u^{[1]} - u_x^{[1]})\chi_{1,2},$$
$$(\chi_{1,2})_t = i(u^{[1]}w^{[1]} + 2\lambda_2^2)\chi_{1,2} - (2i\lambda_2 w^{[1]} + w_x^{[1]})\psi_{1,2}.$$

$$(2.86)$$

As mentioned, the iterated eigenfunctions will be denoted with a set of subindices, such that $\psi_{1,2}$ concerns the eigenfunction associated to the spectral problem for the first iterated solution, with fields $\left\{u^{[1]}, w^{[1]}, m^{[1]}\right\}$, with eigenvalue λ_2. The iterated eigenfunctions $\{\psi_{1,2}, \chi_{1,2}\}$ satisfying (2.86) allow us to construct the corresponding singular manifold $\phi_{1,2}$

$$d\phi_{1,2} = \psi_{1,2}\,\chi_{1,2}\,dx + \left[2\lambda_2\,\psi_{1,2}\,\chi_{1,2} + i\left(\psi_{1,2}\right)_x\chi_{1,2} - i\psi_{1,2}\left(\chi_{1,2}\right)_x\right]dt. \qquad (2.87)$$

The truncated series expansion arising from the application of the SMM constitutes auto-Bäcklund transformations for the corresponding variables. If we consider now the Lax pair (2.86) as a system of nonlinear differential equations that couples the fields $\left\{u^{[1]}, w^{[1]}, m^{[1]}\right\}$ and the eigenfunctions $\{\psi_{1,2}, \chi_{1,2}\}$, both fields and eigenfunctions should be expressed in terms of a similar Painlevé series expansion [83,148,248],

now constructed in a neighbourhood of the singular manifold $\phi_1 = 0$. The truncated series expansion for the fields (2.65) can be rewritten as

$$u^{[1]} = u^{[0]} + i\frac{\psi_1^2}{\phi_1}, \qquad w^{[1]} = w^{[0]} - i\frac{\chi_1^2}{\phi_1}, \qquad m^{[1]} = m^{[0]} + \frac{\psi_1\chi_1}{\phi_1}. \qquad (2.88)$$

Under this perspective, this last series expansion should be accompanied by the corresponding truncated expansion for the eigenfunctions, of the form

$$\psi_{1,2} = \psi_2 - \psi_1\frac{\Delta_{1,2}}{\phi_1}, \qquad \chi_{1,2} = \chi_2 - \chi_1\frac{\Delta_{2,1}}{\phi_1}, \qquad (2.89)$$

where the functions $\Delta_{1,2}(x,t)$, $\Delta_{2,1}(x,t)$ have to be determined.

Analogously, (2.87) may be regarded as a nonlinear differential equation that couples the eigenfunctions $\{\psi_{1,2}, \chi_{1,2}\}$ and the singular manifold $\phi_{1,2}$, so that the latter should also admit a truncated expansion

$$\phi_{1,2} = \phi_2 + \frac{\Omega_{1,2}}{\phi_1}, \qquad (2.90)$$

where the coefficient $\Omega_{1,2}(x,t)$ is expected to be related with $\Delta_{1,2}$, $\Delta_{2,1}$ from the expansion for the eigenfunctions.

By substituting expressions (2.88)-(2.90) into equations (2.86) and (2.87), we get to obtain the explicit expressions for the coefficients $\Delta_{1,2}, \Delta_{2,1}, \Omega_{1,2}$, given by

$$\Delta_{1,2} = \Delta_{2,1} = \frac{i}{2}\left(\frac{\chi_1\psi_2 - \chi_2\psi_1}{\lambda_2 - \lambda_1}\right), \qquad (2.91)$$

with derivatives

$$\begin{aligned}
(\Delta_{1,2})_x &= \frac{1}{2}\left(\psi_1\chi_2 + \psi_2\chi_1\right), \\
(\Delta_{1,2})_t &= i\left(\psi_1\psi_2 w^{[0]} - \chi_1\chi_2 u^{[0]}\right) + (\lambda_1 + \lambda_2)\left(\chi_1\psi_2 + \psi_1\chi_2\right),
\end{aligned} \qquad (2.92)$$

which are clearly symmetric under the permutation $1 \leftrightarrow 2$, and

$$\Omega_{1,2} = -\Delta_{1,2}\Delta_{2,1}. \qquad (2.93)$$

Consequently, expansions (2.88)-(2.90) are viewed as binary Darboux transformations, which map the spectral problem (2.82)-(2.83) into the same spectral problem (2.86), expressed in the new (iterated) variables.

For future calculations, it is convenient to define a 2×2 (symmetric) matrix $\Delta \equiv$

$(\Delta_{i,j})$, $i, j = 1, 2$ whose elements are computed as

$$
\begin{cases}
\Delta_{i,i} = \phi_i & \text{if } i = j \\
\Delta_{i,j} = \dfrac{i}{2} \dfrac{\chi_i \psi_j - \chi_j \psi_i}{\lambda_j - \lambda_i} & \text{if } i \neq j
\end{cases} ,
\tag{2.94}
$$

where the diagonal terms are established through the derivatives (2.92), when compared to (2.84)-(2.85), and the antidiagonal terms coincide with (2.91).

Second iteration and Hirota's τ-function

Darboux transformations constitute the basis of an iterative process that allows to obtain new solutions by recursively applying this procedure. In this regard, the first iterated quantities $\{u^{[1]}, w^{[1]}, m^{[1]}, \psi_{1,2}, \chi_{1,2}, \phi_{1,2}\}$ may be employed to perform a second iteration, such that the new fields $\{u^{[2]}, w^{[2]}, m^{[2]}\}$ can be constructed following expressions (2.88), as

$$
u^{[2]} = u^{[1]} + i\frac{\psi_{1,2}^2}{\phi_{1,2}}, \qquad w^{[2]} = w^{[1]} - i\frac{\chi_{1,2}^2}{\phi_{1,2}}, \qquad m^{[2]} = m^{[1]} + \frac{\psi_{1,2}\chi_{1,2}}{\phi_{1,2}}.
\tag{2.95}
$$

This second iteration can be easily expressed in terms of the different variables associated to the seed solution $\{u^{[0]}, w^{[0]}, m^{[0]}, \psi_1, \chi_1, \psi_2, \chi_2, \phi_1, \phi_2\}$, taking into account their definitions given in (2.88)-(2.90), and resulting in

$$
u^{[2]} = u^{[0]} + i\frac{\Delta_{2,2}\psi_1^2 - 2\Delta_{1,2}\psi_1\psi_2 + \Delta_{1,1}\psi_2^2}{\tau_{1,2}},
$$

$$
w^{[2]} = w^{[0]} - i\frac{\Delta_{2,2}\chi_1^2 - 2\Delta_{2,1}\chi_1\chi_2 + \Delta_{1,1}\chi_2^2}{\tau_{1,2}},
\tag{2.96}
$$

$$
m^{[2]} = m^{[0]} + \frac{(\tau_{1,2})_x}{\tau_{1,2}} = m^{[0]} + \left(\log \tau_{1,2} \right)_x,
$$

where we have introduced the function $\tau_{1,2}$, henceforth referred as τ-function, whose definition reads

$$
\tau_{1,2} = \phi_1 \phi_{1,2} = \phi_1 \phi_2 - \Delta_{1,2}\Delta_{2,1}.
\tag{2.97}
$$

In accordance with the definition of the Δ-matrix given in (2.94), we may write the τ-function as

$$
\tau_{1,2} = \det \Delta
\tag{2.98}
$$

and the second iteration for the fields as

$$
u^{[2]} = u^{[0]} + i\Psi^{\mathsf{T}}\Delta^{-1}\Psi, \quad w^{[2]} = w^{[0]} - i\,\mathrm{X}^{\mathsf{T}}\Delta^{-1}\mathrm{X}, \quad m^{[2]} = m^{[0]} + \left(\log \tau_{1,2} \right)_x,
\tag{2.99}
$$

with $\tau_{1,2} = \det \Delta \neq 0$, Δ^{-1} is the inverse matrix of Δ and where we have defined the two-dimensional eigenvectors $\Psi = (\psi_1, \psi_2)^{\mathsf{T}}$ and $\mathrm{X} = (\chi_1, \chi_2)^{\mathsf{T}}$.

It is worthwhile to remark that the function $\tau_{1,2}$ for the second iteration is not a singular manifold itself, but it can be constructed in terms of two singular manifolds from the previous iterations. This function is closely related to the τ-function, first introduced in [369], of Hirota's bilinear formalism [208, 211, 212] (cf. Subsection 4.4 of this Chapter).

Iterations of nth-order

This procedure may be implemented repeatedly and generalized up to the nth iteration. In general, once this point is reached, we are more concerned about the generation of the nth iteration for the fields, which leads to novel solutions for (2.59), rather than the explicit obtention of the associated eigenfunctions $\{\psi_{1,2,\ldots,n+1}, \chi_{1,2,\ldots,n+1}\}$, with spectral parameter λ_{n+1}, or singular manifold $\phi_{1,2,\ldots,n+1}$.

In order to properly construct this nth iteration for the fields, we should need the following elements:

- A seed solution $\{u^{[0]}, w^{[0]}, m^{[0]}\}$ for the original nonlinear problem (2.59),

- Two sets of n different eigenfunctions $\{\psi_1, \psi_2, \ldots, \psi_n\}$, $\{\chi_1, \chi_2, \ldots, \chi_n\}$ that solve the spectral problem (2.75) and (2.77) for this seed solution with spectral parameters $\{\lambda_1, \lambda_2, \ldots, \lambda_n\}$, respectively. In other words, these eigenfunctions satisfy the following Lax pairs

$$
\begin{aligned}
\psi_{j,x} &= -i\lambda_j \psi_2 + iu^{[0]}\chi_j, \\
\chi_{j,x} &= i\lambda_2 \chi_j - iw^{[0]}\psi_j, \\
\psi_{j,t} &= -i(u^{[0]}w^{[0]} + 2\lambda_j^2)\psi_j + (2i\lambda_j u^{[0]} - u_x^{[0]})\chi_j, \qquad j = 1, \ldots, n, \quad (2.100) \\
\chi_{j,t} &= i(u^{[0]}w^{[0]} + 2\lambda_j^2)\chi_j - (2i\lambda_j w^{[0]} + w_x^{[0]})\psi_j,
\end{aligned}
$$

- n singular manifolds $\{\phi_1, \phi_2, \ldots, \phi_n\}$, defined through (2.81) as

$$
d\phi_j = \psi_j \chi_j \, dx + \left[2\lambda_j \psi_j \chi_j + i\left(\psi_{j,x}\chi_j - \psi_j \chi_{j,x}\right) \right] dt, \qquad j = 1, \ldots, n. \quad (2.101)
$$

In this scenario, the iterated fields of order n $\{u^{[n]}, w^{[n]}, m^{[n]}\}$ may be expressed as

$$
\begin{aligned}
u^{[n]} &= u^{[0]} + i\Psi^{\mathsf{T}}\Delta^{-1}\Psi, \\
w^{[n]} &= w^{[0]} - i\,\mathrm{X}^{\mathsf{T}}\Delta^{-1}\mathrm{X}, \quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad (2.102)
\end{aligned}
$$

$$m^{[n]} = m^{[0]} + \frac{(\tau_{1,2,\ldots,n})_x}{\tau_{1,2,\ldots,n}} = m^{[0]} + \left(\log \tau_{1,2,\ldots,n} \right)_x,$$

in terms of the n-dimensional eigenvectors $\Psi = (\psi_1, \psi_2, \ldots, \psi_n)^\intercal$ and $\mathrm{X} = (\chi_1, \chi_2, \ldots, \chi_n)^\intercal$ and the τ-function

$$\tau_{1,2,\ldots,n} = \phi_1 \phi_{1,2} \cdot \cdots \cdot \phi_{1,2,\ldots,n} = \det \Delta, \tag{2.103}$$

where $\Delta \equiv (\Delta_{i,j})$, $i, j = 1, \ldots, n$ stands now for the generalization of (2.94) to a $n \times n$ (symmetric) matrix of entries

$$\begin{cases} \Delta_{i,i} = \phi_i & \text{if } i = j \\ \Delta_{i,j} = \dfrac{i}{2} \dfrac{\chi_i \psi_j - \chi_j \psi_i}{\lambda_j - \lambda_i} & \text{if } i \neq j \end{cases}, \qquad i, j = 1, \ldots, n. \tag{2.104}$$

It is fundamental to note that the nth iteration for the fields exclusively depends on the elements of the Δ-matrix defined in (2.104). Hence, the resolution of the spectral problem associated to the seed solution suffices to obtain the consecutive iterated solutions for the former nonlinear problem. The binary Darboux transformation approach constitutes an ideal procedure to determine multi-soliton solutions, since the n-soliton solution is closely related to the nth iteration of the fields. In particular, the n-soliton solution for the NLS system (2.59) can be obtained as the nth iteration for the probability density of the fields, *i.e.* the product

$$u^{[n]} w^{[n]} = -m_x^{[n]} = -m_x^{[0]} - \left(\frac{(\tau_{1,2,\ldots,n})_x}{\tau_{1,2,\ldots,n}} \right)_x, \tag{2.105}$$

which remains the usual expression for the n-soliton solution in the Hirota formalism [211, 212].

5.5. Soliton-like solutions for NLS

In view of the previous results, we conclude that the binary Darboux transformation method provides an algorithmic scheme to construct solutions of a given nonlinear problem, which may be summarized as follows:

1. We should choose a seed solution $\left\{ u^{[0]}, w^{[0]}, m^{[0]} \right\}$ for the initial NLS system (2.59). This ansatz will condition the nature of the arising solutions in the iteration process.

2. Secondly, the corresponding spectral problem for this seed solution (2.100) needs to be solved, to obtain the associated eigenfunctions and spectral parameters.

We should try to find as many solutions as desired iterations we would like to implement.

3. The third step entails the computation of the n^2 elements[6] of the Δ-matrix as defined in (2.104).

4. Once the elements $\Delta_{i,j}$, $i, j = 1, \ldots, n$ have been determined, we should now obtain the expression for the τ-function $\tau_{1,2,\ldots,n}$ following (2.103).

5. Finally, the probability density for this solution $u^{[n]}w^{[n]} = -m_x^{[n]}$ is given by (2.105).

Soliton solutions

Thereupon, we will be interested in the one and two soliton solutions for the defocusing NLS system described in (2.59).

1. *Seed solution*

   Since we attempt to find the standard soliton profile, it would be desirable to select exponential seed solutions as

   $$u^{[0]} = j_0 e^{-2ij_0^2 t}, \qquad w^{[0]} = j_0 e^{2ij_0^2 t}, \qquad m^{[0]} = -j_0^2 x, \qquad (2.106)$$

   where j_0 is a free parameter.

2. *Eigenfunctions and singular manifolds*

   We may find two sets of different eigenfunctions $\{\psi_1, \chi_1\}$ and $\{\psi_2, \chi_2\}$ for the spectral problem (2.100) associated to the seed solution (2.106), with spectral parameters $\{\lambda_1, \lambda_2\}$, respectively. These eigenfunctions possess exponential dependence as

   $$\psi_1 = e^{iA_1} e^{j_0 \sin(2A_1)(x+2\lambda_1 t)} e^{-ij_0^2 t}, \quad \chi_1 = e^{-iA_1} e^{j_0 \sin(2A_1)(x+2\lambda_1 t)} e^{ij_0^2 t}, \quad (2.107)$$

   $$\psi_2 = e^{iA_2} e^{j_0 \sin(2A_2)(x+2\lambda_2 t)} e^{-ij_0^2 t}, \quad \chi_2 = e^{-iA_2} e^{j_0 \sin(2A_2)(x+2\lambda_2 t)} e^{ij_0^2 t}, \quad (2.108)$$

   where $A_j, j = 1, 2$ are arbitrary constants and the respective spectral parameters are given by

   $$\lambda_1 = j_0 \cos(2A_1), \qquad \lambda_2 = j_0 \cos(2A_2). \qquad (2.109)$$

   The singular manifolds follow from (2.101), whose explicit expressions read

---

[6]For the NLS system, the Δ-matrix is symmetric, so the number of elements to compute is reduced to $\frac{n(n-1)}{2}$.

$$\phi_1 = \frac{k_1 + E_1^2}{2j_0 \sin{(2A_1)}}, \qquad \phi_2 = \frac{k_2 + E_2^2}{2j_0 \sin{(2A_2)}}, \qquad (2.110)$$

where k_j, $j = 1, 2$ are arbitrary parameters and we have defined

$$E_j = e^{j_0 \sin{(2A_j)}(x + 2j_0 \cos{(2A_j)}t)}, \qquad j = 1, 2. \qquad (2.111)$$

3. *First iteration: one-soliton solution*

   With these ingredients, we are in the position to perform the first iteration for the fields and therefore obtain the one-soliton solution for the NLS system (2.59). Focusing on the field $m_x^{[1]}$ (the density of probability up to a sign), we obtain

$$
\begin{aligned}
m_x^{[1]} = m_x^{[0]} + \left(\frac{(\phi_1)_x}{\phi_1}\right)_x &= -j_0^2 + \frac{4j_0^2 \sin^2{(2A_1)}k_1 E_1^2}{(k_1 + E_1^2)^2} \\
&= j_0^2 \left\{ \sin^2{(2A_1)} \operatorname{sech}^2\left[j_0 \sin{(2A_1)}\left(x + 2j_0 \cos{(2A_1)}\, t \right) - \frac{1}{2}\log k_1 \right] - 1 \right\},
\end{aligned}
$$
$$(2.112)$$

   which is displayed in Figure 2.1. This solution represents, up to a constant background, a travelling solitary wave that propagates with velocity $v = -2j_0 \cos{(2A_1)}$, amplitude $a = j_0^2 \sin^2{(2A_1)}$, wavenumber $k = j_0 \sin{(2A_1)}$ and initial phase $\varphi_0 = -\frac{1}{2}\log k_1$. As it typically happens with soliton solutions, the wave parameters $\{k, v, a\}$ are related through the arbitrary constant A_1, such that $a = k^2, v = \sqrt{1 - 4k^2}$. This solution constitutes the *dark soliton solution* $\left|u^{[1]}\right|^2$ for the defocusing NLS equation (2.58) [201, 432].
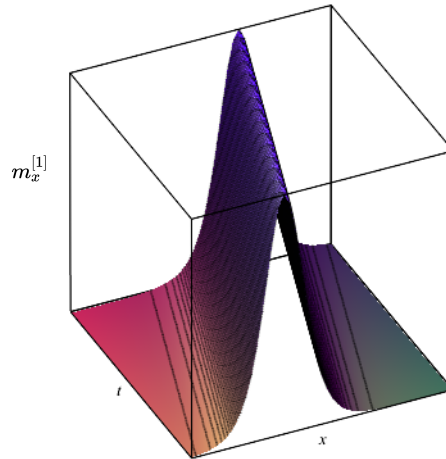
Figure 2.1: *One-soliton solution $m_x^{[1]}$ for NLS, with $j_0 = 1, A_1 = 4, k_1 = 0$.*

4. Δ-matrix and τ-function

According to its definition given in (2.104), we should now compute the different elements of the 2×2 matrix

$$\Delta_{1,1} = \phi_1 = \frac{k_1 + E_1^2}{2j_0 \sin(2A_1)}, \qquad \Delta_{2,2} = \phi_2 = \frac{k_2 + E_2^2}{2j_0 \sin(2A_2)},$$

$$\Delta_{1,2} = \Delta_{2,1} = \frac{\sin(A_1 - A_2)\,E_1 E_2}{j_0(\cos(2A_2) - \cos(2A_1))}, \tag{2.113}$$

where $E_j, j = 1,2$ are defined in (2.111). Hence, the τ-function is written as

$$\tau_{1,2} = \phi_1 \phi_2 - \Delta_{1,2}\Delta_{2,1} = \frac{k_1 k_2 + k_2 E_1^2 + k_1 E_2^2 + A_{1,2} E_1^2 E_2^2}{4j_0^2 \sin(2A_1) \sin(2A_2)}, \tag{2.114}$$

with

$$A_{1,2} = 1 - \frac{\sin(2A_1) \sin(2A_2)}{\sin^2(A_1 + A_2)}. \tag{2.115}$$

5. *Second iteration: two-soliton solution*

Then, the second iteration of the fields straightforwardly yields the two-soliton solution for NLS equation (2.59), as

$$m_x^{[2]} = m_x^{[0]} + \left(\frac{(\tau_{1,2})_x}{\tau_{1,2}} \right)_x, \tag{2.116}$$

when $\tau_{1,2}$ is defined in (2.114).
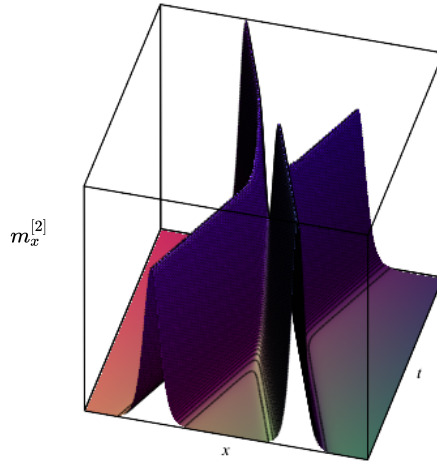
Figure 2.2: *Two-soliton solution $m_x^{[2]}$ for NLS, $j_0 = 1, A_1 = 1, A_2 = 2, k_1 = 0, k_2 = 0$.*

The two-soliton solution has been plotted in Figure 2.2. This picture is displayed in the center of mass reference frame, easily obtained by means of the Galilean transformation $x \to x + j_0[\cos(2A_1) + \cos(2A_2)]t$.

Further solutions

It has been demonstrated that multi-soliton solutions can be successfully obtained by this procedure. Furthermore, a plethora of additional soliton-like solutions may arise for an appropriate choice of both the seed solutions and the associated eigenfunctions, as it has been illustrated in [367]. The obtention of this kind of solutions will be sketched in the following, briefly presented and without extensive details.

Special kinds of breather solutions [19–21, 257, 284] and rogue waves [17, 18, 22, 337] can be easily obtained as particular cases of the two-soliton solution for NLS (2.114). Breather-like solutions are expected to exhibit a localized but oscillatory behaviour in the τ-function, whilst rogue waves are presented as both spatially and temporally localized structures that *appear from nowhere and disappear without a trace* [18]. The latter solutions are typically expressed in terms of explicit rational functions of the independent variables, therefore being known as rational solitons.

Let us consider the case $A_2 = -A_1 + \frac{\pi}{2}$. This choice has been specifically made such that the associated spectral parameters given in (2.109) satisfy $\lambda_2 = -\lambda_1$. The aforementioned selection yields the following expressions for the elements of the Δ-matrix in (2.113)

$$
\begin{aligned}
\Delta_{1,1} = \phi_1 &= \frac{k_1 + e^{2j_0 \sin(2A_1)[x + 2j_0 \cos(2A_1)t]}}{2j_0 \sin(2A_1)}, \\
\Delta_{2,2} = \phi_2 &= \frac{k_2 + e^{2j_0 \sin(2A_1)[x - 2j_0 \cos(2A_1)t]}}{2j_0 \sin(2A_1)}, \\
\Delta_{1,2} = \Delta_{2,1} &= \frac{e^{2j_0 \sin(2A_1)x}}{2j_0},
\end{aligned}
\tag{2.117}
$$

where k_1, k_2 are arbitrary parameters. Taking the additional choice $k_1 = k_2 = \cos(2A_1)$, the resulting τ-function in (2.114) reduces to

$$
\tau_{1,2} = \frac{e^{2j_0 \sin(2A_1)x}}{2j_0^2} \left\{ \frac{\cosh[2j_0 \sin(2A_1)x]}{\tan^2(2A_1)} + \frac{\cos(2A_1)\cosh[2j_0^2 \sin(4A_1)t]}{\sin^2(2A_1)} \right\},
\tag{2.118}
$$

which exclusively depends on the arbitrary parameters j_0 and A_1. For particular combinations of these two constants $\{j_0, A_1\}$, either real or complex, the following classes of solutions are obtained:

1. **Akhmediev breather**: periodic solution in the coordinate x and hyperbolic in t.

   This solution can be easily derived by considering real values of $A_1 \in \mathbb{R}$ and purely imaginary values for $j_0 = ih_0, h_0 \in \mathbb{R}$, such that the hyperbolic trigonometric functions in (2.118) become ordinary circular functions and viceversa, providing

   $$\tau_{1,2} \sim \cos\left(2A_1\right)\cos\left[2h_0\sin\left(2A_1\right)x\right] + \cosh[2h_0^2\sin\left(4A_1\right)t], \qquad (2.119)$$

   which corresponds to the so-called *Akhmediev breather* [19–21]. Figure 2.3 displays the corresponding solution $m_x^{[2]}$ associated to this τ-function through expression (2.116).
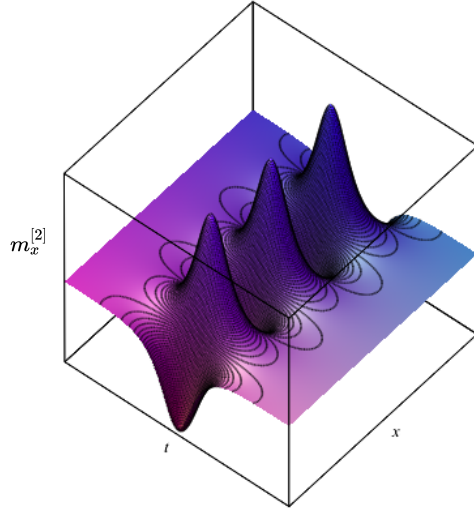
Figure 2.3: *Akhmediev breather $m_x^{[2]}$ for NLS system, with parameters $h_0 = 1, A_1 = 4$.*

2. **Kuznetsov-Ma breather**: periodic solution in the coordinate t and hyperbolic in x.

   A solution with these characteristics may be recovered by setting purely imaginary values of both j_0 and A_1, such that $j_0 = ih_0$, $A_1 = iB_1$ with $h_0, B_1 \in \mathbb{R}$. Then, the τ-function in (2.118) reads

   $$\tau_{1,2} \sim \cosh\left(2A_1\right)\cosh\left[2h_0\sinh\left(2B_1\right)x\right] + \cos\left[2h_0^2\sinh\left(4B_1\right)t\right], \qquad (2.120)$$

   which provides the celebrated *Kuznetsov-Ma breather* [257,284]. The associated

soliton solution is plotted in Figure 2.4.
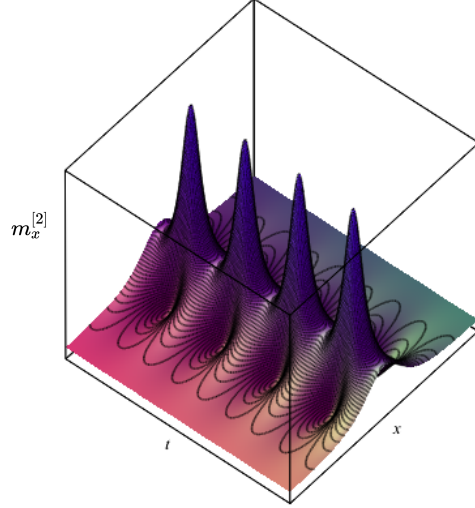
Figure 2.4: *Kuznetsov-Ma breather $m_x^{[2]}$ for NLS system, with parameters $h_0 = 1, B_1 = \frac{1}{2}$.*

3. **Peregrine soliton**: localized solution in both coordinates (x, t).

For the particular ansatz $A_1 = 0$, $A_2 = \frac{\pi}{2}$, $j_0 = ih_0$, $h_0 \in \mathbb{R}$, the different elements of the Δ-matrix become polynomial functions in the variables (x, t). By direct substitution of this choice in equations (2.107)-(2.110), we get that the Δ-matrix read

$$\phi_1 = (x + h_1) + i(2h_0 t + h_2),$$
$$\phi_2 = (x + h_1) - i(2h_0 t + h_2),$$
$$\Delta_{1,2} = -\frac{i}{2h_0},$$

(2.121)

where h_1, h_2 are two arbitrary real constants of integration. In this scenario, the τ-function results in

$$\tau_{1,2} = (x + h_1)^2 + (2h_0 t + h_2)^2 + \left(\frac{1}{2h_0}\right)^2,$$

(2.122)

which does not vanish anywhere. The associated profile $m_x^{[2]}$ yields the well-known *Peregrine soliton* [337], whose explicit representation is given in Figure 2.5.
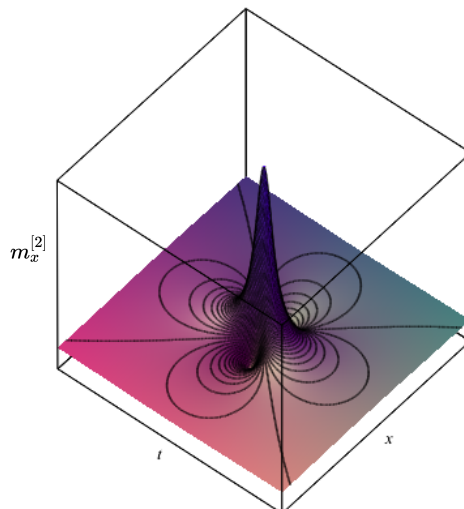
Figure 2.5: *Peregrine soliton $m_x^{[2]}$ for NLS system, with parameters $h_0 = 1, h_1 = 0, h_2 = 0$.*

The analysis performed for the NLS equation constitutes an enlightening example that illustrates the potential of the procedure established in the foundations of this Chapter. Regarding the integrability characterization, the conjunction of the Painlevé analysis and the SMM allows us to straightforwardly obtain the associated spectral problem for the nonlinear system. On the other hand, the (binary) Darboux transformation approach provides an iterative method to construct an extensive variety of solutions of diverse nature. This methodology has proved to be remarkably fruitful when deriving soliton-like solutions for integrable systems, as it has been validated in previous works [141, 142, 148, 150, 159, 161]. In particular, this thesis addresses the analysis of integrable systems under this perspective, focusing on the obtention of rational solitons of diverse kind. A detailed approach to several nonlinear integrable models has been conducted in the ensuing Chapters 3 and 4, devoted to the analysis of systems in $1 + 1$ dimensions or $2 + 1$ dimensions, respectively.

Chapter 3

Applications to PDEs in $1 + 1$ dimensions

The fundamental purpose of this Chapter lies in the different applications of the methodology described in Chapter 2 to the analysis of two integrable models in $1+1$ dimensions arising from the fields of Mathematical Physics, Materials Sciences and Biology. In particular, the two nonlinear evolution systems to be considered are constructed from integrable generalizations of the NLS equation [3, 130].

The approach to adopt, when studying these models, constitutes a direct application of the one already illustrated in the previous Chapter. The characterization of the integrability of these systems will be conducted by means of the Painlevé test, based on the WTC method for partial differential equations [417]. Once the integrability has been identified, we proceed to apply the SMM [410] with the subsequent aim of obtaining the associated spectral problems. Finally, binary Darboux transformations will be properly implemented to derive different families of soliton-like solutions, depending on the case under consideration.

This Chapter is organized in two Sections. Section 1 is devoted to the integrability analysis of a model in $1+1$ dimensions that aims to describe spin transport phenomena in biological helical molecules. This model, proposed in [124, 125], emerges as a theoretical attempt to characterize the physical effect known as *chiral-induced spin selectivity*, reported by numerous experiments on electron transport through helical molecules [113, 198, 300, 310, 354]. The helical conformation of the dipoles induces an unconventional Rashba-like interaction [65, 352] that couples the electron spin with its linear momentum. Besides, an additional nonlinearity arises from the electron-lattice interaction, enabling the formation of a plethora of stable soliton configurations, depending on the focusing or defocusing nature of the nonlinear interaction. The research conducted in this Section is fully covered in publications [26, 124].

Section 2 addresses the study and characterization of the integrability of a modified $(1+1)$-nonlinear Schrödinger equation with derivative-type nonlinearities. This kind of generalizations provides the so-called derivative nonlinear Schrödinger (DNLS)

equations. The Painlevé analysis for this generalized equation additionally leads to other two differential equations of interest: a conservative PDE for the probability density of the initial equation and a nonlocal Boussinesq-like equation. These three equations are deeply analyzed regarding their integrability, providing two equivalent Lax pairs for each equation and successfully deriving rational soliton solutions. The content available in this Section is based on the author's contributions [23, 28].

1. Nonlinear Schrödinger equation in $1+1$ dimensions for a deformable helical molecule

It has been observed in experiments that the helical conformation of certain organic molecules, such as DNA, may induce a sizable chiral-spin selectivity [37, 113, 134, 189, 244, 300, 302, 310, 354, 421]. This effect results from the spin-orbit coupling (SOC) between the electronic momentum and the electric field created by the helical configuration of the molecule. Many theoretical models have been proposed to explain these experimental evidences within different frameworks [68, 135, 196–199, 298, 299, 349, 424], but there is not full agreement with experimental data yet.

Theoretical models usually assume rigid lattices and neglect the local deformation of the molecule about the carrier. However, this assumption seems unrealistic to describe charge transport in molecular systems like DNA [75]. Depending on the energy scales involved, lattice deformation can play a significant role on transport properties [173, 216, 312, 338]. Besides, non negligible molecule deformations, and consequently, the interaction between quasiparticles (electrons or excitons) and the lattice vibrations, have been proved to be particularly useful to describe charge and energy transfer processes in α-helical molecules [126, 246, 288].

In this context, it is mandatory to mention Davydov's soliton theory [81, 121, 122]. Davydov proposed a nonlinear mechanism to describe the energy transfer in α-helix proteins, where their deformability is taken into account. The interaction between the amide-I vibrations and the hydrogen bonds induce a self-trapping phenomenon that enables the formation of collective excitations. These excitations behave as stable and localized quasiparticles which propagate uniformly, the so-called *Davydov solitons*. Besides, it is shown that the continuous limit of Davydov's equations in the adiabatic approximation reduces to NLS equation for the elementary excitations [123].

1.1. Description of the model

In the following, we introduce a nonlinear model in $1 + 1$ dimensions describing the spin dynamics of an electron that propagates along the axis of a deformable helical

molecule [26,124]. From the theoretical point of view, this model is based on two main contributions. The spin-molecule interaction is due to an unconventional Rashba-like SOC over the propagating electronic current along the axis of the molecule, whilst the lattice vibrations give rise to an additional nonlinear interaction.

Spin dynamics in a rigid helical molecule

This candidate generalizes the linear model formerly introduced by Gutiérrez *et al.* [198], and recently revisited by Díaz *et al.* [125]. Firstly, the spin-molecule interaction arises from an unconventional Rashba-like SOC, reflecting the helical symmetry of molecules due to the electron motion in a helical arrangement of peptides dipoles [124, 125]. To be specific, a helical conformation of tangentially oriented dipoles is considered to be spin-orbit coupled to the electron motion directed along the axis.

The Hamiltonian of the propagating electrons can be expressed as

$$\hat{\mathcal{H}} = \frac{\hat{p}^2}{2m} + \mu\boldsymbol{\sigma} \cdot (\hat{\boldsymbol{p}} \times \boldsymbol{E}), \tag{3.1}$$

where $\hat{p}$ is the linear momentum of the carriers and $\hat{\mathcal{H}}_{\text{SOC}} = \mu\boldsymbol{\sigma}\cdot(\hat{\boldsymbol{p}}\times\boldsymbol{E})$ stands for the SOC Hamiltonian. The strength of the SOC is given by $\mu = \frac{e\hbar}{(2mc)^2}$, $\boldsymbol{\sigma} = (\sigma_x, \sigma_y, \sigma_z)$ is a vector whose components are the Pauli matrices and $\boldsymbol{E}$ is the electric field in the rest frame of the electron that induces the magnetic field responsible of the SOC.

Assuming that the helical molecule is oriented along the Z axis, the resulting dimensionless Hamiltonian $\hat{H} = \frac{1}{E_b}\hat{\mathcal{H}}$ reads [124, 125, 198]

$$\hat{H} = -\partial_{\xi\xi} - 2\pi\gamma \begin{pmatrix} 0 & e^{-i2\pi\xi}(i\partial_\xi + \pi) \\ e^{i2\pi\xi}(i\partial_\xi - \pi) & \end{pmatrix}, \tag{3.2}$$

where $E_b = \frac{\hbar^2}{2mb^2}$, with m and b being the electron mass and pitch of the helix, respectively. The (dimensionless) spatial coordinate is given by $\xi = \frac{z}{b}$, $\gamma = \frac{\hbar\mu\mathcal{E}_0}{2\pi bE_b}$ stands for a dimensionless constant that is proportional to the magnitude of the SOC and the subscript indicates differentiation with respect to ξ.

The dimensionless Hamiltonian (3.2) is readily diagonalized since it commutes with the helical operator $\hat{q} = \hat{p} + \pi\sigma_z$. The corresponding normalized eigenfunctions and eigenenergies, which satisfy the associated dimensionless Schrödinger equation $\hat{H}\boldsymbol{\chi}(\xi) = \varepsilon\boldsymbol{\chi}(\xi)$, are found to be

$$\boldsymbol{\chi}(\xi) \equiv \boldsymbol{\chi}_{qs}(\xi) = \begin{pmatrix} \beta_1(s)\,e^{i(q-\pi)\xi} \\ \beta_2(s)\,e^{i(q+\pi)\xi} \end{pmatrix}, \qquad \varepsilon_{qs} = q^2 + \pi^2 - 2\pi s\sqrt{1+\gamma^2}\,q, \tag{3.3}$$

such that $s = \pm 1$, the coefficients $\beta_1(s)$ and $\beta_2(s)$ are given by

$$
\begin{aligned}
\beta_1(s) &= \frac{1}{2}\left[(1+s)\cos\theta + (1-s)\sin\theta\right], \\
\beta_2(s) &= \frac{1}{2}\left[(1-s)\cos\theta - (1+s)\sin\theta\right],
\end{aligned}
\tag{3.4}
$$

satisfying $\beta_1^2(s) + \beta_2^2(s) = 1$, and

$$
\tan\theta = \frac{\gamma}{1 + \sqrt{1+\gamma^2}}.
\tag{3.5}
$$

The helical conformation of the electric dipoles gives rise to an additional effective momentum $q_s = s\pi\sqrt{1+\gamma^2}$, even in the absence of SOC. Previous studies report on this fact as having an impact on the linear optical response as well [127].

The full dynamics of the electron current within the rigid helical molecule can be obtained by solving the associated time-dependent Schrödinger equation

$$
i\partial_t \boldsymbol{\chi}(\xi, t) = \hat{H}\boldsymbol{\chi}(\xi, t).
\tag{3.6}
$$

In particular, publication [124], following [125], addresses the coherent dynamics of an electron wave packet with an initial spatial Gaussian distribution. The spin projection onto the helical axis of different initial configurations have been studied, as well as its asymptotic behaviour, with the sole purpose of better understanding the origin of spin selectivity found in experiments.

Spin dynamics in a deformable helical molecule

According to Davydov's theory [122], the effects of the molecule deformability should be reflected in the appearance of an additional nonlinear term within the adiabatic approximation. In order to describe a deformable helical molecule where the electron dynamics is affected by the lattice vibrations, we will assume that the different dipoles vibrate along the molecule axis independently of each other. After a perturbative analysis developed in [124], we may conclude that the first-order correction to the potential energy of the electron in the electric field created by the dipoles due to the lattice vibrations is given by

$$
\delta V(z) = k\boldsymbol{\chi}^\dagger(z, t) \cdot \boldsymbol{\chi}(z, t),
\tag{3.7}
$$

computed in the adiabatic approximation and the continuum limit, with k being a constant and where the dagger sign $\dagger$ henceforth denotes the conjugate transpose.

This term should be now added to the Hamiltonian $\hat{H}$ discussed in (3.2) in order to describe the electron dynamics in a deformable helical molecule. In such scenario, the dimensionless Schrödinger equation (3.6) becomes a modified NLS equation in $1+1$ dimensions, describing the dynamics of the spinor state $\boldsymbol{\chi}(\xi,t) = [\chi_1(\xi,t), \chi_2(\xi,t)]^{\mathsf{T}}$ by means of the evolution PDE

$$i\partial_t\boldsymbol{\chi}(\xi,t) = \hat{H}\boldsymbol{\chi}(\xi,t) + 2g\left[\boldsymbol{\chi}^\dagger(\xi,t)\cdot\boldsymbol{\chi}(\xi,t)\right]\boldsymbol{\chi}(\xi,t), \tag{3.8}$$

where $\hat{H}$ is given in equation (3.2) and where we have renamed $k = 2g$ in (3.7). g is therefore a free parameter that characterizes the nature of the nonlinear interaction, considering hereafter both cases, $g > 0$ for the defocusing case and $g < 0$ for the focusing case.

Equation (3.8) can be considered as a generalization of the Manakov system [286,345], which is often also called vector NLS system [7,8]. Integrability properties of this Manakov system and the Painlevé Property are described in references [282,404]. Different generalizations of the Manakov system can be found in [389] and, more recently in [435]. Furthermore, equation (3.8) constitutes the only integrable case of a recent model proposed by Kartashov and Konotov [234]. This one-dimensional nonlinear model addresses the dynamics of spatially inhomogeneous Bose-Einstein condensates with helical SOC, where it can be proved that the Gross-Pitaevskii equation for this system reduces to equation (3.8) when the Zeeman splitting is negligible.

Equation (3.8) may be rewritten in autonomous form through the substitution

$$\boldsymbol{\chi}(\xi,t) = N_g \begin{pmatrix} e^{-i\pi(\xi+\pi t)} & 0 \\ 0 & e^{i\pi(\xi-\pi t)} \end{pmatrix} \boldsymbol{\alpha}(\xi,t), \tag{3.9}$$

where $\boldsymbol{\alpha}(\xi,t) = [\alpha_1(\xi,t), \alpha_2(\xi,t)]^{\mathsf{T}}$ constitutes the new spinor state, of components $\alpha_j(\xi,t)$, $j = 1, 2$, and the value of the constant N_g is given by

$$N_g = \begin{cases} \sqrt{\dfrac{1}{g}} & \text{for } g > 0 \text{ (defocusing nonlinear interaction)} \\[3mm] i\sqrt{\dfrac{1}{|g|}} & \text{for } g < 0 \text{ (focusing nonlinear interaction)} \end{cases} . \tag{3.10}$$

In both cases, the change (3.9) yields the following system of differential equations

$$\begin{aligned} \left(i\partial_t + \partial_{\xi\xi} - 2i\pi\partial_\xi - 2\boldsymbol{\alpha}^\dagger\cdot\boldsymbol{\alpha}\right)\alpha_1 + 2i\pi\gamma\,\partial_\xi\alpha_2 &= 0, \\ 2i\pi\gamma\,\partial_\xi\alpha_1 + \left(i\partial_t + \partial_{\xi\xi} + 2i\pi\partial_\xi - 2\boldsymbol{\alpha}^\dagger\cdot\boldsymbol{\alpha}\right)\alpha_2 &= 0. \end{aligned} \tag{3.11}$$

The ensuing Subsections will be devoted to the analysis of this latter model. For subsequent calculations, we should preferably consider both the system (3.11) and its complex conjugate. The integrability analysis arises from the Painlevé test, whilst the Lax pair can be successfully derived by means of the SMM. The Painlevé Property can also be used to derive Darboux transformations and an iterative procedure for obtaining analytic solutions. It is precisely the electron-lattice interaction introduced in (3.7) that enables the formation of stable soliton-like structures, providing an extraordinarily rich dynamics in the spectrum of solutions for both regimes, either in the focusing or the defocusing case.

1.2. Painlevé test and integrability

We shall now apply the Painlevé test [417], based on the WTC algorithm, described in Subsection 1.2.2 of Chapter 2, to analyze the integrability of the nonlinear system (3.11). This procedure requires the following ansatz for the two components of $\boldsymbol{\alpha}(\xi, t)$

$$
\begin{aligned}
\alpha_1 &= \sum_{j=0}^{\infty} a_j \phi^{j-\beta}, \qquad \alpha_1^{\dagger} = \sum_{j=0}^{\infty} a_j^{\dagger} \phi^{j-\beta}, \\
\alpha_2 &= \sum_{j=0}^{\infty} b_j \phi^{j-\delta}, \qquad \alpha_2^{\dagger} = \sum_{j=0}^{\infty} b_j^{\dagger} \phi^{j-\delta},
\end{aligned}
\tag{3.12}
$$

where $a_j(\xi, t)$, $a_j^{\dagger}(\xi, t)$, $b_j(\xi, t)$, $b_j^{\dagger}(\xi, t)$, $\forall j$ are arbitrary coefficients for the series expansion, β, γ are the leading indices and $\phi(\xi, t)$ is the so-called singular manifold. This ansatz means that the solutions are single-valued in a neighbourhood of the singular manifold $\phi(\xi, t) = 0$.

1. The leading-order analysis for the case $j = 0$ trivially yields

$$
\begin{aligned}
\beta &= \delta = 1, \\
a_0 &= A\phi_{\xi}, \quad a_0^{\dagger} = A^{\dagger}\phi_{\xi}, \quad b_0 = B\phi_{\xi}, \quad b_0^{\dagger} = B^{\dagger}\phi_{\xi},
\end{aligned}
\tag{3.13}
$$

such that

$$
AA^{\dagger} + BB^{\dagger} = 1.
\tag{3.14}
$$

The normalization condition (3.14) implies that the coefficients $\{A, A^{\dagger}, B, B^{\dagger}\}$ are not independent, indicating the presence of a resonance in $j = 0$. The number of independent coefficients will be given by the order of the aforementioned resonance.

2. A straightforward calculation provides the following resonance condition

$$j^3(j-3)^3(j-4)(j+1) = 0, \tag{3.15}$$

which retrieves eight resonances, given by the usual resonance in $j = -1$ associated to the arbitrariness of the singular manifold $\phi = 0$, triple resonances in both $j = 0$ and $j = 3$ and a single resonance in $j = 4$. The triple resonance in $j = 0$ indicates that three of the coefficients $\{A, A^\dagger, B, B^\dagger\}$ are independent, which means that one of them can be expressed in terms of the remaining ones.

3. The computation of the resonance conditions for $j = 0, 3, 4$ have been handled with MAPLE, successfully checking that all of them are identically satisfied.

Therefore, we can conclude that the solutions are single-valued around the singularity manifold $\phi = 0$ and the nonlinear system formed by (3.11) and its complex conjugate possesses the Painlevé Property. Hence, the system under consideration has proved to be integrable.

It is worth mentioning that the same Painlevé test, when applied to the model introduced by Kartashov and Konotop in reference [234], is only satisfied when the Zeeman splitting vanishes. Therefore, we are led to the conclusion that (3.11) is the only integrable case of the model proposed by the authors in [234].

1.3. The singular manifold method

The Painlevé Property is usually considered as a proof of the integrability of the equation, especially when it can be used to derive the associated spectral problem. The equivalence between the Painlevé Property and the Lax pair can be achieved through the SMM [410]. The SMM implies the truncation of the Laurent series (3.12) at constant level

$$
\begin{aligned}
\alpha_1^{[1]} &= \frac{A\phi_\xi}{\phi} + \alpha_1^{[0]}, & \left(\alpha_1^{[1]}\right)^\dagger &= \frac{A^\dagger \phi_\xi}{\phi} + \left(\alpha_1^{[0]}\right)^\dagger, \\
\alpha_2^{[1]} &= \frac{B\phi_\xi}{\phi} + \alpha_2^{[0]}, & \left(\alpha_2^{[1]}\right)^\dagger &= \frac{B^\dagger \phi_\xi}{\phi} + \left(\alpha_2^{[0]}\right)^\dagger,
\end{aligned}
\tag{3.16}
$$

such that the normalization condition again implies $AA^\dagger + BB^\dagger = 1$.

Relations (3.16) constitute auto-Bäcklund transformations, where $\boldsymbol{\alpha}^{[0]} = \left(\alpha_1^{[0]}, \alpha_2^{[0]}\right)^\mathsf{T}$ constitutes the seed solution and $\boldsymbol{\alpha}^{[1]} = \left(\alpha_1^{[1]}, \alpha_2^{[1]}\right)^\mathsf{T}$ is the iterated one. $\left(\boldsymbol{\alpha}^{[0]}\right)^\dagger$ and $\left(\boldsymbol{\alpha}^{[1]}\right)^\dagger$, with their respective components, stand for the associated conjugate

transpose elements of $\boldsymbol{\alpha}^{[0]}$ and $\boldsymbol{\alpha}^{[1]}$, respectively.

Substitution of the truncated expansions (3.16) into the starting system (3.11) yields polynomial expressions in powers of the singular manifold ϕ whose coefficients should identically vanish. The standard approach to deal with the resulting equations involves defining the following quantities

$$r = \frac{\phi_t}{\phi_\xi}, \qquad v = \frac{\phi_{\xi\xi}}{\phi_\xi}, \qquad s = v_\xi - \frac{v^2}{2}, \tag{3.17}$$

where r and the Schwarzian derivative s are the so-called homographic invariants.

Expressions of the fields in terms of the singular manifold

With the aid of the definitions given in (3.17), the expressions of the seed fields $\boldsymbol{\alpha}^{[0]}$ are

$$\begin{aligned}
\alpha_1^{[0]} &= -A_\xi + i\pi\left(A - \gamma B\right) - \frac{A}{2}\left(v + ir\right), \\
\alpha_2^{[0]} &= -B_\xi - i\pi\left(B + \gamma A\right) - \frac{B}{2}\left(v + ir\right),
\end{aligned} \tag{3.18a}$$

while the conjugate transpose $\left(\boldsymbol{\alpha}^{[0]}\right)^\dagger$ reads

$$\begin{aligned}
\left(\alpha_1^{[0]}\right)^\dagger &= -A_\xi^\dagger - i\pi\left(A^\dagger - \gamma B^\dagger\right) - \frac{A^\dagger}{2}\left(v - ir\right), \\
\left(\alpha_2^{[0]}\right)^\dagger &= -B_\xi^\dagger + i\pi\left(B^\dagger + \gamma A^\dagger\right) - \frac{B^\dagger}{2}\left(v - ir\right),
\end{aligned} \tag{3.18b}$$

Singular manifold equations

The equations that the singular manifold should satisfy in order to fulfill the truncation ansatz can be written as

$$r = -2\lambda + i\left(A^\dagger A_\xi + B^\dagger B_\xi\right) + \pi\left(AA^\dagger - BB^\dagger\right) - \gamma\pi\left(AB^\dagger + A^\dagger B\right), \tag{3.19a}$$

$$\begin{aligned}
A_t ={}& iA_{\xi\xi} - 2i\pi^2\left(1 + \gamma^2\right)A + A\left(-r_\xi + iv_\xi - \frac{i}{2}r^2 - \frac{i}{2}v^2\right) \\
&+ 2\pi A_\xi - 2\pi\gamma B_\xi + 2Ar\left(AA_\xi^\dagger + BB_\xi^\dagger\right) - 2iA\left(A_\xi A_\xi^\dagger + B_\xi B_\xi^\dagger\right) \\
&+ 2A\pi\gamma\left(AB_\xi^\dagger - B^\dagger A_\xi + BA_\xi^\dagger - A^\dagger B_\xi\right) - 2iA\pi\gamma r\left(AB^\dagger + BA^\dagger\right)
\end{aligned} \tag{3.19b}$$

$$- 2A\pi \left[ir \left(BB^\dagger - AA^\dagger \right) + 2 \left(B^\dagger B_\xi + AA^\dagger_\xi \right) \right],$$

$$
\begin{aligned}
A^\dagger_t &= -iA^\dagger_{\xi\xi} + 2i\pi^2 \left(1 + \gamma^2 \right) A^\dagger + A^\dagger \left(-r_\xi - iv_\xi + \frac{i}{2} r^2 + \frac{i}{2} v^2 \right) \\
&\quad + 2\pi A^\dagger_\xi - 2\pi\gamma B^\dagger_\xi + 2A^\dagger r \left(A^\dagger A_\xi + B^\dagger B_\xi \right) + 2iA^\dagger \left(A_\xi A^\dagger_\xi + B_\xi B^\dagger_\xi \right) \\
&\quad - 2A^\dagger \pi\gamma \left(AB^\dagger_\xi - B^\dagger A_\xi + BA^\dagger_\xi - A^\dagger B_\xi \right) + 2iA^\dagger \pi\gamma r \left(AB^\dagger + BA^\dagger \right) \\
&\quad - 2A^\dagger \pi \left[-ir \left(BB^\dagger - AA^\dagger \right) + 2 \left(BB^\dagger_\xi + A^\dagger A_\xi \right) \right],
\end{aligned}
\tag{3.19c}
$$

$$
\begin{aligned}
B_t &= iB_{\xi\xi} - 2i\pi^2 \left(1 + \gamma^2 \right) B + B \left(-r_\xi + iv_\xi - \frac{i}{2} r^2 - \frac{i}{2} v^2 \right) \\
&\quad - 2\pi B_\xi - 2\pi\gamma A_\xi + 2Br \left(AA^\dagger_\xi + BB^\dagger_\xi \right) - 2iB \left(A_\xi A^\dagger_\xi + B_\xi B^\dagger_\xi \right) \\
&\quad + 2B\pi\gamma \left(AB^\dagger_\xi - B^\dagger A_\xi + BA^\dagger_\xi - A^\dagger B_\xi \right) - 2iB\pi\gamma r \left(AB^\dagger + BA^\dagger \right) \\
&\quad + 2B\pi \left[-ir \left(BB^\dagger - AA^\dagger \right) + 2 \left(A^\dagger A_\xi + BB^\dagger_\xi \right) \right],
\end{aligned}
\tag{3.19d}
$$

$$
\begin{aligned}
B^\dagger_t &= -iB^\dagger_{\xi\xi} + 2i\pi^2 \left(1 + \gamma^2 \right) B^\dagger + B^\dagger \left(-r_\xi - iv_\xi + \frac{i}{2} r^2 + \frac{i}{2} v^2 \right) \\
&\quad - 2\pi B^\dagger_\xi - 2\pi\gamma A^\dagger_\xi + 2B^\dagger r \left(A^\dagger A_\xi + B^\dagger B_\xi \right) + 2iB^\dagger \left(A_\xi A^\dagger_\xi + B_\xi B^\dagger_\xi \right) \\
&\quad - 2B^\dagger \pi\gamma \left(AB^\dagger_\xi - B^\dagger A_\xi + BA^\dagger_\xi - A^\dagger B_\xi \right) + 2iB^\dagger \pi\gamma r \left(AB^\dagger + BA^\dagger \right) \\
&\quad + 2B^\dagger \pi \left[ir \left(BB^\dagger - AA^\dagger \right) + 2 \left(AA^\dagger_\xi + B^\dagger B_\xi \right) \right],
\end{aligned}
\tag{3.19e}
$$

where the subscript t denotes a time derivative and the arbitrary constant 2λ[1] arises from the integration of the invariant r.

Moreover, we should not forget the existing relation between $\{v, r\}$ arising from the compatibility condition of their definitions given in (3.17), such that

$$v_t = (r_\xi + rv)_\xi. \tag{3.20}$$

1.4. Spectral problem

The equations for the fields (3.18) and the singular manifold equations (3.19) can be linearized by introducing three new function $\{\psi(\xi, t), \omega(\xi, t), \eta(\xi, t)\}$ and their

---

[1]The multiplicative factor 2 is written for convenience in view of the forthcoming calculations.

conjugates $\{\psi^\dagger(\xi, t), \omega^\dagger(\xi, t), \eta^\dagger(\xi, t)\}$. Following the prescription described in [151, 410], we introduced the definitions

$$A = \frac{\omega}{\psi}, \qquad A^\dagger = \frac{\omega^\dagger}{\psi^\dagger}, \qquad B = \frac{\eta}{\psi}, \qquad B^\dagger = \frac{\eta^\dagger}{\psi^\dagger}, \qquad (3.21)$$

that allow us to write the normalization condition $AA^\dagger + BB^\dagger = 1$ as

$$\omega\omega^\dagger + \eta\eta^\dagger - \psi\psi^\dagger = 0. \qquad (3.22)$$

Therefore, the variables $\{v, r\}$ can be written as

$$v = \frac{\psi_\xi}{\psi} + \frac{\psi_\xi^\dagger}{\psi^\dagger}, \qquad r = -2\lambda - i\left(\frac{\psi_\xi}{\psi} - \frac{\psi_\xi^\dagger}{\psi^\dagger}\right). \qquad (3.23)$$

Spatial part of the Lax pair

If we define the eigenvectors

$$\boldsymbol{\Psi} = \begin{pmatrix} \psi \\ \omega \\ \eta \end{pmatrix}, \qquad \boldsymbol{\Psi}^\dagger = \begin{pmatrix} \psi^\dagger \\ \omega^\dagger \\ \eta^\dagger \end{pmatrix}, \qquad (3.24)$$

the definitions introduced in (3.21) in combination with (3.22) allow us to straight-forwardly linearize the expressions for the seed fields given in (3.18), yielding the spatial part for the spectral problem for (3.11), of the form

$$\boldsymbol{\Psi}_\xi = V_1\left[\alpha_1^{[0]}, \alpha_2^{[0]}\right]\boldsymbol{\Psi} + i\lambda V_2\boldsymbol{\Psi} + i\pi V_3(\gamma)\boldsymbol{\Psi}, \qquad (3.25a)$$

and its complex conjugate

$$\boldsymbol{\Psi}_\xi^\dagger = V_1\left[\left(\alpha_1^{[0]}\right)^\dagger, \left(\alpha_2^{[0]}\right)^\dagger\right]\boldsymbol{\Psi}^\dagger - i\lambda V_2\boldsymbol{\Psi}^\dagger - i\pi V_3(\gamma)\boldsymbol{\Psi}^\dagger, \qquad (3.25b)$$

where the parameter λ acts as the spectral parameter and the 3×3 matrices V_1, V_2, V_3 have the expressions

$$V_1\left[\alpha_1^{[0]}, \alpha_2^{[0]}\right] = \begin{pmatrix} 0 & -\left(\alpha_1^{[0]}\right)^\dagger & -\left(\alpha_2^{[0]}\right)^\dagger \\ -\alpha_1^{[0]} & 0 & 0 \\ -\alpha_2^{[0]} & 0 & 0 \end{pmatrix}, \qquad (3.26)$$

$$V_2 = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \qquad V_3(\gamma) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & -\gamma \\ 0 & -\gamma & -1 \end{pmatrix},$$

such that the dependence of the spatial Lax pair with the parameter γ is encoded in the matrix V_3.

Temporal part of the Lax pair

A similar result can be found for the time derivatives of $\{\boldsymbol{\Psi}, \boldsymbol{\Psi}^\dagger\}$, obtained through the linearization process of the singular manifold equations (3.19) and (3.20),

$$\begin{aligned}
\boldsymbol{\Psi}_t &= iU_1\left[\alpha_1^{[0]}, \alpha_2^{[0]}\right]\boldsymbol{\Psi} + \pi U_2\left[\gamma, \alpha_1^{[0]}, \alpha_2^{[0]}\right]\boldsymbol{\Psi} - i\pi^2 U_3(\gamma)\boldsymbol{\Psi} \\
&\quad - 2\lambda\left(V_1\left[\alpha_1^{[0]}, \alpha_2^{[0]}\right] + i\lambda V_2\right)\boldsymbol{\Psi},
\end{aligned} \tag{3.27a}$$

and its complex conjugate

$$\begin{aligned}
\boldsymbol{\Psi}_t^\dagger &= -iU_1\left[\left(\alpha_1^{[0]}\right)^\dagger, \left(\alpha_2^{[0]}\right)^\dagger\right]\boldsymbol{\Psi}^\dagger + \pi U_2\left[\gamma, \left(\alpha_1^{[0]}\right)^\dagger, \left(\alpha_2^{[0]}\right)^\dagger\right]\boldsymbol{\Psi}^\dagger \\
&\quad + i\pi^2 U_3(\gamma)\boldsymbol{\Psi}^\dagger - 2\lambda\left(V_1\left[\left(\alpha_1^{[0]}\right)^\dagger, \left(\alpha_2^{[0]}\right)^\dagger\right] - i\lambda V_2\right)\boldsymbol{\Psi}^\dagger,
\end{aligned} \tag{3.27b}$$

where the matrices V_1, V_2 are given in (3.26) and we have introduced the additional 3×3 matrices U_1, U_2, U_3 as

$$U_1\left[\alpha_1^{[0]}, \alpha_2^{[0]}\right] = \begin{pmatrix} \alpha_1^{[0]}\left(\alpha_1^{[0]}\right)^\dagger + \alpha_2^{[0]}\left(\alpha_2^{[0]}\right)^\dagger & \partial_\xi\left[\left(\alpha_1^{[0]}\right)^\dagger\right] & \partial_\xi\left[\left(\alpha_2^{[0]}\right)^\dagger\right] \\ -\partial_\xi\left[\alpha_1^{[0]}\right] & -\alpha_1^{[0]}\left(\alpha_1^{[0]}\right)^\dagger & -\alpha_1^{[0]}\left(\alpha_2^{[0]}\right)^\dagger \\ -\partial_\xi\left[\alpha_2^{[0]}\right] & -\alpha_2^{[0]}\left(\alpha_1^{[0]}\right)^\dagger & -\alpha_2^{[0]}\left(\alpha_2^{[0]}\right)^\dagger \end{pmatrix},$$

$$U_2\left[\gamma, \alpha_1^{[0]}, \alpha_2^{[0]}\right] = \begin{pmatrix} 0 & \gamma\left(\alpha_2^{[0]}\right)^\dagger - \left(\alpha_1^{[0]}\right)^\dagger & \gamma\left(\alpha_1^{[0]}\right)^\dagger + \left(\alpha_2^{[0]}\right)^\dagger \\ \gamma\alpha_2^{[0]} - \alpha_1^{[0]} & 0 & 0 \\ \gamma\alpha_1^{[0]} + \alpha_2^{[0]} & 0 & 0 \end{pmatrix},$$

$$U_3(\gamma) = \begin{pmatrix} 1+\gamma^2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

$$\tag{3.28}$$

where $\partial_\xi \left[\alpha_j^{[0]} \right]$, $j = 1, 2$ denotes the partial derivative of the fields $\alpha_j^{[0]}$ with respect to the coordinate ξ.

Equations (3.25) and (3.27) therefore constitute a three-component Lax pair [8,389], whose compatibility condition yields (3.11) (and its complex conjugate).

Eigenfunctions and singular manifold

According to definitions (3.17) and the linearization ansatz (3.23), the singular manifold ϕ can be obtained by integration of the differential

$$d\phi = \psi\psi^\dagger d\xi - \left[2\lambda\psi\psi^\dagger + i \left(\psi_\xi \psi^\dagger - \psi\psi_\xi^\dagger \right) \right] dt, \tag{3.29}$$

which is exclusively written in terms of the eigenfunctions $\{\psi, \psi^\dagger\}$, their derivatives and the spectral parameter λ.

1.5. Darboux transformations

One of the main advantages of the SMM is that it allows to construct an iterative procedure to obtain highly nontrivial solutions by means of the eigenfunctions of a trivial seed solutions. Let

$$\boldsymbol{\alpha}^{[0]} = \begin{pmatrix} \alpha_1^{[0]} \\ \alpha_2^{[0]} \end{pmatrix} \tag{3.30}$$

be such seed solution for the starting NLS system (3.11), and let

$$\boldsymbol{\Psi}_j = \begin{pmatrix} \psi_j \\ \omega_j \\ \eta_j \end{pmatrix}, \qquad j = 1, 2 \tag{3.31}$$

be two eigenvectors of the spectral problem associated to $\boldsymbol{\alpha}^{[0]}$ with eigenvalues λ_j, given by (3.25) and (3.27). These Lax pairs explicitly read

$$
\begin{aligned}
(\boldsymbol{\Psi}_j)_\xi &= V_1 \left[\boldsymbol{\alpha}^{[0]} \right] \boldsymbol{\Psi}_j + i\lambda_j V_2 \, \boldsymbol{\Psi}_j + i\pi V_3(\gamma) \boldsymbol{\Psi}_j, \\
(\boldsymbol{\Psi}_j)_t &= i\, U_1 \left[\boldsymbol{\alpha}^{[0]} \right] \boldsymbol{\Psi}_j + \pi U_2 \left[\gamma, \boldsymbol{\alpha}^{[0]} \right] \boldsymbol{\Psi}_j - i\pi^2 U_3(\gamma) \boldsymbol{\Psi}_j \\
&\quad - 2\lambda_j \left(V_1 \left[\boldsymbol{\alpha}^{[0]} \right] + i\lambda_j V_2 \right) \boldsymbol{\Psi}_j,
\end{aligned}
\tag{3.32}
$$

for $j = 1, 2$, where the notation $(\boldsymbol{\Psi}_j)_\xi \equiv \partial_\xi \boldsymbol{\Psi}_j$, $(\boldsymbol{\Psi}_j)_t \equiv \partial_t \boldsymbol{\Psi}_j$ denote the derivatives of the eigenvector $\boldsymbol{\Psi}_j$ with respect to the coordinates ξ, t, respectively. The complex

conjugate of (3.32) should be taken into account as well, but we omit its explicit expression for simplicity.

The associated singular manifolds are defined through the following exact derivatives (3.29)

$$d\phi_j = \psi_j \psi_j^\dagger \left(d\xi - 2\lambda_j dt \right) - i \left[\left(\psi_j \right)_\xi \psi_j^\dagger - \left(\psi_j^\dagger \right)_\xi \psi_j \right] dt, \qquad j = 1, 2. \qquad (3.33)$$

First iteration

According to the truncated expressions (3.16), we can use the first eigenvector $\boldsymbol{\Psi}_1 = (\psi_1, \omega_1, \eta_1)^\mathsf{T}$, its complex conjugate $\boldsymbol{\Psi}_1^\dagger$ and the singular manifold ϕ_1 to construct the iterated solution $\boldsymbol{\alpha}^{[1]}$, which combined with the linearization ansatz (3.21) yields

$$
\begin{aligned}
\alpha_1^{[1]} &= \alpha_1^{[0]} + \frac{\omega_1 \psi_1^\dagger}{\phi_1}, & \left(\alpha_1^{[1]} \right)^\dagger &= \left(\alpha_1^{[0]} \right)^\dagger + \frac{\omega_1^\dagger \psi_1}{\phi_1}, \\
\alpha_2^{[1]} &= \alpha_2^{[0]} + \frac{\eta_1 \psi_1^\dagger}{\phi_1}, & \left(\alpha_2^{[1]} \right)^\dagger &= \left(\alpha_2^{[0]} \right)^\dagger + \frac{\eta_1^\dagger \psi_1}{\phi_1}.
\end{aligned}
\qquad (3.34)
$$

Hence, after some algebraic manipulations regarding relations (3.34) and (3.25), we obtain the following expressions for the modulus of the iterated solution $\boldsymbol{\alpha}^{[1]}$

$$
\begin{aligned}
\left(\alpha_1^{[1]} \right)^\dagger \left(\alpha_1^{[1]} \right) &= \left(\alpha_1^{[0]} \right) \left(\alpha_1^{[0]} \right)^\dagger + \frac{i\pi\gamma \left(\omega_1 \eta_1^\dagger - \omega_1^\dagger \eta_1 \right)}{\phi_1} - \left(\frac{\omega_1 \omega_1^\dagger}{\phi_1} \right)_\xi, \\
\left(\alpha_2^{[1]} \right)^\dagger \left(\alpha_2^{[1]} \right) &= \left(\alpha_2^{[0]} \right) \left(\alpha_2^{[0]} \right)^\dagger - \frac{i\pi\gamma \left(\omega_1 \eta_1^\dagger - \omega_1^\dagger \eta_1 \right)}{\phi_1} - \left(\frac{\eta_1 \eta_1^\dagger}{\phi_1} \right)_\xi, \\
\left(\boldsymbol{\alpha}^{[1]} \right)^\dagger \cdot \left(\boldsymbol{\alpha}^{[1]} \right) &= \left(\boldsymbol{\alpha}^{[0]} \right)^\dagger \cdot \left(\boldsymbol{\alpha}^{[0]} \right) - \left(\frac{(\phi_1)_\xi}{\phi_1} \right)_\xi.
\end{aligned}
\qquad (3.35)
$$

We shall now construct the spectral problem associated to the iterated solution $\boldsymbol{\alpha}^{[1]}$. Let

$$\boldsymbol{\Psi}_{1,2} = \begin{pmatrix} \psi_{1,2} \\ \omega_{1,2} \\ \eta_{1,2} \end{pmatrix} \qquad (3.36)$$

be an eigenvector for $\boldsymbol{\alpha}^{[1]}$ with spectral parameter λ_2 such that

$$(\boldsymbol{\Psi}_{1,2})_\xi = V_1 \left[\boldsymbol{\alpha}^{[1]} \right] \boldsymbol{\Psi}_{1,2} + i\lambda_2 V_2 \, \boldsymbol{\Psi}_{1,2} + i\pi V_3(\gamma) \boldsymbol{\Psi}_{1,2},$$

$$(\boldsymbol{\Psi}_{1,2})_t = i\, U_1 \left[\boldsymbol{\alpha}^{[1]}\right] \boldsymbol{\Psi}_{1,2} + \pi U_2 \left[\gamma, \boldsymbol{\alpha}^{[1]}\right] \boldsymbol{\Psi}_{1,2} - i\pi^2 U_3(\gamma) \boldsymbol{\Psi}_{1,2} \tag{3.37}$$
$$- 2\lambda_2 \left(V_1 \left[\boldsymbol{\alpha}^{[1]}\right] + i\lambda_2 V_2\right) \boldsymbol{\Psi}_{1,2},$$

where we obtain a similar expression for the complex conjugate eigenvector $\boldsymbol{\Psi}_{1,2}^{\dagger}$ associated to the iterated solution $\left(\boldsymbol{\alpha}^{[1]}\right)^{\dagger}$ with spectral parameter λ_2. The introduction of such eigenfunctions allows us to construct the corresponding singular manifold $\phi_{1,2}$, by integrating

$$d\phi_{1,2} = \left(\psi_{1,2}\right)\left(\psi_{1,2}^{\dagger}\right)\left(d\xi - 2\lambda_2 dt\right) - i\left[\left(\psi_{1,2}\right)_{\xi}\psi_{1,2}^{\dagger} - \left(\psi_{1,2}^{\dagger}\right)_{\xi}\psi_{1,2}\right]dt. \tag{3.38}$$

The Lax pair (3.37) can be understood as a system of nonlinear equations that couples the field $\boldsymbol{\alpha}^{[1]}$ and the eigenvector $\boldsymbol{\Psi}_{1,2}$. This implies that the Painlevé expansion (3.34) for the fields should be accompanied by a similar expansion for the eigenfunctions that can be written in the following form

$$\psi_{1,2} = \psi_2 - \psi_1 \frac{\Delta_{1,2}}{\phi_1}, \qquad \omega_{1,2} = \omega_2 - \omega_1 \frac{\Delta_{1,2}}{\phi_1}, \qquad \eta_{1,2} = \eta_2 - \eta_1 \frac{\Delta_{1,2}}{\phi_1}, \tag{3.39a}$$

where their complex conjugates are given by

$$\psi_{1,2}^{\dagger} = \psi_2^{\dagger} - \psi_1^{\dagger} \frac{\Delta_{1,2}^{\dagger}}{\phi_1}, \qquad \omega_{1,2}^{\dagger} = \omega_2^{\dagger} - \omega_1^{\dagger} \frac{\Delta_{1,2}^{\dagger}}{\phi_1}, \qquad \eta_{1,2}^{\dagger} = \eta_2^{\dagger} - \eta_1^{\dagger} \frac{\Delta_{1,2}^{\dagger}}{\phi_1}. \tag{3.39b}$$

Substitution of expressions (3.39) in their respective spectral problems, in combination with (3.34) and (3.32), yields

$$\Delta_{i,j} = \Delta\left(\boldsymbol{\Psi}_i, \boldsymbol{\Psi}_j\right) = i\frac{\omega_i^{\dagger}\omega_j + \eta_i^{\dagger}\eta_j - \psi_i^{\dagger}\psi_j}{2(\lambda_i - \lambda_j)},$$
$$\Delta_{i,j}^{\dagger} = \Delta\left(\boldsymbol{\Psi}_i^{\dagger}, \boldsymbol{\Psi}_j^{\dagger}\right) = -i\frac{\omega_j^{\dagger}\omega_i + \eta_j^{\dagger}\eta_i - \psi_j^{\dagger}\psi_i}{2(\lambda_i - \lambda_j)} = \Delta_{j,i}, \tag{3.40}$$

for $i,j = 1,2, i \neq j$, where $\boldsymbol{\Psi}_j$ is the eigenvector for the seed solution $\boldsymbol{\alpha}^{[0]}$ with eigenvalue λ_j as defined in (3.32).

It is easy to see that a similar expansion could be applied to the singular manifold $\phi_{1,2}$ in (3.38), retrieving the result

$$\phi_{1,2} = \phi_2 - \frac{\Delta_{1,2}\Delta_{1,2}^{\dagger}}{\phi_1}. \tag{3.41}$$

The corresponding Δ-matrix of dimension 2 can be therefore defined as

$$
\begin{cases}
\Delta_{i,i} = \phi_i & \text{if } i = j \\[2mm]
\Delta_{i,j} = \dfrac{i}{2} \dfrac{\omega_i^\dagger \omega_j + \eta_i^\dagger \eta_j - \psi_i^\dagger \psi_j}{\lambda_i - \lambda_j}, \qquad \Delta_{j,i} = \Delta_{i,j}^\dagger & \text{if } i \neq j
\end{cases}
\tag{3.42}
$$

Second iteration and τ-function

As far as $\phi_{1,2}$ is a singular manifold for $\boldsymbol{\alpha}^{[1]}$, we can now iterate expressions (3.34) in order to construct the second iteration for the fields $\boldsymbol{\alpha}^{[2]}$ as

$$
\alpha_1^{[2]} = \alpha_1^{[1]} + \frac{\left(\omega_{1,2}\right)\left(\psi_{1,2}^\dagger\right)}{\phi_{1,2}}, \qquad \left(\alpha_1^{[2]}\right)^\dagger = \left(\alpha_1^{[1]}\right)^\dagger + \frac{\left(\omega_{1,2}^\dagger\right)\left(\psi_{1,2}\right)}{\phi_{1,2}},
$$

$$
\alpha_2^{[2]} = \alpha_2^{[1]} + \frac{\left(\eta_{1,2}\right)\left(\psi_{1,2}^\dagger\right)}{\phi_{1,2}}, \qquad \left(\alpha_2^{[2]}\right)^\dagger = \left(\alpha_2^{[1]}\right)^\dagger + \frac{\left(\eta_{1,2}^\dagger\right)\left(\psi_{1,2}\right)}{\phi_{1,2}},
\tag{3.43}
$$

which combined with (3.34) yields

$$
\alpha_1^{[2]} = \alpha_1^{[0]} + \frac{\omega_1 \psi_1^\dagger}{\phi_1} + \frac{\left(\omega_{1,2}\right)\left(\psi_{1,2}^\dagger\right)}{\phi_{1,2}}, \quad \left(\alpha_1^{[2]}\right)^\dagger = \left(\alpha_1^{[0]}\right)^\dagger + \frac{\omega_1^\dagger \psi_1}{\phi_1} + \frac{\left(\omega_{1,2}^\dagger\right)\left(\psi_{1,2}\right)}{\phi_{1,2}},
$$

$$
\alpha_2^{[2]} = \alpha_2^{[0]} + \frac{\eta_1 \psi_1^\dagger}{\phi_1} + \frac{\left(\eta_{1,2}\right)\left(\psi_{1,2}^\dagger\right)}{\phi_{1,2}}, \quad \left(\alpha_2^{[2]}\right)^\dagger = \left(\alpha_2^{[0]}\right)^\dagger + \frac{\eta_1^\dagger \psi_1}{\phi_1} + \frac{\left(\eta_{1,2}^\dagger\right)\left(\psi_{1,2}\right)}{\phi_{1,2}}.
\tag{3.44}
$$

By inserting definitions (3.39) and (3.41) in the expressions above, we obtain the recursion relations for the second iteration in terms of the eigenfunctions of the seed equations as

$$
\alpha_1^{[2]} = \alpha_1^{[0]} + \frac{\psi_2^\dagger \Delta_{1,1}\, \omega_2 + \psi_1^\dagger \Delta_{2,2}\, \omega_1 - \psi_2^\dagger \Delta_{1,2}\, \omega_1 - \psi_1^\dagger \Delta_{1,2}^\dagger\, \omega_2}{\tau_{1,2}},
$$

$$
\alpha_2^{[2]} = \alpha_2^{[0]} + \frac{\psi_2^\dagger \Delta_{1,1}\, \eta_2 + \psi_1^\dagger \Delta_{2,2}\, \eta_1 - \psi_2^\dagger \Delta_{1,2}\, \eta_1 - \psi_1^\dagger \Delta_{1,2}^\dagger\, \eta_2}{\tau_{1,2}},
\tag{3.45a}
$$

and the conjugate fields

$$
\left(\alpha_1^{[2]}\right)^\dagger = \left(\alpha_1^{[0]}\right)^\dagger + \frac{\omega_2^\dagger \Delta_{1,1}\, \psi_2 + \omega_1^\dagger \Delta_{2,2}\, \psi_1 - \omega_1^\dagger \Delta_{1,2}^\dagger\, \psi_2 - \omega_2^\dagger \Delta_{1,2}\, \psi_1}{\tau_{1,2}},
\tag{3.45b}
$$

$$\left(\alpha_2^{[2]}\right)^\dagger = \left(\alpha_2^{[0]}\right)^\dagger + \frac{\eta_2^\dagger \Delta_{1,1}\, \psi_2 + \eta_1^\dagger \Delta_{2,2}\, \psi_1 - \eta_1^\dagger \Delta_{1,2}^\dagger\, \psi_2 - \eta_2^\dagger \Delta_{1,2}\, \psi_1}{\tau_{1,2}},$$

where we have defined the τ-function $\tau_{1,2}$ as

$$\tau_{1,2} = \phi_1 \phi_{1,2} = \phi_1 \phi_2 - \Delta_{1,2} \Delta_{1,2}^\dagger, \tag{3.46}$$

with coincides with $\tau_{1,2} = \det \Delta$ by virtue of definition (3.42).

In conclusion, $\boldsymbol{\alpha}^{[0]}$ and its eigenvectors $\boldsymbol{\Psi}_j$, $j = 1, 2$ allow us to directly obtain the first iterated solution $\boldsymbol{\alpha}^{[1]}$ as well as the second one $\boldsymbol{\alpha}^{[2]}$, through relations (3.34) and (3.45), respectively.

1.6. Soliton-like solutions

This Section is devoted to the use of the procedure described above to build up a plethora of soliton-like solutions for the modified NLS equation that describes the spin dynamics in a deformable helical molecule (3.8). Solutions in this fashion may be algorithmically constructed following analogous steps to the ones stated in Subsection 5.5 of the previous Chapter.

1. *Seed solution*

   We start with the following trivial seed solution

$$\boldsymbol{\alpha}^{[0]} = j_0 e^{-2ij_0^2 t} \begin{pmatrix} \beta_1 \\ \beta_2 \end{pmatrix}, \tag{3.47}$$

   where j_0 is an arbitrary constant and the coefficients β_1 and β_2 are parametrized as in (3.4), of the form

$$\begin{pmatrix} \beta_1 \\ \beta_2 \end{pmatrix} = \frac{1}{2} \begin{pmatrix} (1+s)\cos(\theta_0) + (1-s)\sin(\theta_0) \\ (1-s)\cos(\theta_0) - (1+s)\sin(\theta_0) \end{pmatrix}, \tag{3.48}$$

   depending on the arbitrary parameter $\theta_0 \in [0, 2\pi)$ and $s = \pm 1$. Since $\beta_1^2 + \beta_2^2 = 1$, it is immediate to see that the norm of the the seed solution (3.47) is $\left|\boldsymbol{\alpha}^{[0]}\right|^2 = \left(\boldsymbol{\alpha}^{[0]}\right)^\dagger \cdot \left(\boldsymbol{\alpha}^{[0]}\right) = \left|\alpha_1^{[2]}\right|^2 + \left|\alpha_2^{[2]}\right|^2 = |j_0|^2$.

   To deal with both the focusing and the defocusing cases together, j_0 ought to remain as a free parameter. Actually, j_0 should be real in the defocusing case ($g > 0$) and purely imaginary $j_0 = ih_0$, $h_0 \in \mathbb{R}$ in the focusing one ($g < 0$).

2. *Eigenfunctions and singular manifolds*

We shall now find two set of different eigenvectors $\{\boldsymbol{\Psi}_1, \boldsymbol{\Psi}_2\}$ of the form (3.31), such that they solve the corresponding Lax pair (3.32) for the seed solution (3.47) with respective spectral parameters $\{\lambda_1, \lambda_2\}$. Such solutions read

$$
\begin{aligned}
\psi_j &= e^{k_j(\xi + c_j t)} e^{\frac{im_0\pi(\xi - m_0\pi t)}{2}} e^{ij_0^2 t}, \\
\omega_j &= d_j e^{k_j(\xi + c_j t)} e^{\frac{im_0\pi(\xi - m_0\pi t)}{2}} e^{-ij_0^2 t}, \\
\eta_j &= h_j e^{k_j(\xi + c_j t)} e^{\frac{im_0\pi(\xi - m_0\pi t)}{2}} e^{-ij_0^2 t},
\end{aligned}
\tag{3.49}
$$

for $j = 1, 2$, and where the parameters satisfy

$$
\gamma = \tan(2\theta_0), \qquad m_0 = \frac{s}{\cos(2\theta_0)}.
\tag{3.50}
$$

Furthermore, the constants c_j, λ_j, k_j, $j = 1, 2$ are linked via the relations

$$
c_j = m_0\pi - 2\lambda_j, \qquad k_j^2 + \left(\lambda_j^2 + \frac{m_0\pi}{2}\right)^2 = j_0^2.
\tag{3.51}
$$

These expressions allow us to introduce a new parameter, the angle θ_j, such that

$$
\lambda_j = -\frac{m_0\pi}{2} + j_0\cos(\theta_j), \qquad k_j = j_0\sin(\theta_j),
\tag{3.52}
$$

for every $j = 1, 2$. Finally, the coefficients d_j, h_j, $j = 1, 2$ in (3.49) are

$$
d_j = -i\beta_1 e^{-i\theta_j}, \qquad h_j = -i\beta_2 e^{-i\theta_j},
\tag{3.53}
$$

where the coefficients β_1, β_2 are given in (3.48).

The associated singular manifolds are therefore computed by using expression (3.33), resulting in

$$
\phi_1 = \frac{a_1 + E_1^2}{2j_0\sin(\theta_1)}, \qquad \phi_2 = \frac{a_2 + E_2^2}{2j_0\sin(\theta_2)},
\tag{3.54}
$$

where a_1 and a_2 are arbitrary constants of integration and

$$
E_j = e^{j_0\sin(\theta_j)[\xi + 2(m_0\pi - j_0\cos(\theta_j))\,t]}, \qquad j = 1, 2.
\tag{3.55}
$$

3. Δ-*matrix and τ-function*

Equation (3.40) easily provides the antidiagonal terms of the Δ-matrix as

$$\Delta_{1,2} = \frac{\sin(\theta_2 - \theta_1) + i\left[\cos(\theta_2 - \theta_1) - 1\right]}{2j_0(\cos\theta_1 - \cos\theta_2)} E_1 E_2,$$

$$\Delta_{1,2}^\dagger = \frac{\sin(\theta_2 - \theta_1) - i\left[\cos(\theta_2 - \theta_1) - 1\right]}{2j_0(\cos\theta_1 - \cos\theta_2)} E_1 E_2,$$

(3.56)

such that $\Delta_{1,1} = \phi_1$, $\Delta_{2,2} = \phi_2$ in (3.54) according to (3.42), and $E_j, j = 1, 2$ are defined in (3.55).

The τ-function defined in (3.46) can be explicitly written as

$$\tau_{1,2} = \frac{a_1 a_2 + a_2 E_1^2 + a_1 E_2^2 + A_{1,2} E_1^2 E_2^2}{4j_0^2 \sin(\theta_1)\sin(\theta_2)},$$

(3.57)

where

$$A_{1,2} = 1 - \frac{2\sin(\theta_1)\sin(\theta_2)\left[1 - \cos(\theta_1 - \theta_2)\right]}{(\cos\theta_1 - \cos\theta_2)^2}$$

(3.58)

and a_1, a_2 are free constants.

4. *First and second iterations for the fields*

The first iteration can now be obtained through (3.35) as

$$\left|\alpha_1^{[1]}\right|^2 = \beta_1^2 j_0^2 \left(1 - \frac{1}{j_0^2}\left[\frac{(\phi_1)_\xi}{\phi_1}\right]_\xi\right),$$

$$\left|\alpha_2^{[1]}\right|^2 = \beta_2^2 j_0^2 \left(1 - \frac{1}{j_0^2}\left[\frac{(\phi_1)_\xi}{\phi_1}\right]_\xi\right),$$

(3.59)

whilst the second iteration is deduced from (3.45) as

$$\left|\alpha_1^{[2]}\right|^2 = \beta_1^2 j_0^2 \left(1 - \frac{1}{j_0^2}\left[\frac{(\tau_{1,2})_\xi}{\tau_{1,2}}\right]_\xi\right),$$

$$\left|\alpha_2^{[2]}\right|^2 = \beta_2^2 j_0^2 \left(1 - \frac{1}{j_0^2}\left[\frac{(\tau_{1,2})_\xi}{\tau_{1,2}}\right]_\xi\right).$$

(3.60)

The arising solution essentially depends on up to four arbitrary parameters, $\{j_0, \theta_0, \theta_1, \theta_2\}$. Different combinations for the values of these parameters will retrieve diverse solutions, as already illustrated in the prior Chapter, Subsection 5.5. In particular, we will be able to find the following kinds of solitary waves, defined in distinct regimes:

- Dark solitons $(g > 0)$

- Breathers $(g < 0)$

- Rogue waves $(g < 0)$

These solutions may be regarded as generalizations of the results previously derived in (2.114), (2.119) and (2.120) or (2.122) for this novel NLS system (3.8) describing the electron dynamics in a deformable helical molecule.

1.6.1. Dark solitons. Defocusing case $(g > 0)$

In the defocusing regime, where $g > 0$, j_0 should be a real parameter such that $\left|\boldsymbol{\alpha}^{[j]}\right|^2 > 0$, $j = 0, 1, 2$. According to expression (3.9), the first iteration (3.59) then provides

$$
\left|\chi_1^{[1]}\right|^2 = j_0^2 \frac{1 + s\cos(2\theta_0)}{2g} \left(1 - \frac{1}{j_0^2}\left[\frac{(\phi_1)_\xi}{\phi_1}\right]_\xi\right),
$$
$$
\left|\chi_2^{[1]}\right|^2 = j_0^2 \frac{1 - s\cos(2\theta_0)}{2g} \left(1 - \frac{1}{j_0^2}\left[\frac{(\phi_1)_\xi}{\phi_1}\right]_\xi\right),
$$

(3.61)

and the second iteration (3.60) yields

$$
\left|\chi_1^{[2]}\right|^2 = j_0^2 \frac{1 + s\cos(2\theta_0)}{2g} \left(1 - \frac{1}{j_0^2}\left[\frac{(\tau_{1,2})_\xi}{\tau_{1,2}}\right]_\xi\right),
$$
$$
\left|\chi_2^{[2]}\right|^2 = j_0^2 \frac{1 - s\cos(2\theta_0)}{2g} \left(1 - \frac{1}{j_0^2}\left[\frac{(\tau_{1,2})_\xi}{\tau_{1,2}}\right]_\xi\right),
$$

(3.62)

such that j_0 is a real free parameter, $g > 0$ and the singular manifold ϕ_1 and the τ-function $\tau_{1,2}$ are respectively given in (3.54) and (3.57). Regarding the initial parametrization for the seed solution, the angle θ_0 is related to the parameter of the SOC through (3.50) and $s = \pm 1$.

This choice of the parameters provides dark solitons for the NLS system (3.8), which are represented in Figures 3.1 and 3.2. The first iteration retrieves the one-soliton solution, which reveals a travelling dark soliton that propagates along the direction $\xi + 2\left[\frac{s\pi}{\cos(\theta_0)} - j_0\cos(\theta_1)\right]t = 0$, as evidenced in Figure 3.1.
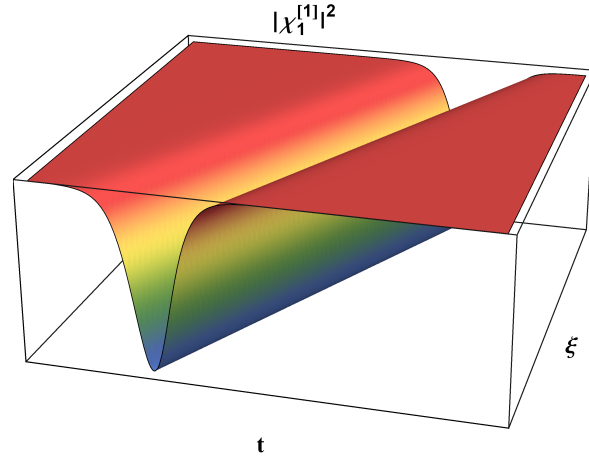
Figure 3.1: *One-soliton solution for the upper component $\left| \chi_1^{[1]} \right|^2$, with parameters $g = 2$, $\theta_0 = 0.5$, $\theta_1 = 1$, $s = 1$, $j_0 = 1$, $a_1 = 1$.*

Regarding the second iteration, Figure 3.2 displays the upper component $\left| \chi_1^{[2]} \right|^2$ of a two-soliton solution in the center-of-mass reference frame obtained through the Galilean transformation $\xi \to \xi - \frac{(c_1 + c_2)\, t}{2}$, where the velocities are given in (3.51).
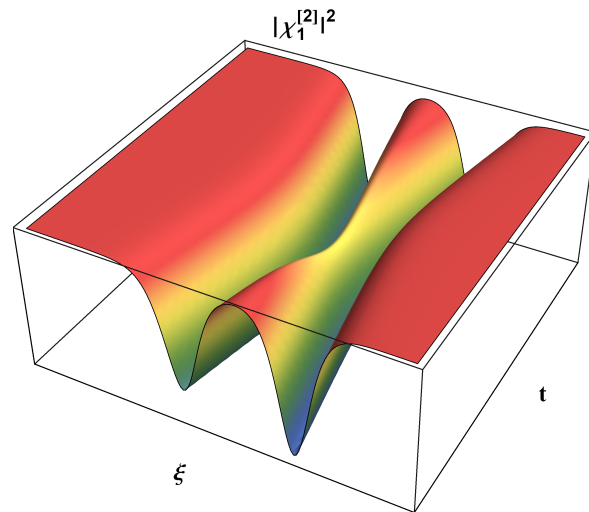
Figure 3.2: *Two-soliton solution for the upper component $\left| \chi_1^{[2]} \right|^2$. Parameters are $g = 2$, $\theta_0 = 0.5$, $\theta_1 = 1$, $\theta_2 = 1.2$, $s = 1$, $j_0 = 1$, and $a_1 = a_2 = 1$.*

Notice that the lower component and the upper component in both cases are related through the identity

$$\left|\chi_2^{[j]}\right|^2 = \frac{1-s\cos(2\theta_0)}{1+s\cos(2\theta_0)}\left|\chi_1^{[j]}\right|^2, \qquad j=1,2, \tag{3.63}$$

so that the component $\left|\chi_2^{[j]}\right|^2$, $j=1,2$ possesses a complete analogous profile to the one displayed in Figures 3.1 and 3.2 for $\left|\chi_1^{[j]}\right|^2$, $j=1,2$, respectively.

1.6.2. Breathers. Focusing case $(g < 0)$

In a similar fashion as proceeded in the quest for breather solutions for the standard $(1+1)$-NLS equation (2.59) (cf. Subsection 5.5 of Chapter 2), we should consider the ansatz

$$\theta_2 = \pi - \theta_1 \tag{3.64}$$

such that $\lambda_2 + \frac{m_0\pi}{2} = -\left(\lambda_1 + \frac{m_0\pi}{2}\right)$ in accordance with (3.52). Moreover, the additional consideration $a_1 = a_2 = \cos\theta_1$, together with this choice for the parameters θ_1, θ_2, allows us to write the τ-function in (3.57) as the particular expression

$$\tau_{1,2} \sim \cosh\left[2j_0^2\sin(2\theta_1)t\right] + \cos\theta_1\cosh\left[2j_0\sin\theta_1\left(\xi + \frac{2s\pi}{\cos(2\theta_0)}t\right)\right]. \tag{3.65}$$

As it has been mentioned before, in order to derive solutions in the focusing regime $(g < 0)$, j_0 should be purely imaginary, which means $j_0 = ih_0$ with $h_0 \in \mathbb{R}$. By imposing this condition, the second iteration (3.62) yields bright solitons

$$\left|\chi_1^{[2]}\right|^2 = h_0^2\frac{1+s\cos(2\theta_0)}{2|g|}\left(1+\frac{1}{h_0^2}\left[\frac{(\tau_{1,2})_\xi}{\tau_{1,2}}\right]_\xi\right),$$

$$\left|\chi_2^{[2]}\right|^2 = h_0^2\frac{1-s\cos(2\theta_0)}{2|g|}\left(1+\frac{1}{h_0^2}\left[\frac{(\tau_{1,2})_\xi}{\tau_{1,2}}\right]_\xi\right), \tag{3.66}$$

where the equivalent of expression (3.65) now reads

$$\tau_{1,2} \sim \cosh\left[2h_0^2\sin(2\theta_1)t\right] + \cos\theta_1\cos\left[2h_0\sin(\theta_1)\left(\xi + \frac{2s\pi}{\cos(2\theta_0)}t\right)\right]. \tag{3.67}$$

At this moment, the latter τ-function may provide oscillatory but localized solutions in either the spatial coordinate ξ or the temporal coordinate t, yielding the well-

known breather solutions for the system (3.8). It is possible then to distinguish between two cases:

1. **Akhmediev breather**

   A generalization of the Akhmediev breather [19–21, 241, 345] may be obtained by considering real values of the angle $\theta_1 \in \mathbb{R}$. Hence, the corresponding τ-function is directly given by expression (3.67), which is a solution periodic in ξ and hyperbolic in t. The associated graphical representation is given in Figure 3.3.
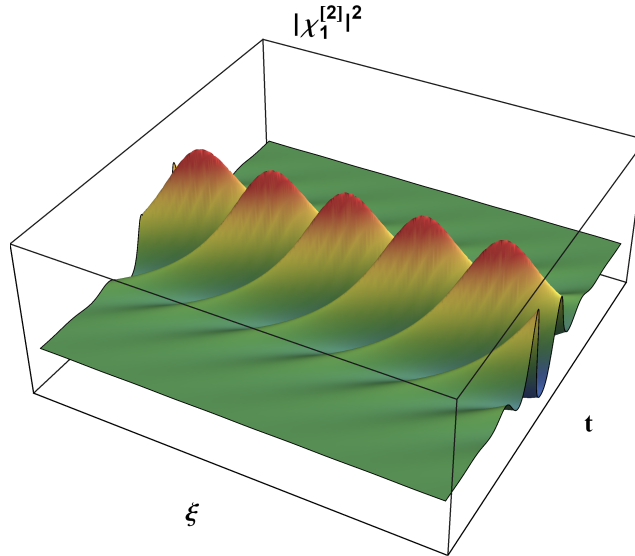
Figure 3.3: *Generalization of the Akhmediev breather* $\left| \chi_1^{[2]} \right|^2$ *with parameters* $g = -2$, $\theta_0 = 5$, $\theta_1 = 2$, $s = 1$, $h_0 = 3$.

2. **Kuznetsov-Ma breather**

   Besides considering purely imaginary values of j_0, we should now set imaginary values for the angle θ_1, such that $\theta_1 = i\hat{\theta}_1$, with $\hat{\theta}_1 \in \mathbb{R}$. Then, expression (3.67) becomes

   $$\tau_{1,2} \sim \cos \left[2h_0^2 \sinh(2\hat{\theta}_1)t \right] + \cosh \hat{\theta}_1 \cosh \left[2h_0 \sinh \hat{\theta}_1 \left(\xi + \frac{2s\pi}{\cos(2\theta_0)} t \right) \right], \tag{3.68}$$

   which is a solution periodic in t and hyperbolic in ξ. It is actually a generalization of the breather introduced by Kuznetsov and Ma [242, 257, 284].
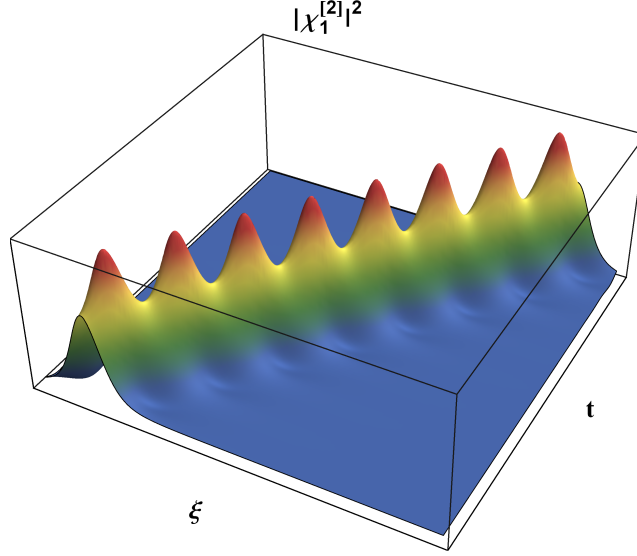
Figure 3.4: *Generalization of the Kuznetsov-Ma breather* $\left|\chi_1^{[2]}\right|^2$ *with parameters* $g = -2$, $\theta_0 = 5$, $\hat{\theta}_1 = -2$, $s = 1$, $h_0 = 3$.

1.6.3. Rogue waves I. Focusing case $(g < 0)$

In the last years, rogue waves have been described as a curious type of waves that appears from nowhere and disappear without a trace. The well known Peregrine soliton [241, 337] is an example of a rogue wave for the focusing NLS equation. In [345], rogue waves for the Manakov system have been obtained. In this Subsection, we will derive this type of solutions for the modified NLS system (3.8).

It is immediate to observe that there exist limiting cases of expression (3.54) when $k_j = 0$, or in other words $\sin(\theta_j) = 0$, $j = 1, 2$. These cases arise when either $\theta_j = 0$ or $\theta_j = \pi$, for any $j = 1, 2$. The corresponding eigenfunctions are then

$$\psi_1 = e^{\frac{im_0\pi(\xi - m_0\pi t)}{2}} e^{ij_0^2 t},$$

$$\omega_1 = -i\beta_1 e^{\frac{im_0\pi(\xi - m_0\pi t)}{2}} e^{-ij_0^2 t}, \tag{3.69}$$

$$\eta_1 = -i\beta_2 e^{\frac{im_0\pi(\xi - m_0\pi t)}{2}} e^{-ij_0^2 t}$$

when $\theta_1 = 0$, $\lambda_1 = -\frac{m_0\pi}{2} + j_0$ and $c_1 = 2(m_0\pi - j_0)$, whilst

$$\psi_2 = e^{\frac{im_0\pi(\xi - m_0\pi t)}{2}} e^{ij_0^2 t},$$

$$\omega_2 = i\beta_1 e^{\frac{im_0\pi(\xi-m_0\pi t)}{2}} e^{-ij_0^2 t}, \tag{3.70}$$

$$\eta_2 = i\beta_2 e^{\frac{im_0\pi(\xi-m_0\pi t)}{2}} e^{-ij_0^2 t}$$

emerge after considering $\theta_2 = \pi$, $\lambda_2 = -\frac{m_0\pi}{2} - j_0$ and $c_2 = 2(m_0\pi + j_0)$.

The analysis of solutions in the focusing regime requires $j_0 = ih_0, h_0 \in \mathbb{R}$. We can now compute the associated singular manifolds $\{\phi_1, \phi_2\}$ and the τ-function following expressions (3.33), (3.40) and (3.46), respectively. We obtain as results the expected rational functions in the coordinates (ξ, t)

$$\phi_1 = \xi + 2\pi m_0 t - 2ih_0 t,$$
$$\phi_2 = \xi + 2\pi m_0 t + 2ih_0 t, \tag{3.71}$$
$$\tau_{1,2} = (\xi + 2\pi m_0 t)^2 + 4h_0^2 t^2 + \frac{1}{h_0^2}.$$

Solutions are then obtained through expressions (3.66). The behaviour of the upper component $\left|\chi_1^{[2]}\right|^2$ for the above value of $\tau_{1,2}$ is presented in Figure 3.5. The lower component is obtained after the proper rescaling of the upper component. This solution can be considered as a generalization of the Peregrine soliton [241, 337].
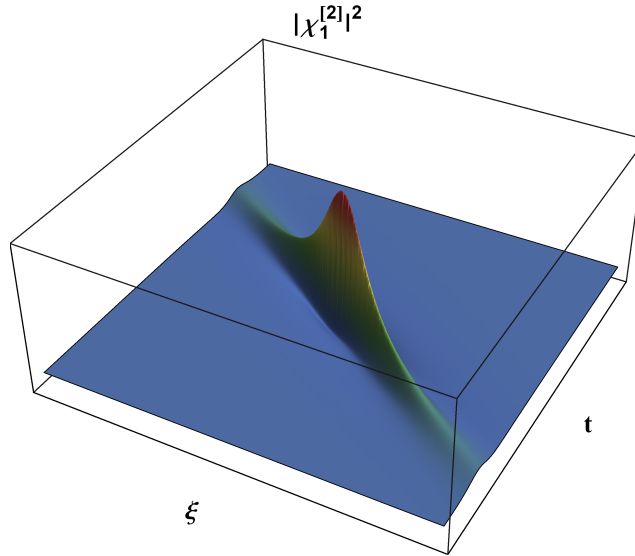
Figure 3.5: *Generalization of the Peregrine soliton* $\left|\chi_1^{[2]}\right|^2$ *with parameters* $g = -2$, $\theta_0 = 0.5$, $\theta_1 = 0$, $\theta_2 = \pi$, $s = 1$ *and* $h_0 = 1.5$.

1.6.4. Rogue waves II. Focusing case ($g < 0$)

Preserving the ansatz $\theta_1 = 0$, $\theta_2 = \pi$, it is straightforward to prove that there exists a slightly more complicated solution for the Lax pair (3.32). The eigenfunctions are constructed as products of polynomial functions in (ξ, t) and exponentials, of the form

$$\psi_1 = \left[\xi + 2\left(m_0\pi - j_0\right)t + \frac{i}{2j_0}\right] e^{\frac{im_0\pi(\xi - m_0\pi t)}{2}} e^{ij_0^2 t},$$

$$\omega_1 = -i\beta_1 \left[\xi + 2\left(m_0\pi - j_0\right)t - \frac{i}{2j_0}\right] e^{\frac{im_0\pi(\xi - m_0\pi t)}{2}} e^{-ij_0^2 t}, \qquad (3.72)$$

$$\eta_1 = -i\beta_2 \left[\xi + 2\left(m_0\pi - j_0\right)t - \frac{i}{2j_0}\right] e^{\frac{im_0\pi(\xi - m_0\pi t)}{2}} e^{-ij_0^2 t}$$

when $\theta_1 = 0$, $\lambda_1 = -\frac{m_0\pi}{2} + j_0$ and $c_1 = 2\left(m_0\pi - j_0\right)$, and

$$\psi_2 = \left[\xi + 2\left(m_0\pi + j_0\right)t - \frac{i}{2j_0}\right] e^{\frac{im_0\pi(\xi - m_0\pi t)}{2}} e^{ij_0^2 t},$$

$$\omega_2 = i\beta_1 \left[\xi + 2\left(m_0\pi + j_0\right)t + \frac{i}{2j_0}\right] e^{\frac{im_0\pi(\xi - m_0\pi t)}{2}} e^{-ij_0^2 t}, \qquad (3.73)$$

$$\eta_2 = i\beta_2 \left[\xi + 2\left(m_0\pi + j_0\right)t + \frac{i}{2j_0}\right] e^{\frac{im_0\pi(\xi - m_0\pi t)}{2}} e^{-ij_0^2 t}$$

for $\theta_2 = \pi$, $\lambda_2 = -\frac{m_0\pi}{2} - j_0$ and $c_2 = 2\left(m_0\pi + j_0\right)$.

For the focusing case, we should choose $j_0 = ih_0$, $h_0 \in \mathbb{R}$, which yields the following results through (3.33), (3.40) and (3.46)

$$\phi_1 = (\xi + 2\pi m_0 t)\left[\frac{1}{3}(\xi + 2\pi m_0 t)^2 - 4h_0^2 t^2 - \frac{1}{4h_0^2}\right]$$

$$+ ih_0\left[-2t(\xi + 2\pi m_0 t)^2 + \frac{3t}{2h_0^2} + \frac{8}{3}h_0^2 t^3\right],$$

$$\phi_2 = (\xi + 2\pi m_0 t)\left[\frac{1}{3}(\xi + 2\pi m_0 t)^2 - 4h_0^2 t^2 - \frac{1}{4h_0^2}\right]$$

$$\qquad (3.74)$$

$$- ih_0\left[-2t(\xi + 2\pi m_0 t)^2 + \frac{3t}{2h_0^2} + \frac{8}{3}h_0^2 t^3\right],$$

$$\tau_{1,2} = (\xi + 2\pi m_0 t)^2\left[\frac{1}{3}(\xi + 2\pi m_0 t)^2 - 4h_0^2 t^2 - \frac{1}{4h_0^2}\right]^2$$

$$+ h_0^2\left[-2t(\xi + 2\pi m_0 t)^2 + \frac{3t}{2h_0^2} + \frac{8}{3}h_0^2 t^3\right]^2 + h_0^2\left[(\xi + 2\pi m_0 t)^2 + 4h_0^2 t^2 + \frac{1}{4h_0^2}\right]^2.$$

Figure 3.6 displays the upper component of the solution $\left|\chi_1^{[2]}\right|^2$ corresponding to the substitution of the τ-function (3.74) in (3.66).
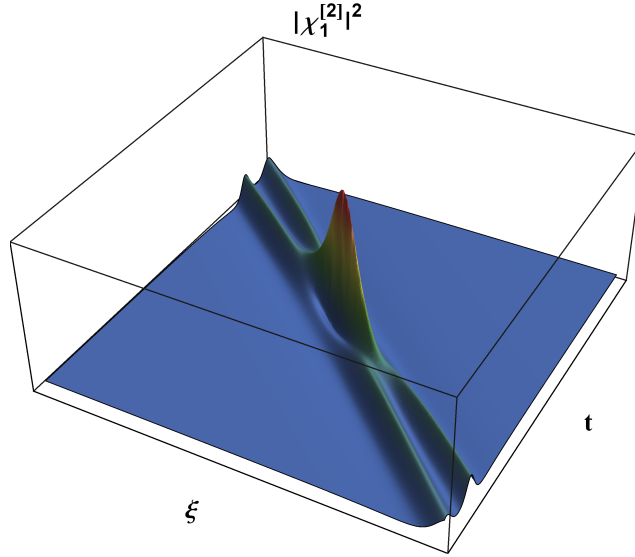
Figure 3.6: *Two-soliton solution for the upper component* $\left|\chi_1^{[2]}\right|^2$ *of a rogue wave II. Parameters are* $g = -2$, $\theta_0 = 0.5, \theta_1 = 0$, $\theta_2 = \pi$, $s = 1$ *and* $h_0 = 0.7$.

1.7. Further solutions. Bright solitons. Focusing case $(g < 0)$

In the previous Subsection we have successfully derived a plethora of soliton solutions through the combination of the SMM, the Lax pair formalism and the binary Darboux transformation approach. The conjunction of these three procedures has proved to be an extremely valuable technique to obtain this kind of solutions. Nevertheless, it is also possible to determine new classes of soliton-like solution by means of alternative procedures. To illustrate this point, this Subsection is devoted to the computation of elliptic solutions for the starting NLS system (3.8).

Elliptic solutions may be easily obtained by considering the following ansatz for a seed solution of (3.11),

$$\boldsymbol{\alpha} = e^{-i\varphi\xi,t)} F(z) \begin{pmatrix} \beta_1 \\ \beta_2 \end{pmatrix}, \tag{3.75}$$

where $\varphi(\xi, t)$ and $F(z)$ are functions to be determined and the coefficients $\{\beta_1, \beta_2\}$ are given in (3.48). We have introduced the new variable $z = \xi + ct$, which plays the

role of a reduced variable[2] in the quest for travelling wave solutions for (3.11) of the form (3.75).

Substitution of the ansatz (3.75) in the system of PDEs (3.11) provides the following expression for the function $\varphi(\xi, t)$,

$$\varphi(\xi, t) = \frac{c}{2}\left(\xi + \frac{c}{2}t\right) - k^2 t - \pi m_0(\xi + \pi m_0 t), \tag{3.76}$$

and the elliptic ODE to the satisfied for $F(z)$,

$$F_{zz} - 2F^3 - k^2 F = 0, \tag{3.77}$$

where k is an arbitrary constant.

The aforementioned ODE (3.77) corresponds to the canonical equation VIII in [228] and can be easily integrated in terms of Jacobi and Weierstrass elliptic functions [66, 207, 418], giving rise to the solution

$$F(z) = \frac{km}{\sqrt{1 - 2m^2}} \, \mathrm{cn}\left(\frac{kz}{\sqrt{2m^2 - 1}}, m\right), \tag{3.78}$$

where $\mathrm{cn}(\cdot)$ is the Jacobi elliptic cosine and m is an arbitrary constant of integration. The associated general solution for the focusing case (3.9) therefore reads

$$\boldsymbol{\chi}(\xi, t) = \frac{km \, e^{-i\varphi(\xi, t)}}{\sqrt{|g|(1 - 2m^2)}} \, \mathrm{cn}\left[\frac{kz}{\sqrt{2m^2 - 1}}, m\right] \begin{pmatrix} e^{-i\pi(\xi + \pi t)} \, \beta_1 \\ e^{i\pi(\xi - \pi t)} \, \beta_2 \end{pmatrix}. \tag{3.79}$$

Bright solitons for this regime emerge in the hyperbolic limit $m = 1$ of the previous solution, yielding the result

$$\boldsymbol{\chi}(\xi, t) = \sqrt{\frac{1}{|g|}} \, \frac{k \, e^{-i\varphi(\xi, t)}}{\cosh\left[k(\xi + ct)\right]} \begin{pmatrix} e^{-i\pi(\xi + \pi t)} \, \beta_1 \\ e^{i\pi(\xi - \pi t)} \, \beta_2 \end{pmatrix}, \tag{3.80}$$

where k still remains as an arbitrary parameter.

This solution can be straightforwardly normalized by imposing

$$\int_{-\infty}^{\infty} \boldsymbol{\chi}^{\dagger}(\xi, t) \cdot \boldsymbol{\chi}(\xi, t) \, d\xi = \frac{2k}{|g|} = 1. \tag{3.81}$$

Therefore, this expression implies $k = \frac{|g|}{2}$, such that the normalized solution can be

---

[2]For further information regarding this methodology, we refer the reader to Subsection 5.3 from Chapter 5 of this manuscript.

finally written as

$$\boldsymbol{\chi}(\xi, t) = \frac{\sqrt{|g|}}{2} \frac{e^{i\Omega(\xi,t)}}{\cosh\left[\frac{|g|}{2}\left(\xi + ct\right)\right]} \begin{pmatrix} e^{-i\pi\xi}\,\beta_1 \\ e^{i\pi\xi}\,\beta_2 \end{pmatrix}, \tag{3.82}$$

where

$$\Omega(\xi, t) = -\left[\frac{c}{2}\left(\xi + \frac{c}{2}t\right) - \frac{g^2}{4}t\right] + \pi\left[\frac{s\xi}{\cos(2\theta_0)} + \pi\gamma^2 t\right] \tag{3.83}$$

and the coefficients $\{\beta_1, \beta_2\}$ are given in (3.48), with $s = \pm 1$.

The behaviour of the square modulus of each component follows the expression

$$|\chi_j|^2 = \frac{|g|}{4} \frac{\beta_j^2}{\cosh^2\left[\frac{|g|}{2}\left(\xi + ct\right)\right]}, \qquad j = 1, 2, \tag{3.84}$$

such that both components are linked via a relation analogous to (3.63).

This solution results in a travelling solitary wave with a bright soliton profile propagating along the straight line $\xi + ct = 0$, plotted in Figure 3.7.
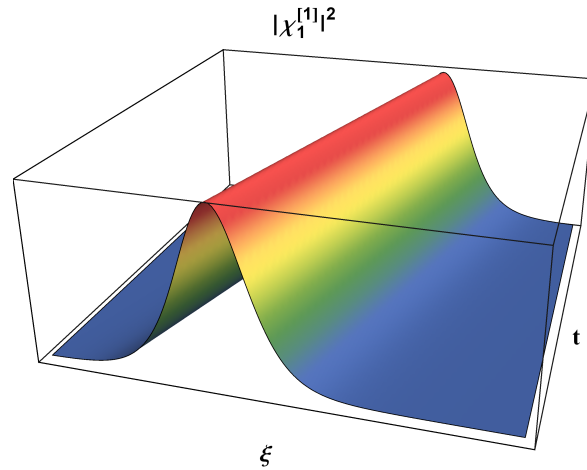
Figure 3.7: *Generalization of the Davydov soliton $|\chi_1|^2$ with parameters $g = -2$, $\theta_0 = 0.5$, $s = 1$ and $c = -\frac{1}{4}$.*

Solution (3.82) may be regarded as the generalization of the Davydov soliton [122], which also appears in reference [234]. Besides, the existence of two different bright solitons due to the arbitrary choice of the constant $s = \pm 1$ is known as the Kramer

doublet and it is directly related to the preservation of the time-reversal symmetry in the model [124].

From the physical point of view, it is important to notice that the magnitude of the SOC is relevant only in the expression of the phase $\Omega(\xi, t)$. In particular, it was found that the generalized Davydov soliton (3.82) presents a well defined spin projection onto the molecule axis, and most importantly, it is preserved during its motion, in spite of the fact that the electron spin is not a constant of motion of the linear Hamiltonian ($g = 0$). This spin polarization is found to be

$$|\mathrm{SP}(t)| = \int_{-\infty}^{\infty} \boldsymbol{\chi}^{\dagger}(\xi, t)\, \sigma_z \boldsymbol{\chi}(\xi, t) d\xi = |\beta_1^2 - \beta_2^2| = \frac{1}{\sqrt{1+\gamma^2}}, \tag{3.85}$$

which clearly does not depend on the temporal coordinate t and it is proved to be stable alongside the propagation of the soliton. Moreover, the value of the spin polarization exclusively depends on the strength of the SOC encoded in γ. A thorough study of the dynamics of this solution, as well as a deeper analysis about the impact of the model and connection with experiments have been conducted in [124].

2. Derivative nonlinear Schrödinger equation in $1+1$ dimensions

Derivative nonlinear Schrödinger (DNLS) equations constitute a generalization of the ubiquitous NLS equation with derivative-type nonlinearities. The most famous integrable representative of this class was first introduced by Kaup and Newell in [236], also denoted as the Kaup-Newell (KN) system,

$$im_t - m_{xx} - i\left(|m|^2 m\right)_x = 0, \tag{3.86}$$

where $m = m(x, t)$ is a complex-valued function and the subscripts x, t denote partial derivatives.

The nonlinear dispersive equation (3.86) arises from the field of Physics and it constitutes a particularly relevant model in plasma physics [78, 233, 363–365, 375] and nonlinear optics [15, 32, 390]. From the integrability point of view, equation (3.86) is known to possess an isospectral Lax pair and it is solvable by the IST method [236]. KN system has attracted attention and it has been extensively studied in literature during decades [224, 227, 232, 237, 238, 399]. It is also well established that it presents a plethora of soliton-like solutions in diverse scenarios [223, 224, 227, 237, 311, 379, 405, 422, 423].

It has been shown by Kundu [256] that equation (3.86) is equivalent via a $U(1)$-gauge transformation to other two celebrated DNLS equations, *i.e.* the Chen-Lee-Liu (CLL) equation [80],

$$im_t - m_{xx} - i\,|m|^2\,m_x = 0, \tag{3.87}$$

and the Gerdjikov-Ivanov (GI) equation [185]

$$im_t - m_{xx} + im^2\overline{m}_x - \frac{1}{2}\,|m|^4\,m = 0, \tag{3.88}$$

where $\overline{m}$ denotes the complex conjugate of m. It is easy to demonstrate that if $m(x,t)$ is a solution of the KN system (3.86), then the new field $\hat{m}(x,t)$

$$\hat{m}(x,t) = m(x,t)\,e^{\frac{i\mu}{2}\,\theta(x,t)}, \quad \text{with} \quad \theta_x = |m|^2, \quad \theta_t = i(m\overline{m}_x - \overline{m}m_x) + \frac{3}{2}\,|m|^4, \tag{3.89}$$

satisfies the CLL equation (3.87) for $\mu = 1$ and the GI equation (3.88) for $\mu = 2$. Gauge transformations constitute a useful tool to connect integrable evolution equations in soliton theory, since they provide Bäcklund transformations between those equations as well as the relation of their associated linear problems [400]. In fact, DNLS equations are also linked via gauge transformations [232,400] to several other notorious integrable equations, such as the Ablowitz-Kaup-Newell-Segur (AKNS) system [6], the NLS equation or the Wadati-Konno-Ichikawa (WKI) system [396,397].

In the light of the gauge equivalence for the aforementioned equations, we propose the following generalized DNLS equation in $1+1$ dimensions

$$im_t - m_{xx} + i(\gamma - 2)\,|m|^2\,m_x + i(\gamma - 1)m^2\,\overline{m}_x - \frac{1}{4}\gamma(\gamma - 1)\,|m|^4\,m = 0, \tag{3.90}$$

depending on a real parameter γ. This equation includes the three former systems as particular cases ($\gamma = 0$ for the KN system, $\gamma = 1$ for the CLL equation and $\gamma = 2$ for the GI equation) and allows us to mix the derivative-type nonlinearites for any other arbitrary value of γ. Besides, (3.90) is also gauge-equivalent to the previous equations and the parameter γ can be easily removed by any $U(1)$-gauge transformation. This equation can be regarded as a particular case for the generalization proposed by Clarkon and Cosgrove in [96] with $a = -(\gamma - 2)$, $b = -(\gamma - 1)$ and $c = \frac{1}{4}\gamma(\gamma - 1)$. Similar generalizations for DNLS equations can be found in [232,256].

The integrability of (3.90) via Painlevé analysis has been explored in [96], with the introduction of a potential. After some algebraic manipulations, this equation is proved to be integrable in the Painlevé sense. In this thesis we will conduct a different approach. We will review the Painlevé integrability for (3.90) in an alternative fashion, searching for the proper scenario where the SMM is applicable such that we

can derive the associated Lax pair. In this process, three differential equations of physical interest naturally emerge. By means of the link among the aforementioned equations and with the aid of the SMM, we will be able to find the corresponding spectral problems for each system, as well as soliton-like solutions of rational kind for the starting generalized DNLS equation (3.90).

2.1. Painlevé test and integrability: Miura transformation

The generalized DNLS equation proposed in (3.90) may be written as a system of coupled PDEs of the form

$$
\begin{aligned}
im_t - m_{xx} + i(\gamma - 2)\, m\,\overline{m}\, m_x + i(\gamma - 1)m^2\,\overline{m}_x - \frac{1}{4}\gamma(\gamma - 1)m^3\,\overline{m}^2 = 0, \\
-i\overline{m}_t - \overline{m}_{xx} - i(\gamma - 2)\, m\,\overline{m}\,\overline{m}_x - i(\gamma - 1)\overline{m}^2 m_x - \frac{1}{4}\gamma(\gamma - 1)\overline{m}^3\, m^2 = 0,
\end{aligned}
\tag{3.91}
$$

where $\overline{m} = \overline{m}(x,t)$ is the complex conjugate of m and $|m|^2 = m \cdot \overline{m}$.

The Painlevé test [417] has proved to be a powerful criterion for the identification of the integrability of PDEs. This procedure requires that the fields $\{m, \overline{m}\}$ should be expressed as generalized Laurent expansions of the form

$$
m(x,t) = \sum_{j=0}^{\infty} a_j(x,t)\,\phi(x,t)^{j-p}, \qquad \overline{m}(x,t) = \sum_{j=0}^{\infty} b_j(x,t)\,\phi(x,t)^{j-q}, \tag{3.92}
$$

where $p, q \in \mathbb{N}$ are the leading indices and $\phi(x,t)$ is so-called singular manifold.

Nevertheless, a cursory leading-order analysis provides noninteger indices for the expansions, $p = q = \frac{1}{2}$. Consequently, the Painlevé test is unable to check the integrability of DNLS equation (3.91), at least when it is expressed in terms of these variables.

This fact allows us to introduce two new real fields $\alpha(x,t)$, $\beta(x,t)$ of the form

$$
m = \sqrt{2\alpha_x}\, e^{\frac{i}{2}\beta}, \qquad\qquad \overline{m} = \sqrt{2\alpha_x}\, e^{-\frac{i}{2}\beta}. \tag{3.93}
$$

The introduction of (3.93) in the system (3.91) retrieves the following expressions for the derivatives of the new field $\alpha(x,t)$

$$
\alpha_x = \frac{1}{2}\,|m|^2, \qquad \alpha_t = \frac{i}{2}\,(m\overline{m}_x - \overline{m}m_x) - \frac{1}{4}(2\gamma - 3)\,|m|^4, \tag{3.94}
$$

whilst $\beta(x, t)$ shall obey

$$\beta = (2\gamma - 3)\alpha + \int \frac{\alpha_t}{\alpha_x} dx. \tag{3.95}$$

Taking these identities into account, the aforementioned ansatz yields a differential equation for α that can be written in conservative form as

$$\left[\alpha_x^2 - \alpha_t\right]_t = \left[\alpha_{xxx} + \alpha_x^3 - \frac{\alpha_t^2 + \alpha_{xx}^2}{\alpha_x}\right]_x. \tag{3.96}$$

Notice that $|m|^2 = 2\alpha_x$ is the density of probability for the DNLS equation (3.91) and therefore, α_x is essentially the physically relevant field in the model under consideration. This representative equation for the probability density will constitute the second PDE of interest to be considered in this Section, after the former DNLS system (3.91) itself.

From the point of view of the Painlevé Property [417], the Painlevé test can be applied to (3.96) due to the fact that the leading terms for α_x and α_t are

$$\alpha_x \sim \pm i \frac{\phi_x}{\phi}, \qquad \alpha_t \sim \pm i \frac{\phi_t}{\phi}, \tag{3.97}$$

where $\phi(x, t)$ is the singular manifold. Nevertheless, the duality in the signs indicates the existence of two Painlevé branches, which might be an inconvenient when the singular manifold method is applied [141]. The restriction to just one of the two possible signs means that we are loosing a lot of information about equation (3.96). In compliance with previous studies [150, 157], the best method to confront this problem requires the splitting of the field α as

$$\alpha = i(u - \overline{u}), \tag{3.98}$$

where $u = u(x, t)$ denotes a new field and $\overline{u} = \overline{u}(x, t)$ stands for its complex conjugate. According to (3.96), the additional relation arises

$$\alpha_x^2 - \alpha_t = u_{xx} + \overline{u}_{xx}. \tag{3.99}$$

The combination of (3.98) and (3.99) implies the Miura transformations [141]

$$u_{xx} = \frac{1}{2}\left(\alpha_x^2 - \alpha_t - i\alpha_{xx}\right), \tag{3.100a}$$

$$\overline{u}_{xx} = \frac{1}{2}\left(\alpha_x^2 - \alpha_t + i\alpha_{xx}\right), \tag{3.100b}$$

as well as a coupling condition between the field $\{u, \overline{u}\}$, which reads

$$iu_t + u_{xx} - i\overline{u}_t + \overline{u}_{xx} + (u_x - \overline{u}_x)^2 = 0. \tag{3.101}$$

The derivation of equations (3.100) with respect to t yields

$$u_{xt} = \frac{1}{2}\left(\alpha_{xxx} + \alpha_x^3 - \frac{\alpha_t^2 + \alpha_{xx}^2}{\alpha_x}\right) - \frac{i}{2}\alpha_{xt}, \tag{3.102a}$$

$$\overline{u}_{xt} = \frac{1}{2}\left(\alpha_{xxx} + \alpha_x^3 - \frac{\alpha_t^2 + \alpha_{xx}^2}{\alpha_x}\right) + \frac{i}{2}\alpha_{xt}, \tag{3.102b}$$

where equation (3.96) has been used to perform an integration in x. In order to get the equation to be satisfied by $u(x, t)$, we can use (3.100a) and (3.102a) to obtain

$$\alpha_t = \alpha_x^2 - i\alpha_{xx} - 2u_{xx},$$

$$\alpha_{xxx} = -\alpha_x^3 + \frac{\alpha_t^2 + \alpha_{xx}^2}{\alpha_x} + i\alpha_{xt} + 2u_{xt}, \tag{3.103}$$

$$\alpha_{xx} = i\left(u_{xx} - \alpha_x^2\right) + \frac{\alpha_x}{2u_{xx}}\left(u_{xxx} + iu_{xt}\right).$$

Therefore, the compatibility condition $(\alpha_t)_{xx} = (\alpha_{xx})_t$ provides the following non-linear partial differential equation to be satisfied for the field $u(x, t)$

$$\left[u_{tt} + u_{xxxx} + 2u_{xx}^2 - \frac{u_{xt}^2 + u_{xxx}^2}{u_{xx}}\right]_x = 0. \tag{3.104}$$

Following exactly the same argument with equations (3.100b) and (3.102b), we can easily prove that $\overline{u}(x, t)$ should satisfy the same equation (3.104). To summarize, $u(x, t)$ and $\overline{u}(x, t)$ are both solutions of the same PDE (3.104), being additionally related by the Bäcklund transformation defined in (3.101).

Equation (3.104) is known as the nonlocal Boussinesq equation [261,419] and its connection to DNLS equations, such as the Kaup system, has been extensively studied in [141] from the point of view of the singular manifold method. This differential equation represents the third PDE of physical relevance to be analyzed throughout the present Section.

Painlevé test for (3.104)

We should now address the Painlevé integrability analysis for equation (3.104). By imposing the usual Painlevé series

$$u(x,t) = \sum_{j=0}^{\infty} a_j(x,t) \phi(x,t)^{j-p}, \tag{3.105}$$

the leading-order analysis retrieves $p = 0$ as the leading index. As already mentioned, the Painlevé test for PDEs based on the WTC method [417] is not able to manage differential equations whose leading indices are not purely integers. There exist distinct approaches to overcome this inconvenience, which customarily imply the conception of generalized series expansion for the fields [170, 253, 340]. For null leading indices, it has been suggested [89, 90, 341] that the modified series expansion should include (a finite number of) linear logarithmic terms.

Following the prescription introduced in [90], we seek an expansion of the form

$$u(x,t) = \tilde{a}_0 \log(\phi) + \sum_{j=0}^{\infty} a_j \phi^j, \tag{3.106}$$

where $\tilde{a}_0(x,t)$ and $a_j(x,t), \forall j$ are analytic functions in the neighbourhood of the singularity manifold $\phi = 0$.

1. A leading-order analysis straightforwarly provides

$$\tilde{a}_0(x,t) = 1, \tag{3.107}$$

   which is uniquely determined. Therefore, the series (3.106) possesses a single branch of expansion.

2. By introducing expansion (3.106) into the equation (3.104) and equating to zero the different coefficients in powers of ϕ, it is easy to obtain the following resonance condition

$$j(j+1)(j-1)(j-2) = 0, \tag{3.108}$$

   which retrieves four resonances, consistent with the fourth-order PDE (3.104). The usual resonance in $j = -1$ is associated to the arbitrariness of the function ϕ, while the additional single resonances in $j = 0, 1, 2$ correspond to the fact that the coefficients a_0, a_1, a_2 in expansion (3.106) should be arbitrary.

3. The resonance conditions for $j = 0, 1, 2$ are identically satisfied and the coefficients a_0, a_1, a_2 are found to be arbitrary. Besides, coefficients $a_j, j \geq 3$ can be uniquely determined by the corresponding recursion relation series.

Thus, we shall conclude that equation (3.104) passes the Painlevé test and then, it possesses the Painlevé Property.

The underlying observation is that the expansion (3.106) for u only contains a single logarithmic term, which can be easily removed by differentiation. In fact, regarding the differential equation (3.104), one may observe that it is exclusively defined in terms of the derivatives u_x, u_t. It is precisely these first derivatives the ones that are expressed in terms of the Laurent series in a neighbourhood of the singular manifold $\phi = 0$, rather than the solution u [89,90,141]. Then, the appearance of a logarithmic term does not contradict the assertion of the Painlevé conjecture in this case.

Alternatively, if we introduce the nonlocal variables $v = u_x$ and $w = u_t$, then (3.104) is transformed into the system

$$
\begin{aligned}
w_x &= v_t, \\
\left[w_t + v_{xxx} + 2v_x^2 - \frac{w_x^2 + v_{xx}^2}{v_x} \right]_x &= 0,
\end{aligned}
\tag{3.109}
$$

which has the Painlevé Property, where $\{v, w\}$ follow the expansions

$$
v(x,t) = \frac{\phi_x}{\phi} + \sum_{j=1}^{\infty} v_j(x,t)\phi^{j-1}, \qquad w(x,t) = \frac{\phi_t}{\phi} + \sum_{j=1}^{\infty} w_j(x,t)\phi^{j-1}.
\tag{3.110}
$$

The resonances are located in $j = \{-1, 1 \text{ (double)}, 2\}$, such that the coefficients v_1, w_1, w_2 in (3.110) are found to be arbitrary. Then, expansions (3.110) can be trivially integrated providing a series expansion for u of the form (3.106).

In view of the aforementioned arguments, we conclude that equation (3.104) has therefore the Painlevé Property and hence, it is conjectured integrable.

2.2. The singular manifold method

The advantage of equation (3.104) is that besides having the Painlevé Property, it also has just one Painlevé branch [141]. This fact allows us to easily perform the singular manifold method in order to derive many of the properties associated to a nonlinear partial differential equation, sumarized as

- The singular manifold equations

- The Lax pair and its eigenfunctions

- Darboux transformations of the Lax pair

- τ-functions

- Iterative method for the construction of solutions

The SMM requires the truncation of the Painlevé expansion for u (3.106) at constant level, which implies

$$u^{[1]} = u^{[0]} + \log(\phi). \tag{3.111}$$

This truncation acquires the form of an auto-Bäcklund transformation between two solutions $u^{[0]}$ and $u^{[1]}$ of the same equation (3.104). Besides that, the manifold $\phi(x,t)$ is not longer an arbitrary function. The singular manifold equations to be satisfied for ϕ can be obtained by direct substitution of (3.111) in (3.104).

Expression of the field in terms of the singular manifold

The expressions for the derivatives of the seed field $u^{[0]}$ read as

$$u_{xx}^{[0]} = -\frac{1}{4}\left[v^2 + (r + 2\lambda)^2\right], \tag{3.112a}$$

$$u_{xt}^{[0]} = \frac{1}{2}\left[(r + 2\lambda)v_x - vr_x - (r + \lambda)\left(v^2 + (r + 2\lambda)^2\right)\right], \tag{3.112b}$$

where λ is an arbitrary constant and $\{v, r\}$ are the usual functions related to the singular manifold through the definitions

$$v = \frac{\phi_{xx}}{\phi_x}, \qquad r = \frac{\phi_t}{\phi_x}. \tag{3.113}$$

Singular manifold equations

The equations to be satisfied by the singular manifold can be written as the system

$$r_t = \left(-v_x + \frac{v^2}{2} + \frac{3\,r^2}{2} + 4\lambda r\right)_x, \tag{3.114a}$$

$$v_t = (r_x + rv)_x, \tag{3.114b}$$

where the second equation (3.114b) is trivially obtained from the compatibility condition $(\phi_{xx})_t = (\phi_t)_{xx}$, arising from definitions (3.113).

It is relevant to remark that the singular manifold equations are easily related to the Kaup system [235]. Actually, we can write system (3.114) as

$$\gamma_t = -\eta_{xx} + 2\gamma\gamma_x, \tag{3.115a}$$

$$\eta_t = \gamma_{xx} + 2\gamma\eta_x, \tag{3.115b}$$

through the following change of variables

$$\gamma = r + \lambda, \qquad \eta_x = v_x - \frac{v^2}{2} - \frac{(r + 2\lambda)^2}{2}. \tag{3.116}$$

If the singular manifold equations can be considered as the intrinsic canonical form of a PDE, we can conclude that our original DNLS equation (3.91) is nothing but a different form of the Kaup system via Miura transformation.

2.3. Spectral problem

The SMM provides a direct procedure to construct spectral problems by means of the linearization of the singular manifold equations. The treatment in this fashion of equations (3.114) will successfully retrieve the Lax pair associated to the nonlinear equation (3.104). Once this spectral problem is known, we might invert the different changes of variables we have performed, *i.e.* relations (3.93) and (3.98), to straightforwardly derive the associated spectral problems for the former equations (3.96) for the field $\alpha(x,t)$ and the original DNLS equation (3.91), respectively.

2.3.1. Spectral problem for $u(x,t)$

The question concerning the linearization of the singular manifold equation is not trivial. This system of equations (3.114) can be understood as a coupled nonlinear system of PDEs expressed in the variables $\{v, r\}$. If we propose similar Painlevé expansions in a neighbourhood of a new singular manifold $\psi = 0$ for these variables, the dominant terms

$$v \sim \frac{v_0}{\psi^p}, \qquad r \sim \frac{r_0}{\psi^q} \tag{3.117}$$

behave as

$$p = 1, \qquad q = 1, \qquad v_0 = \psi_x, \qquad r_0 = \pm i\psi_x. \tag{3.118}$$

The duality in the sign of r_0 implies the existence of two expansion branches in the series, confirming the well known fact that the Kaup system [235] does also present these two branches [108, 141, 149]. As illustrated in [141], an optimal procedure to overcome this situation requires the introduction two singular manifolds $\{\psi, \chi\}$, such that the fields v and r are written as

$$v = \frac{\psi_x}{\psi} + \frac{\chi_x}{\chi},$$

$$r = i\left(\frac{\psi_x}{\psi} - \frac{\chi_x}{\chi}\right) - 2\lambda, \tag{3.119}$$

where the term -2λ in (3.119) is not essential, but it is useful to simplify the forthcoming results.

Spatial part of the Lax pair

Definitions (3.119) constitute the linearization ansatz that will yield the appropriate result when applied to equations (3.112). The new functions $\psi(x,t), \chi(x,t)$ will play the role of the associated eigenfunctions and λ will act as the spectral parameter. Without further ado, substitution of (3.119) in (3.112) gives rise to

$$u_{xx}^{[0]} + \frac{\psi_x}{\psi}\frac{\chi_x}{\chi} = 0, \tag{3.120a}$$

$$u_{xt}^{[0]} + \frac{\psi_x}{\psi}\frac{\chi_x}{\chi}\left(i\frac{\psi_{xx}}{\psi_x} - i\frac{\chi_{xx}}{\chi_x} + i\frac{\psi_x}{\psi} - i\frac{\chi_x}{\chi} - 2\lambda\right) = 0, \tag{3.120b}$$

that can easily be combined in order to obtain the spatial part of the spectral problem, as

$$\psi_{xx} = \left(\frac{u_{xxx}^{[0]} - iu_{xt}^{[0]}}{2u_{xx}^{[0]}} - i\lambda\right)\psi_x - u_{xx}^{[0]}\psi, \tag{3.121a}$$

$$\chi_{xx} = \left(\frac{u_{xxx}^{[0]} + iu_{xt}^{[0]}}{2u_{xx}^{[0]}} + i\lambda\right)\chi_x - u_{xx}^{[0]}\chi. \tag{3.121b}$$

Temporal part of the Lax pair

Besides that, the corresponding substitution in the singular manifold equations (3.114) provides the temporal part of the Lax pair

$$\psi_t = i\psi_{xx} - 2\lambda\psi_x + i\left(2u_{xx}^{[0]} + \lambda^2\right)\psi, \tag{3.122a}$$

$$\chi_t = -i\chi_{xx} - 2\lambda\chi_x - i\left(2u_{xx}^{[0]} + \lambda^2\right)\chi. \tag{3.122b}$$

It is immediate to verify that equations (3.121) and (3.122) constitute a Lax pair for equation (3.104). Notice that the two eigenfuntions are also related by (3.120a). This fact allows us to construct two equivalent and complementary Lax pairs for (3.104). The elementary Lax pair for (3.104) is given by the conjunction of (3.121b) and (3.122b)

$$\chi_{xx} = \left(\frac{u_{xxx}^{[0]} + i u_{xt}^{[0]}}{2 u_{xx}^{[0]}} + i\lambda \right) \chi_x - u_{xx}^{[0]} \chi,$$

$$\chi_t = -i\chi_{xx} - 2\lambda \chi_x - i \left(2 u_{xx}^{[0]} + \lambda^2 \right) \chi, \tag{3.123}$$

while the alternative Lax pair arises from the combination of (3.121a) and (3.122a)

$$\psi_{xx} = \left(\frac{u_{xxx}^{[0]} - i u_{xt}^{[0]}}{2 u_{xx}^{[0]}} - i\lambda \right) \psi_x - u_{xx}^{[0]} \psi,$$

$$\psi_t = i\psi_{xx} - 2\lambda \psi_x + i \left(2 u_{xx}^{[0]} + \lambda^2 \right) \psi, \tag{3.124}$$

such that $\{\chi, \psi\}$ are two complex conjugate eigenfunctions satisfying $\frac{\chi_x}{\chi} \frac{\psi_x}{\psi} + u_{xx}^{[0]} = 0$ and λ is the spectral parameter.

In the following, we will focus our attention in the spectral problem defined by means of the eigenfunction $\chi(x,t)$ (3.123), whereas the eigenfunction $\psi(x,t)$ may be easily obtained through (3.120a). Obviously, the complex conjugate form of (3.123) and (3.124) retrieves two equivalent Lax pairs for $\overline{u}^{[0]}(x,t)$, which also satisfies (3.104).

Eigenfunctions and singular manifold

Definitions (3.119) can be easily combined with (3.113) to provide the singular manifold through the exact derivative

$$d\phi = \psi \chi dx + \left[-2\lambda \psi \chi + i \left(\chi \psi_x - \psi \chi_x \right) \right] dt, \tag{3.125}$$

whose integration allows us to obtain the iterated solution (3.111).

2.3.2. Spectral problem for $\alpha(x,t)$

We can now use our previous results to derive a Lax pair for the field $\alpha(x,t)$. Combination of (3.98) and the Painlevé expansion (3.111) gives a truncated expansion for α of the form

$$\alpha^{[1]} = i \left(u^{[1]} - \overline{u}^{[1]} \right) = \alpha^{[0]} + i \log \left(\frac{\phi}{\overline{\phi}} \right), \tag{3.126}$$

where $\overline{\psi}$ and $\overline{\chi}$ should be introduced as the complex conjugates of ψ and χ, respectively, in order to have the complex conjugate of (3.125). Besides that, the coupling condition (3.101) should be fulfilled for $u^{[0]}$ and $\overline{u}^{[0]}$. It imposes an additional condition for the singular manifold $\phi(x,t)$ and its complex conjugate $\overline{\phi}(x,t)$ that can be written as

$$\frac{\phi_x}{\phi}\left(\frac{\overline{\chi}_x}{\overline{\chi}} + i\alpha_x^{[0]} + i\lambda\right) + \frac{\overline{\phi}_x}{\overline{\phi}}\left(\frac{\chi_x}{\chi} - i\alpha_x^{[0]} - i\lambda\right) = 1. \tag{3.127}$$

A Lax pair for $\alpha(x,t)$ can be easily obtained from (3.123) with the aid of (3.100) and (3.102). The result is

$$\chi_{xx} = i\left[\frac{\alpha_t^{[0]} + \left(\alpha_x^{[0]}\right)^2 - i\alpha_{xx}^{[0]}}{2\alpha_x^{[0]}} + \lambda\right]\chi_x + \left[\frac{\alpha_t^{[0]} - \left(\alpha_x^{[0]}\right)^2 + i\alpha_{xx}^{[0]}}{2}\right]\chi,$$

$$\chi_t = -i\chi_{xx} - 2\lambda\chi_x + i\left[\alpha_t^{[0]} - \left(\alpha_x^{[0]}\right)^2 + i\alpha_{xx}^{[0]} - \lambda^2\right]\chi. \tag{3.128}$$

The alternative Lax pair can be obtained by substitution of (3.100) and (3.102) in (3.124), whose explicit expression is

$$\psi_{xx} = \frac{\psi_x}{2\alpha_x^{[0]}}\left[\frac{i\left(\alpha_x^{[0]}\right)^4 + 2i\alpha_{xxx}^{[0]}\alpha_x^{[0]} - 2\left(\alpha_x^{[0]}\right)^2\alpha_{xx}^{[0]} - i\left(\alpha_t^{[0]}\right)^2 - i\left(\alpha_{xx}^{[0]}\right)^2 + 2\alpha_{xt}^{[0]}\alpha_x^{[0]}}{\alpha_t^{[0]} - \left(\alpha_x^{[0]}\right)^2 + i\alpha_{xx}^{[0]}}\right]$$

$$- i\lambda\psi_x + \psi\left[\frac{\alpha_t^{[0]} - \left(\alpha_x^{[0]}\right)^2 + i\alpha_{xx}^{[0]}}{2}\right],$$

$$\psi_t = i\psi_{xx} - 2\lambda\psi_x - i\left[\alpha_t^{[0]} - \left(\alpha_x^{[0]}\right)^2 + i\alpha_{xx}^{[0]} - \lambda^2\right]\psi. \tag{3.129}$$

The corresponding counterpart of equation (3.120a) that relates the eigenfunctions $\{\chi,\psi\}$ in terms of $\alpha^{[0]}$ now reads

$$2\frac{\psi_x\chi_x}{\psi\chi} + \left(\alpha_x^{[0]}\right)^2 - \alpha_t^{[0]} - i\alpha_{xx}^{[0]} = 0. \tag{3.130}$$

2.3.3. Spectral problem for DNLS equation

The derivatives of $\alpha^{[0]}$ in (3.94) can be now expressed in terms of the seed solutions $\{m^{[0]}, \overline{m}^{[0]}\}$ as

$$\alpha_x^{[0]} = \frac{1}{2}\left|m^{[0]}\right|^2, \qquad \alpha_t^{[0]} = \frac{i}{2}\left(m^{[0]}\overline{m}_x^{[0]} - \overline{m}^{[0]}m_x^{[0]}\right) - \frac{1}{4}(2\gamma - 3)\left|m^{[0]}\right|^4, \tag{3.131}$$

which finally allow us to write the Lax pair (3.128) for the initial system (3.91) as

$$\chi_{xx} = i \left[\lambda - \frac{\gamma - 2}{2} \left| m^{[0]} \right|^2 - i \frac{m_x^{[0]}}{m^{[0]}} \right] \chi_x + \frac{1}{2} \left[i m^{[0]} \overline{m}_x^{[0]} - \frac{\gamma - 1}{2} \left| m^{[0]} \right|^4 \right] \chi, \quad (3.132)$$

$$\chi_t = -i \chi_{xx} - 2\lambda \chi_x + i \left[i m^{[0]} \overline{m}_x^{[0]} - \frac{\gamma - 1}{2} \left| m^{[0]} \right|^4 - \lambda^2 \right] \chi.$$

Analogously, the alternative Lax pair for $m(x,t)$ is found to be

$$\psi_{xx} = \left[-i\lambda + \frac{i(\gamma - 2)}{2} |m|^2 + \frac{\left((\gamma - 1) |m|^2 \, \overline{m} - 2i\overline{m}_x \right)_x}{(\gamma - 1) |m|^2 \, \overline{m} - 2i\overline{m}_x} \right] \psi_x$$

$$+ \frac{1}{2} \left[i m^{[0]} \overline{m}_x^{[0]} - \frac{\gamma - 1}{2} \left| m^{[0]} \right|^4 \right] \psi, \quad (3.133)$$

$$\psi_t = i \psi_{xx} - 2\lambda \psi_x - i \left[i m^{[0]} \overline{m}_x^{[0]} - \frac{\gamma - 1}{2} \left| m^{[0]} \right|^4 - \lambda^2 \right] \chi,$$

such that the relation between the two eigenfunctions χ and ψ becomes

$$\frac{\psi_x \chi_x}{\psi \chi} - \frac{i}{2} m^{[0]} \, \overline{m}_x^{[0]} + \frac{\gamma - 1}{4} \left| m^{[0]} \right|^4 = 0. \quad (3.134)$$

2.4. Darboux transformations

As it has been previously shown, once the Lax pair has been obtained for a given PDE by means of the SMM, a binary Darboux transformation can be constructed in order to derive recursive solutions. Since the SMM has been applied to equation (3.104), consequently yielding the spectral problems (3.123) and (3.124), the forthcoming calculations regarding the binary Darboux transformation approach should be performed over this linear problem.

Let $u^{[0]}$ be a seed solution for the nonlocal Boussinesq equation (3.104). Let χ_j, $j = 1, 2$ also be two different eigenfunctions for the associated spectral problem (3.123) for this seed field $u^{[0]}$, corresponding to two different eigenvalues λ_j, $j = 1, 2$. Therefore, we have

$$(\chi_j)_{xx} = \left(\frac{u_{xxx}^{[0]} + i u_{xt}^{[0]}}{2 u_{xx}^{[0]}} + i\lambda_j \right) (\chi_j)_x - u_{xx}^{[0]} \chi_j,$$

$$(\chi_j)_t = -i (\chi_j)_{xx} - 2\lambda_j (\chi_j)_x - i \left(2 u_{xx}^{[0]} + \lambda_j^2 \right) \chi_j, \quad (3.135)$$

for $j = 1, 2$, and where the notation $(\chi_j)_x$, $(\chi_j)_{xx}$, $(\chi_j)_t$ represents the derivatives of the eigenfunction χ_j with respect to the corresponding coordinates.

Furthermore, two additional eigenfunctions ψ_j, $j = 1, 2$ arise from expression (3.120a), in the form

$$u_{xx}^{[0]} + \frac{(\psi_j)_x}{\psi_j} \frac{(\chi_j)_x}{\chi_j} = 0, \tag{3.136}$$

which means that we can introduce two different singular manifolds ϕ_j, $j = 1, 2$ defined by (3.125) through the exact derivative

$$d\phi_j = \psi_j \chi_j \, dx + \left\{ -2\lambda_j \psi_j \chi_j + i \left[\chi_j (\psi_j)_x - \psi_j (\chi_j)_x \right] \right\} dt. \tag{3.137}$$

First iteration

As we have seen in the previous Subsection, the truncated Painlevé expansion (3.111) can be considered as an auto-Bäcklund transformation

$$u^{[1]} = u^{[0]} + \log(\phi_1), \tag{3.138}$$

which allows us to obtain an iterated field $u^{[1]}$. Analogously, an iterated Lax pair can be defined for this iterated field $u^{[1]}$ with associated spectral parameter λ_2 in the form

$$(\chi_{1,2})_{xx} = \left(\frac{u_{xxx}^{[1]} + i u_{xt}^{[1]}}{2 u_{xx}^{[1]}} + i\lambda_2 \right) (\chi_{1,2})_x - u_{xx}^{[1]} \chi_{1,2},$$

$$(\chi_{1,2})_t = -i(\chi_{1,2})_{xx} - 2\lambda_2 (\chi_{1,2})_x - i \left(2 u_{xx}^{[1]} + \lambda_2^2 \right) \chi_{1,2}, \tag{3.139}$$

such that

$$u_{xx}^{[1]} + \frac{(\psi_{1,2})_x}{\psi_{1,2}} \frac{(\chi_{1,2})_x}{\chi_{1,2}} = 0. \tag{3.140}$$

Consequently, an iterated singular manifold $\phi_{1,2}$ can also be defined for the field $u^{[1]}$ though expression (3.125)

$$d\phi_{1,2} = \psi_{1,2} \chi_{1,2} \, dx + \left\{ -2\lambda_2 \psi_{1,2} \chi_{1,2} + i \left[\chi_{1,2} (\psi_{1,2})_x - \psi_{1,2} (\chi_{1,2})_x \right] \right\} dt. \tag{3.141}$$

A Lax pair is usually considered a system of equations which is linear in the eigenfuctions. A different point of view lies in the consideration of the equations (3.139) as nonlinear relations between the field $u^{[1]}$ and the eigenfunction $\chi_{1,2}$. According to this new perspective, the Painlevé expansion for the field (3.138) should be accompanied by a similar truncated expansion for the eigenfunctions, performed in a neighbourhood of the singular manifold $\phi_1 = 0$. Let

$$\chi_{1,2} = \chi_2 - \chi_1 \frac{\Delta_{1,2}}{\phi_1}, \qquad \psi_{1,2} = \psi_2 - \psi_1 \frac{\Sigma_{1,2}}{\phi_1}, \tag{3.142}$$

be such expansion, where $\Delta_{1,2}(x,t)$ and $\Sigma_{1,2}(x,t)$ are functions to be determined later. Besides that, (3.141) implies that we can also provide a Painlevé expansion for the singular manifold in the form

$$\phi_{1,2} = \phi_2 + \frac{\Omega_{1,2}}{\phi_1}, \tag{3.143}$$

in terms of the coefficient $\Omega_{1,2}(x,t)$.

Substitution of (3.138), (3.142) and (3.143) in the spectral problem (3.139) and the expression for the singular manifold (3.141) yields the following results

$$\Sigma_{1,2} = \Delta_{2,1}, \qquad\qquad \Omega_{1,2} = -\Delta_{1,2}\Delta_{2,1},$$
$$\Delta_{i,j} = i\psi_i \frac{\chi_i(\chi_j)_x - \chi_j(\chi_i)_x}{(\lambda_i - \lambda_j)(\chi_i)_x}. \tag{3.144}$$

Therefore, we can conclude that (3.138) and (3.142) are binary Darboux transformations for the Lax pair (3.139). These transformations exclusively depend on two sets of eigenfunctions $\{\chi_j, \psi_j\}$, $j = 1, 2$ for the seminal solution $u^{[0]}$, and these will be the only tools required to construct the iterated solution (3.138).

Second iteration and τ-function

The singular manifold $\phi_{1,2}$, as defined in (3.141), can be used to perform a second iteration such that a new field $u^{[2]}$ can be constructed as

$$u^{[2]} = u^{[1]} + \log(\phi_{1,2}), \tag{3.145}$$

that can be expressed in terms of the seed solution $u^{[0]}$ as

$$u^{[2]} = u^{[0]} + \log(\tau_{1,2}), \tag{3.146}$$

where we have introduced the τ-function $\tau_{1,2}$ through the definition

$$\tau_{1,2} = \phi_1\phi_2 - \Delta_{1,2}\Delta_{2,1}. \tag{3.147}$$

nth iteration

This procedure may be implemented repeatedly and generalized up to the nth iteration, which reads as

$$u^{[n]} = u^{[0]} + \log(\phi_1 \phi_{1,2} \cdots \phi_{1,2,\ldots,n}) = u^{[0]} + \log(\tau_{1,2,\ldots,n}). \qquad (3.148)$$

The τ-function for the nth iteration can be computed as

$$\tau_{1,2,\ldots,n} = \det \Delta, \qquad (3.149)$$

where $\Delta \equiv (\Delta_{i,j})_n$ denotes the $n \times n$ matrix of entries

$$\begin{cases} \Delta_{i,i} = \phi_i & \text{for} \quad i = j \\ \Delta_{i,j} = i\psi_i \dfrac{\chi_i(\chi_j)_x - \chi_j(\chi_i)_x}{(\lambda_i - \lambda_j)(\chi_i)_x} & \text{for} \quad i \neq j \end{cases} \qquad (3.150)$$

that may be exclusively expressed in terms of n different pairs of eigenfunctions $\{\chi_k, \psi_k\}$ of eigenvalues λ_k, for the seminal Lax pairs (3.135) and n singular manifolds ϕ_k given by (3.137), $k = 1, \ldots, n$.

2.5. Rational soliton solutions

In this Subsection, rational soliton-like solutions for the DNLS equation (3.91) are obtained by applying the procedure described above. Solutions in this fashion may be algorithmically constructed following analogous steps to the ones previously considered in Subsection 1.6 of this Chapter.

The density of probability for the DNLS equations, the relevant physical field associated to the formation of solitons, may be expressed as

$$|m|^2 = m \cdot \overline{m} = 2i\,(u_x - \overline{u}_x). \qquad (3.151)$$

1. *Seed solution*

Let us consider the following seed solution for (3.91),

$$\begin{aligned} m^{[0]} &= j_0\, e^{\frac{i}{2} j_0^2 (z_0^2 - 1)\left[x + \frac{j_0^2}{2}(z_0^2 + 1)t\right]}, \\ \overline{m}^{[0]} &= j_0\, e^{-\frac{i}{2} j_0^2 (z_0^2 - 1)\left[x + \frac{j_0^2}{2}(z_0^2 + 1)t\right]}, \end{aligned} \qquad (3.152)$$

where j_0, z_0 are arbitrary constants. This seed solution leads to a polynomial solution in u and $\overline{u}$ for (3.104) as

$$u^{[0]} = -\frac{j_0^2}{4}\left[j_0^2 z_0^2 x \left(\frac{x}{2} + j_0^2(z_0^2 + 1)t\right) + i\left(x + j_0^2\left(z_0^2 + \frac{1}{2}\right)t\right)\right], \qquad (3.153)$$

$$\overline{u}^{[0]} = -\frac{j_0^2}{4}\left[j_0^2 z_0^2 x\left(\frac{x}{2} + j_0^2(z_0^2+1)\,t\right) - i\left(x + j_0^2\left(z_0^2 + \frac{1}{2}\right)t\right)\right],$$

where condition (3.101) is identically satisfied.

2. *Eigenfunctions and singular manifolds*

We shall seek now a pair of eigenfunctions that solve the Lax pair (3.123) and (3.124) for the seed solution (3.153). Such solutions may be constructed as

$$
\begin{aligned}
\chi_\sigma &= e^{\frac{i}{2}j_0^2 z_0 \sigma\left[x + j_0^2\left(-\frac{\sigma}{2z_0}(z_0^4 + 7z_0^2 + 1) + 3(z_0^2 + 1)\right)t\right]}, \\
\psi_\sigma &= e^{-\frac{i}{2}j_0^2 z_0 \sigma\left[x + j_0^2\left(-\frac{\sigma}{2z_0}(z_0^4 + 7z_0^2 + 1) + 3(z_0^2 + 1)\right)t\right]},
\end{aligned}
\tag{3.154}
$$

where these eigenfunctions depend on an additional binary real parameter σ, such that $\sigma^2 = 1$. The spectral parameter associated to these eigenfunctions is written as

$$\lambda_\sigma = \frac{j_0^2}{2}\left(2\sigma z_0 - (z_0^2 + 1)\right), \qquad \sigma = \pm 1. \tag{3.155}$$

By means of equation (3.137), we get the singular manifold, which also depends on σ,

$$\phi_\sigma = x - j_0^2\left(\sigma z_0 - (z_0^2 + 1)\right)t - \frac{i}{j_0^2 z_0(\sigma - z_0)}, \qquad \sigma = \pm 1. \tag{3.156}$$

3. *First iteration and one-soliton solution*

Then, it is possible to compute the first iteration through (3.138),

$$
\begin{aligned}
u_\sigma^{[1]} &= -\frac{j_0^2}{4}\left[j_0^2 z_0^2 x\left(\frac{x}{2} + j_0^2(z_0^2+1)\,t\right) + i\left(x + j_0^2\left(z_0^2 + \frac{1}{2}\right)t\right)\right] \\
&\quad + \log\left[x - j_0^2\left(\sigma z_0 - (z_0^2+1)\right)t - \frac{i}{j_0^2 z_0(\sigma - z_0)}\right], \\
\overline{u}_\sigma^{[1]} &= -\frac{j_0^2}{4}\left[j_0^2 z_0^2 x\left(\frac{x}{2} + j_0^2(z_0^2+1)\,t\right) - i\left(x + j_0^2\left(z_0^2 + \frac{1}{2}\right)t\right)\right] \\
&\quad + \log\left[x - j_0^2\left(\sigma z_0 - (z_0^2+1)\right)t + \frac{i}{j_0^2 z_0(\sigma - z_0)}\right],
\end{aligned}
\tag{3.157}
$$

where we can check that $u_\sigma^{[1]}$ and $\overline{u}_\sigma^{[1]}$ are complex conjugates.

Hence, the density of probability for the first iteration is deduced from (3.151) as

$$\begin{aligned}
\left| m_\sigma^{[1]} \right|^2 &= \left| m_\sigma^{[0]} \right|^2 - 4\,\mathrm{Im}\left[\frac{(\phi_\sigma)_x}{\phi_\sigma} \right] \\
&= j_0^2 - \frac{4}{j_0^2 z_0 (\sigma - z_0)\left[(x - v_\sigma t)^2 + \frac{1}{j_0^4 z_0^2 (\sigma - z_0)^2} \right]},
\end{aligned} \tag{3.158}$$

where $\mathrm{Im}(\cdot)$ refers to the imaginary part of the quantity in brackets, and $\sigma = \pm 1$. This solution corresponds to a travelling rational soliton-like wave along the direction $x - v_\sigma t = 0$, of speed

$$v_\sigma = j_0^2 \left(\sigma z_0 - (z_0^2 + 1) \right), \qquad \sigma = \pm 1, \tag{3.159}$$

and constant amplitude

$$a_\sigma = -j_0^2 \left(4 z_0 (\sigma - z_0) - 1 \right), \qquad \sigma = \pm 1. \tag{3.160}$$

One may observe that, depending on the values of the parameters $\sigma = \pm 1$ and z_0, it is possible to obtain either bright or dark rational solitons.

These one-soliton solutions $\left| m_\sigma^{[1]} \right|^2$ are displayed in Figure 3.8 at different times, where a bright rational soliton is obtained for $\sigma = -1$, and a dark one, for $\sigma = 1$. The dynamics of a travelling rational soliton may be alternatively appreciated in Figure 3.9, where a bright soliton has been displayed, whilst the corresponding dark soliton presents a complete analogous profile.
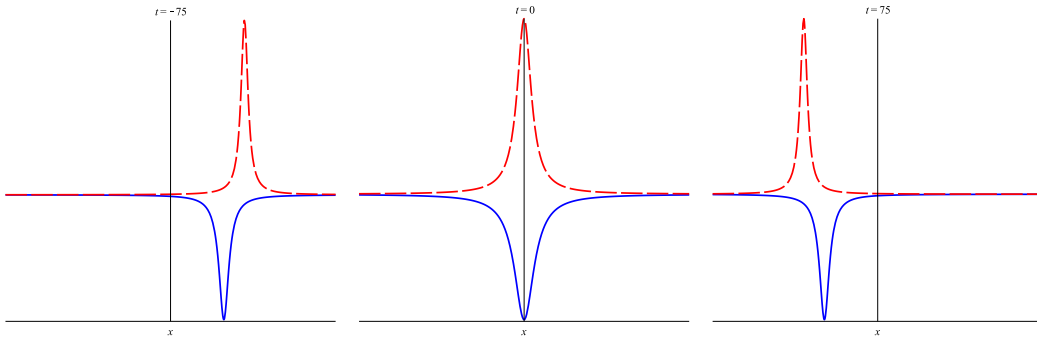
Figure 3.8: *One rational soliton solution* $\left| m_\sigma^{[1]} \right|^2$ *at times* $t = -75,\ 0,\ 75$. *The solid blue line represents a dark soliton for* $\sigma = 1$, $j_0 = 1$, $z_0 = \frac{1}{6}$, *and the dashed red line displays a bright soliton for* $\sigma = -1$, $j_0 = 1$, $z_0 = \frac{1}{6}$.
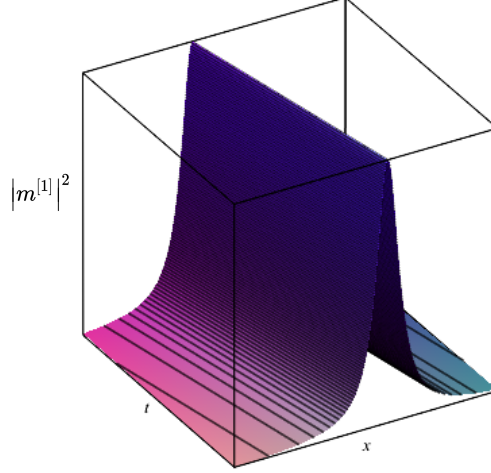
Figure 3.9: *Bright rational soliton solution* $\left|m^{[1]}\right|^2$ *for parameters* $\sigma = -1$, $j_0 = 1$, $z_0 = \frac{1}{6}$.

4. Δ-*matrix and* τ-*function*

In the following, we will identify the first set of functions $\{\chi_1, \psi_1, \lambda_1, \phi_1\}$ as those with $\sigma = 1$ in equations (3.154)-(3.156), and the second set, $\{\chi_2, \psi_2, \lambda_2, \phi_2\}$, with $\sigma = -1$, respectively. After this identification, it is immediate to obtain the Δ-matrix and the τ-function through equations (3.144) and (3.147), of final expressions

$$
\begin{aligned}
\Delta_{1,2} &= -\frac{i}{j_0^2 z_0}\, e^{-ij_0^2 z_0 \left[x + 3j_0^2\left(z_0^2 + 1\right)t\right]}, \\
\Delta_{2,1} &= \frac{i}{j_0^2 z_0}\, e^{ij_0^2 z_0 \left[x + 3j_0^2\left(z_0^2 + 1\right)t\right]}
\end{aligned}
\tag{3.161}
$$

and

$$
\tau_{1,2} = \left(x + j_0^2(z_0^2 + 1)\,t\right)^2 - j_0^4 z_0^2 t^2 + \frac{2ij_0^2\left(x + j_0^2(z_0^2 + 2)\,t\right) - 1}{j_0^4(z_0^2 - 1)}.
\tag{3.162}
$$

5. *Second iteration and two-soliton solution*

Therefore, the second iteration for u and $\overline{u}$ is given by (3.146)

$$
u^{[2]} = -\frac{j_0^2}{4}\left[j_0^2 z_0^2 x\left(\frac{x}{2} + j_0^2(z_0^2 + 1)\,t\right) + i\left(x + j_0^2\left(z_0^2 + \frac{1}{2}\right)t\right)\right]
$$

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$$+ \log \left[\left(x + j_0^2 (z_0^2 + 1) \, t \right)^2 - j_0^4 z_0^2 t^2 + \frac{2i j_0^2 \left(x + j_0^2 (z_0^2 + 2) \, t \right) - 1}{j_0^4 (z_0^2 - 1)} \right],$$

$$\overline{u}^{[2]} = -\frac{j_0^2}{4} \left[j_0^2 z_0^2 x \left(\frac{x}{2} + j_0^2 (z_0^2 + 1) \, t \right) - i \left(x + j_0^2 \left(z_0^2 + \frac{1}{2} \right) t \right) \right]$$

$$+ \log \left[\left(x + j_0^2 (z_0^2 + 1) \, t \right)^2 - j_0^4 z_0^2 t^2 - \frac{2i j_0^2 \left(x + j_0^2 (z_0^2 + 2) \, t \right) + 1}{j_0^4 (z_0^2 - 1)} \right],$$

$$(3.163)$$

and the density of probability $\left| m^{[2]} \right|^2$ acquires the final form

$$\left| m^{[2]} \right|^2 = \left| m^{[0]} \right|^2 - 4 \operatorname{Im} \left[\frac{(\tau_{1,2})_x}{\tau_{1,2}} \right]$$

$$= j_0^2 + \frac{8 \left[\left(x + j_0^2 (z_0^2 + 2) \, t \right)^2 + j_0^4 (z_0^2 - 1) \, t^2 + \frac{1}{j_0^4 (z_0^2 - 1)} \right]}{j_0^2 (z_0^2 - 1) \left[\left(\left(x + j_0^2 (z_0^2 + 1) \, t \right)^2 - j_0^4 z_0^2 t^2 - \frac{1}{j_0^4 (z_0^2 - 1)} \right)^2 + \frac{4 \left(x + j_0^2 (z_0^2 + 2) \, t \right)^2}{j_0^4 (z_0^2 - 1)^2} \right]}.$$

$$(3.164)$$

6. *Asymptotic behaviour*

   This solution asymptotically yields two rational solitons moving along the lines $x - v_\sigma t = 0$ of the form (3.158), with speed (3.159) for $\sigma = \pm 1$, respectively. In order to enlighten this point, the asymptotic behaviour for each rational soliton may be performed. Let us consider the following transformation

   $$X_1 = x - v_1 t, \qquad v_1 = -j_0^2 \left(z_0^2 - z_0 + 1 \right), \qquad (3.165)$$

   that allows to write the limit of $\left| m^{[2]} \right|^2$ at $t \to \pm \infty$ as the static rational soliton

   $$\left| m^{[2]} \right|^2 \sim j_0^2 + \frac{4}{j_0^2 z_0 (z_0 - 1) \left[X_1^2 + \frac{1}{j_0^4 z_0^2 (z_0 - 1)^2} \right]}, \qquad (3.166)$$

   which correspond to the first iterated solution (3.158) for $\sigma = 1$.

   A complete analogous analysis can be consider for the second soliton, by means of the transformation

   $$X_2 = x - v_2 t, \qquad v_2 = -j_0^2 \left(z_0^2 + z_0 + 1 \right),$$

   $$\left| m^{[2]} \right|^2 \sim j_0^2 + \frac{4}{j_0^2 z_0 (z_0 + 1) \left[X_2^2 + \frac{1}{j_0^4 z_0^2 (z_0 + 1)^2} \right]}, \qquad (3.167)$$

that leads to a similar profile to (3.158) with $\sigma = -1$.

Figure 3.10 displays the two-soliton solution $\left|m^{[2]}\right|^2$ at different times and in Figure 3.11, a spatio-temporal plot of the two-soliton solution is also presented. In both graphics, the scattering between the bright and the dark rational solitons is explicitly appreciated.
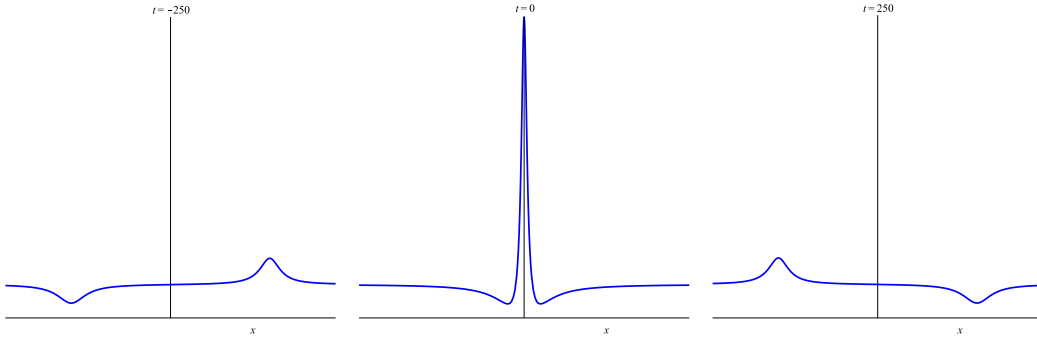
Figure 3.10: *Two rational soliton solution* $\left|m^{[2]}\right|^2$ *at times* $t = -250, 0, 250,$ *for* $j_0 = 1$, $z_0 = \frac{1}{6}$.
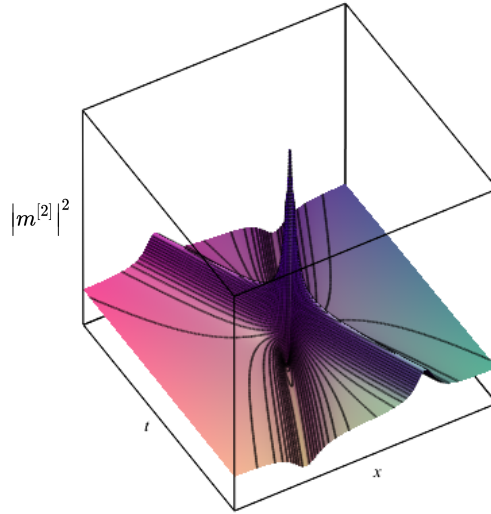
Figure 3.11: *Two rational soliton solution* $\left|m^{[2]}\right|^2$ *for parameters* $j_0 = 1$, $z_0 = \frac{1}{6}$.

Both Figures 3.10 and 3.11 have been plotted in the center-of-mass reference frame of the two colliding rational solitons, which may be achieved after the Galilean transformation $x = X_{\mathrm{CM}} + \frac{1}{2}\left(v_1 + v_2\right)t$. In this system of reference, the two

rational solitons move with equal and opposite velocities $c = \frac{1}{2}(v_1 - v_2)$ along the lines $X_{\mathrm{CM}} - \sigma c t = 0$.

This Chapter has successfully addressed the study of nonlinear integrable systems in 1+1 dimensions, demonstrating therefore the effectiveness of this procedure to obtain the spectral problem and a plethora of soliton-like solutions for such models. The continuation of this research is depicted in the next Chapter, where this theoretical machinery has been generalized, extended and adapted in order to study nonlinear integrable systems with an additional spatial dimension, *i.e.* in $2 + 1$ dimensions.

Chapter 4

Applications to PDEs in $2+1$ dimensions

This Chapter constitutes a continuation of the preceding ones, dedicated to the application of the theoretical foundations exhibited in Chapter 2 for various integrable models in $2+1$ dimensions.

While the treatment concerning integrability and the obtention of spectral problems is analogously developed to the case of $1+1$ dimensions, the incorporation of extra dimensions exerts a significant effect on the characterization of soliton solutions for such kind of systems. Additional spatial dimensions enable the formation of a richer class of novel soliton-like structures. More precisely, we find of particular interest the solutions known as *lumps* or *lump solitons*. Generally speaking, lump solutions are framed within the category of rational-like solitons, and they behave as localized wave configurations that decay to an asymptotic value in all the spatial directions. Lump solutions were first reported in the Kadomtsev–Petviashvili I (KP-I) equation [287], and later studied in [168, 230, 283, 370]. Subsequently, countless standard PDEs in soliton theory have been found to admit this kind of solutions, such as several generalizations of the NLS equation in $2+1$ dimensions [161, 370, 393], the BKP equation [188], the Davey–Stewartson II (DS-II) equation [38], multidimensional sine-Gordon equation [316], the Ishimori I equation [226] or the Toda lattice in $2+1$ dimensions [277].

This Chapter is devoted to the study of two integrable systems in $2+1$ dimensions and the characterization of their soliton solutions of lump-type. Section 1 addresses the integrability analysis of a multi-component NLS equation in $2+1$ dimensions. This PDE can be regarded either as one possible generalization of the Manakov system [286] to $2+1$ dimensions or as a vector generalization for the Fokas system [167]. Its integrability is fully explored by the obtention of the associated spectral problem. Moreover, lump solitons can be directly constructed, with the consequent analysis of their dynamics. These solutions are proved to exhibit an analogous behaviour to the soliton configuration in $1+1$ dimensions, either in terms of propagation or inter-

action. The research associated to this Section is partially enclosed in the author's publication [25]. The corresponding Lie symmetries and similarity reductions for this nonlinear system, obtained in [24], have also been studied in this manuscript, which can be consulted in Subsection 2.2 of Chapter 6.

Section 2 aims to investigate the so-called generalized Nizhnik-Novikov-Veselov (NNV) equation, which constitutes a symmetric generalization of the KdV equation to $2+1$ dimensions [53, 54, 317, 391, 392]. This equation, as other generalizations to higher dimensions of KdV, such as the Kadomtsev–Petviashvili equation [231], is integrable via the IST method [3, 53, 54] and present a wide spectrum of soliton-like solutions [255, 280, 336, 347, 436]. The integrability of this equation is reviewed by means of the Painlevé test and a Lax pair is easily obtained as a consequence of the application of the SMM. We then exploit this formalism and employ the Darboux transformation approach to generate lump soliton solutions of diverse kinds, whose dynamics is deeply investigated. This Section is based on the research presented in [29].

1. Multi-component NLS equation in $2+1$ dimensions

This first Section will be focused in the study of the $(2+1)$-dimensional multi-component nonlinear Schrödinger equation

$$
\begin{aligned}
i\boldsymbol{\alpha}_t + \boldsymbol{\alpha}_{xx} + 2\,m_x\,\boldsymbol{\alpha} &= 0, \\
-i\boldsymbol{\alpha}_t^\dagger + \boldsymbol{\alpha}_{xx}^\dagger + 2\,m_x\,\boldsymbol{\alpha}^\dagger &= 0, \\
\left(m_y + \boldsymbol{\alpha}\,\boldsymbol{\alpha}^\dagger\right)_x &= 0,
\end{aligned}
\tag{4.1}
$$

where $\boldsymbol{\alpha}(x,y,t) = [\alpha_1(x,y,t),\alpha_2(x,y,t)]^\mathsf{T}$ is a two-dimensional vector whose components are complex functions depending on two spatial dimensions (x,y) and one temporal dimension t. The vector $\boldsymbol{\alpha}^\dagger$ denotes the conjugate transpose of $\boldsymbol{\alpha}$ and $m(x,y,t)$ is a real scalar function related to the probability density $\boldsymbol{\alpha}\cdot\boldsymbol{\alpha}^\dagger$ through the third equation in (4.1).

The reduction $x = y$ in (4.1) yields the Manakov system [286], which is often called as vector NLS system [8]. Integrability properties of this Manakov system and its Painlevé Property are described in references [282, 404]. Different generalizations of this Manakov system with linear derivative-type terms and their solutions have been recently studied by the author of this manuscript in the previous Chapter, Section 1 and references [26, 124].

Furthermore, (4.1) constitutes a multi-component generalization of a system that

has been discussed by several authors [76, 167, 348], sometimes referred as the Fokas system, whose lump solutions have been extensively studied in [160, 161, 351, 393].

1.1. Painlevé test and integrability

It is easy to check whether or not (4.1) has the Painlevé Property. The existence of such property requires that all the solutions of (4.1) are single-valued in the initial conditions. This requirement means that the fields $\{\boldsymbol{\alpha}, \boldsymbol{\alpha}^\dagger, m\}$ should be expanded as the Laurent expansion

$$
\begin{aligned}
\alpha_1 &= \sum_{j=0}^{\infty} a_j \phi^{j-\beta}, & \alpha_1^\dagger &= \sum_{j=0}^{\infty} a_j^\dagger \phi^{j-\beta}, \\
\alpha_2 &= \sum_{j=0}^{\infty} b_j \phi^{j-\gamma}, & \alpha_2^\dagger &= \sum_{j=0}^{\infty} b_j^\dagger \phi^{j-\gamma}, \\
m &= \sum_{j=0}^{\infty} c_j \phi^{j-\delta},
\end{aligned}
\tag{4.2}
$$

where $\phi(x, y, t) = 0$ is the manifold of movable singularities and the coefficients $a_j(x, y, t), a_j^\dagger(x, y, t), b_j(x, y, t), b_j^\dagger(x, y, t), c_j(x, y, t), \forall j$ are still to be determined, together with the indices of the expansion β, γ, δ.

Substitution of (4.2) in (4.1) provides polynomials in powers of ϕ that should vanish. It results in five recursion relations for the coefficients of the expansion $\{a_j, a_j^\dagger, b_j, b_j^\dagger, c_j\}$, for a proper value of j.

1. A leading-order analysis easily provides

$$
\beta = \gamma = \delta = 1,
$$
$$
a_0 = A_1 \phi_x, \qquad a_0^\dagger = A_1^\dagger \phi_x^\dagger, \qquad b_0 = A_2 \phi_x, \qquad b_0^\dagger = A_2^\dagger \phi_x^\dagger, \qquad c_0 = \phi_x,
$$

such that

$$
A_1 A_1^\dagger + A_2 A_2^\dagger = \frac{\phi_y}{\phi_x}.
\tag{4.3}
$$

The latter expression among the different coefficients indicates the presence of a resonance at $j = 0$, as it will be proved later.

2. The recursion relations among the coefficients retrieve the following resonance condition

$$
j^3 (j - 1)(j - 3)^3 (j - 4)(j + 1) = 0,
\tag{4.4}
$$

which yields eight resonances, located in $j = \{-1, 0 \text{ (triple)}, 1, 3 \text{ (triple)}, 4\}$. The usual resonance in $j = -1$ reasserts the arbitrariness of ϕ, whilst the triple resonance in $j = 0$ indicates that three of the four coefficients at constant level $\{A_1, A_1^\dagger, A_2, A_2^\dagger\}$ should be arbitrary.

3. The calculation regarding the validation of the resonance conditions can be easily performed with MAPLE. It is found that $\{a_0, a_0^\dagger, b_0, a_1, a_3, a_3^\dagger, b_3, a_4\}$ are arbitrary coefficients and then the resonance conditions are identically satisfied. Therefore, we can conclude that the system of PDEs (4.1) has the Painlevé Property.

1.2.   The singular manifold method

The relation between integrability and Painlevé Property is an extremely interesting topic. The existence of a Lax pair is usually considered as the best proof of the integrability of a nonlinear PDE. The issue concerning the identification of such spectral problem is a nontrivial matter that can be approached through the SMM [410]. The SMM is based on the truncation of equation (4.2) at constant level, of the form

$$\alpha_1^{[1]} = \alpha_1^{[0]} + \frac{A_1 \phi_x}{\phi}, \qquad \left(\alpha_1^{[1]}\right)^\dagger = \left(\alpha_1^{[0]}\right)^\dagger + \frac{A_1^\dagger \phi_x}{\phi},$$

$$\alpha_2^{[1]} = \alpha_2^{[0]} + \frac{A_2 \phi_x}{\phi}, \qquad \left(\alpha_2^{[1]}\right)^\dagger = \left(\alpha_2^{[0]}\right)^\dagger + \frac{A_2^\dagger \phi_x}{\phi}, \qquad (4.5)$$

$$m^{[1]} = m^{[0]} + \frac{\phi_x}{\phi},$$

such that $A_1 A_1^\dagger + A_2 A_2^\dagger = \frac{\phi_y}{\phi_x}$. The vector $\boldsymbol{\alpha}^{[0]} = \left(\alpha_1^{[0]}, \alpha_2^{[0]}\right)^{\mathsf{T}}$ and the scalar field $m^{[0]}$ are the seed fields, whilst $\boldsymbol{\alpha}^{[1]} = \left(\alpha_1^{[1]}, \alpha_2^{[1]}\right)^{\mathsf{T}}$ and $m^{[1]}$ are the iterated ones. The truncation (4.5) can be therefore considered as an auto-Bäcklund transformation between two solutions of the same initial problem (4.1).

Substitution of (4.5) in (4.1) yields five polynomials in powers of ϕ, where each coefficient should identically hold. At this point, it is extremely convenient to introduce the following quantities [141, 148]

$$v = \frac{\phi_{xx}}{\phi_x}, \qquad r = \frac{\phi_t}{\phi_x}, \qquad q = \frac{\phi_y}{\phi_x}, \qquad (4.6)$$

that generalize to $2 + 1$ dimensions the homographic invariants defined in (2.22) and (2.23). The associated compatibility conditions for those definitions, given by the identification of the cross derivatives $\phi_{xxt} = \phi_{txx}$, $\phi_{xxy} = \phi_{yxx}$ and $\phi_{yt} = \phi_{ty}$,

trivially yield

$$v_t = (r_x + r\,v)_x\,, \qquad v_y = (q_x + q\,v)_x\,, \qquad r_y = q_t + q r_x - q_x r, \qquad (4.7)$$

which can be understood as an extension of relations (2.25) to $2+1$ dimensions.

Expressions of the fields in terms of the singular manifold

The introduction of (4.6) allows us to write, after performing some symbolic computations with MAPLE, the seed fields $\boldsymbol{\alpha}^{[0]}$ in terms of the singular manifold in the following form

$$\alpha_1^{[0]} = -\left(A_1\right)_x - A_1\left(\frac{v+ir}{2}\right), \qquad \left(\alpha_1^{[0]}\right)^\dagger = -\left(A_1^\dagger\right)_x - A_1^\dagger\left(\frac{v-ir}{2}\right), \qquad (4.8a)$$

$$\alpha_2^{[0]} = -\left(A_2\right)_x - A_2\left(\frac{v+ir}{2}\right), \qquad \left(\alpha_2^{[0]}\right)^\dagger = -\left(A_2^\dagger\right)_x - A_2^\dagger\left(\frac{v-ir}{2}\right), \qquad (4.8b)$$

$$m_x^{[0]} = -\frac{1}{4}\left(v_x + \frac{v^2+r^2}{2} + \int r_t\,dx\right). \qquad (4.8c)$$

Singular manifold equations

Truncation (4.5) therefore implies that the coefficients $A_1, A_1^\dagger, A_2, A_2^\dagger$ and ϕ should now satisfy some nontrivial equations, the so-called singular manifold equations. The expression for the coefficients can be listed as

$$-i\left(A_1\right)_t = \left(A_1\right)_{xx} + A_1\left(v_x + i\,r_x + 2\,m_x^{[0]}\right), \qquad (4.9a)$$

$$i\left(A_1^\dagger\right)_t = \left(A_1^\dagger\right)_{xx} + A_1^\dagger\left(v_x - i\,r_x + 2\,m_x^{[0]}\right), \qquad (4.9b)$$

$$-i\left(A_2\right)_t = \left(A_2\right)_{xx} + A_2\left(v_x + i\,r_x + 2\,m_x^{[0]}\right), \qquad (4.9c)$$

$$i\left(A_2^\dagger\right)_t = \left(A_2^\dagger\right)_{xx} + A_2^\dagger\left(v_x - i\,r_x + 2\,m_x^{[0]}\right), \qquad (4.9d)$$

such that the relations for the homographic invariants $\{r,q\}$ read

$$q = A_1 A_1^\dagger + A_2 A_2^\dagger, \qquad (4.10a)$$

$$\int r_y\,dx = -q\,r + i\left[\left(A_1\right)_x A_1^\dagger - A_1\left(A_1\right)_x^\dagger + \left(A_2\right)_x A_2^\dagger - A_2\left(A_2\right)_x^\dagger\right]. \qquad (4.10b)$$

1.3. Spectral problem

Equations (4.8) and (4.10) can be linearized (cf. [138]) through the introduction of three complex functions $\{\psi(x,y,t), \chi(x,y,t), \rho(x,y,t)\}$ and their conjugates $\{\psi^\dagger(x,y,t), \chi^\dagger(x,y,t), \rho^\dagger(x,y,t)\}$, such that

$$A_1 = \frac{\chi}{\psi}, \qquad A_1^\dagger = \frac{\chi^\dagger}{\psi^\dagger}, \qquad A_2 = \frac{\rho}{\psi}, \qquad A_2^\dagger = \frac{\rho^\dagger}{\psi^\dagger}. \tag{4.11}$$

Then, the expression of the variables $\{v, r, q\}$ in terms of this new triad of functions $\{\psi, \chi, \rho\}$ can be written as

$$v = \frac{\psi_x^\dagger}{\psi^\dagger} + \frac{\psi_x}{\psi}, \qquad r = i\left(\frac{\psi_x^\dagger}{\psi^\dagger} - \frac{\psi_x}{\psi}\right), \qquad q = \frac{\chi\chi^\dagger + \rho\rho^\dagger}{\psi\psi^\dagger}. \tag{4.12}$$

Spatial part of the Lax pair

Substitution of the definitions (4.11)-(4.12) in equations (4.8) and (4.10b) trivially yields the spatial part for the spectral problem for (4.1), as

$$\psi_y = -\left(\alpha_1^{[0]}\right)^\dagger \chi - \left(\alpha_2^{[0]}\right)^\dagger \rho,$$
$$\chi_x = -\alpha_1^{[0]}\psi, \tag{4.13a}$$
$$\rho_x = -\alpha_2^{[0]}\psi,$$

and its complex conjugate

$$\psi_y^\dagger = -\alpha_1^{[0]}\chi^\dagger - \alpha_2^{[0]}\rho^\dagger,$$
$$\chi_x^\dagger = -\left(\alpha_1^{[0]}\right)^\dagger \psi^\dagger, \tag{4.13b}$$
$$\rho_x^\dagger = -\left(\alpha_2^{[0]}\right)^\dagger \psi^\dagger.$$

Temporal part of the Lax pair

A similar result arises from the substitution of equations (4.11)-(4.12) in (4.7) and (4.9), which in combination with (4.13) successfully provides the temporal part of the Lax pair as

$$\psi_t = -i\psi_{xx} - 2im_x^{[0]}\psi,$$

$$\chi_t = -i \left[\alpha_1^{[0]} \right]_x \chi + i \alpha_1^{[0]} \psi_x, \tag{4.14a}$$

$$\rho_t = -i \left[\alpha_2^{[0]} \right]_x \rho + i \alpha_2^{[0]} \psi_x,$$

and its complex conjugate

$$\psi_t^\dagger = i \psi_{xx}^\dagger + 2i m_x^{[0]} \psi^\dagger,$$

$$\chi_t^\dagger = i \left[\left(\alpha_1^{[0]} \right)^\dagger \right]_x \chi^\dagger - i \left(\alpha_1^{[0]} \right)^\dagger \psi_x^\dagger, \tag{4.14b}$$

$$\rho_t^\dagger = i \left[\left(\alpha_2^{[0]} \right)^\dagger \right]_x \rho^\dagger - i \left(\alpha_1^{[0]} \right)^\dagger \psi_x^\dagger.$$

It is immediate to test that the compatibility among equations (4.13) and (4.14) retrieves the initial $(2 + 1)$-NLS system (4.1). Thus, the system of linear PDEs (4.13)-(4.14) can be regarded as a three-component Lax pair for (4.1).

Eigenfunctions and the singular manifold

The relation between the singular manifold ϕ and the eigenfunctions $\{\psi, \chi, \rho\}$ can be easily established by combining (4.6) and (4.12), providing the following exact derivative

$$d\phi = \psi \, \psi^\dagger \, dx + \left(\chi \, \chi^\dagger + \rho \, \rho^\dagger \right) \, dy + i \left(\psi \, \psi_x^\dagger - \psi^\dagger \, \psi_x \right) \, dt. \tag{4.15}$$

The role of the spectral parameter

It is worthwhile to notice the absence of a spectral parameter in this spectral problem. This circumstance is as a direct consequence of the fact that no additional arbitrary element (that could act as the spectral parameter) arises in the linearization process of the singular manifold equations (4.9) and (4.10). Nevertheless, the explicit presence of a spectral parameter in the associated linear problem is not mandatory for Lax pairs in higher dimensions, as it is evidenced in the literature [247]. Neither does this situation affect the computation of solutions through the Darboux transformation approach, as it has been extensively illustrated in [138, 148, 151, 159, 161].

As it was shown in Section 3 from Chapter 2, Lax equations are invariant under similarity transformations and it is possible to construct gauge transformations such that the spectral problem is properly introduced. The inner question concerning the *removability* of the spectral parameter or the consideration of non-parametric

Lax pairs is a controversial topic that has been addressed by several authors [291] and the references therein. Another different approach regarding this matter lies in the use of group techniques. The introduction of a spectral parameter in a linear problem without spectral parameter through one-parameter groups of Lie symmetries was proposed by Levi *et al.* [270, 271]. This group interpretation of the spectral parameter has been subsequently studied by several authors [84, 85, 87, 252, 290, 291]. In particular, it is possible to combine these techniques with the SMM approach to introduce the spectral parameter in linear problems in $2+1$ dimensions [149, 152].

1.4. Darboux transformations

Let the vector $\boldsymbol{\alpha}^{[0]} = \left(\alpha_1^{[0]}, \alpha_2^{[0]}\right)^{\mathsf{T}}$ and the scalar field $m^{[0]}$ be a seed solution for the initial multi-component NLS system (4.1). Let $\{\psi_j, \chi_j, \rho_j\}$, $j = 1, 2$ also be two different sets of eigenfunctions for the linear problem (4.13)-(4.14) associated to this seed solution, which explicitly reads

$$
\begin{aligned}
\left(\psi_j\right)_y &= -\left(\alpha_1^{[0]}\right)^{\dagger} \chi_j - \left(\alpha_2^{[0]}\right)^{\dagger} \rho_j, & \left(\psi_j\right)_t &= -i\left(\psi_j\right)_{xx} - 2im_x^{[0]} \psi_j, \\
\left(\chi_j\right)_x &= -\alpha_1^{[0]} \psi_j, & \left(\chi_j\right)_t &= -i\left[\alpha_1^{[0]}\right]_x \chi_j + i\alpha_1^{[0]} \left(\psi_j\right)_x, \quad (4.16) \\
\left(\rho_j\right)_x &= -\alpha_2^{[0]} \psi_j, & \left(\rho_j\right)_t &= -i\left[\alpha_2^{[0]}\right]_x \rho_j + i\alpha_2^{[0]} \left(\psi_j\right)_x,
\end{aligned}
$$

and where its complex conjugate is given by

$$
\begin{aligned}
\left(\psi_j^{\dagger}\right)_y &= -\alpha_1^{[0]} \chi_j^{\dagger} - \alpha_2^{[0]} \rho_j^{\dagger}, & \left(\psi_j^{\dagger}\right)_t &= i\left[\psi_j^{\dagger}\right]_{xx} + 2im_x^{[0]} \psi_j^{\dagger}, \\
\left(\chi_j^{\dagger}\right)_x &= -\left(\alpha_1^{[0]}\right)^{\dagger} \psi_j^{\dagger}, & \left(\chi_j^{\dagger}\right)_t &= i\left[\left(\alpha_1^{[0]}\right)^{\dagger}\right]_x \chi_j^{\dagger} - i\left(\alpha_1^{[0]}\right)^{\dagger} \left[\psi_j^{\dagger}\right]_x, \quad (4.17) \\
\left(\rho_j^{\dagger}\right)_x &= -\left(\alpha_2^{[0]}\right)^{\dagger} \psi_j^{\dagger}, & \left(\rho_j^{\dagger}\right)_t &= i\left[\left(\alpha_2^{[0]}\right)^{\dagger}\right]_x \rho_j^{\dagger} - i\left(\alpha_1^{[0]}\right)^{\dagger} \left[\psi_j^{\dagger}\right]_x,
\end{aligned}
$$

with $j = 1, 2$ and the subindices $\{x, y, t, xx\}$ denote the partial derivatives with respect to the corresponding variables.

We are also able to introduce two singular manifolds ϕ_j, $j = 1, 2$ following (4.15), as

$$
d\phi_j = \psi_j \, \psi_j^{\dagger} \, dx + \left(\chi_j \, \chi_j^{\dagger} + \rho_j \, \rho_j^{\dagger}\right) dy + i\left[\psi_j \left(\psi_j^{\dagger}\right)_x - \psi_j^{\dagger} \left(\psi_j\right)_x\right] dt. \qquad (4.18)
$$

First iteration

The Painlevé truncated series (4.5) may be expressed in terms of the seed eigenfunctions $\{\psi_j, \chi_j, \rho_j\}, j = 1, 2$ and the singular manifold ϕ_1, giving rise to the first iterated fields $\{\boldsymbol{\alpha}^{[1]}, m^{[1]}\}$ as

$$\alpha_1^{[1]} = \alpha_1^{[0]} + \frac{\chi_1 \psi_1^\dagger}{\phi_1}, \qquad \left(\alpha_1^{[1]}\right)^\dagger = \left(\alpha_1^{[0]}\right)^\dagger + \frac{\chi_1^\dagger \psi_1}{\phi_1},$$

$$\alpha_2^{[1]} = \alpha_2^{[0]} + \frac{\rho_1 \psi_1^\dagger}{\phi_1}, \qquad \left(\alpha_2^{[1]}\right)^\dagger = \left(\alpha_2^{[0]}\right)^\dagger + \frac{\rho_1^\dagger \psi_1}{\phi_1}, \qquad (4.19)$$

$$m^{[1]} = m^{[0]} + \frac{(\phi_1)_x}{\phi_1}.$$

Proceeding in a similar fashion to the previous Chapter, we may construct the spectral problem associated to the iterated solution $\{\boldsymbol{\alpha}^{[1]}, m^{[1]}\}$ defined in (4.19). Let us define the corresponding eigenfunctions $\{\psi_{1,2}, \chi_{1,2}, \rho_{1,2}\}$, and their complex conjugates $\{\psi_{1,2}^\dagger, \chi_{1,2}^\dagger, \rho_{1,2}^\dagger\}$, such that the associated Laix pair reads

$$\left(\psi_{1,2}\right)_y = -\left(\alpha_1^{[1]}\right)^\dagger \chi_{1,2} - \left(\alpha_2^{[1]}\right)^\dagger \rho_{1,2}, \qquad \left(\psi_{1,2}\right)_t = -i \left(\psi_{1,2}\right)_{xx} - 2im_x^{[1]} \psi_{1,2},$$

$$\left(\chi_{1,2}\right)_x = -\alpha_1^{[1]} \psi_{1,2}, \qquad\qquad \left(\chi_{1,2}\right)_t = -i \left[\alpha_1^{[1]}\right]_x \chi_{1,2} + i\alpha_1^{[1]} \left(\psi_{1,2}\right)_x,$$

$$\left(\rho_{1,2}\right)_x = -\alpha_2^{[1]} \psi_{1,2}, \qquad\qquad \left(\rho_{1,2}\right)_t = -i \left[\alpha_2^{[1]}\right]_x \rho_{1,2} + i\alpha_2^{[1]} \left(\psi_{1,2}\right)_x,$$

$$(4.20)$$

and

$$\left(\psi_{1,2}^\dagger\right)_y = -\alpha_1^{[1]} \chi_{1,2}^\dagger - \alpha_2^{[1]} \rho_{1,2}^\dagger, \qquad \left(\psi_{1,2}^\dagger\right)_t = i \left[\psi_{1,2}^\dagger\right]_{xx} + 2im_x^{[1]} \psi_{1,2}^\dagger,$$

$$\left(\chi_{1,2}^\dagger\right)_x = -\left(\alpha_1^{[1]}\right)^\dagger \psi_{1,2}^\dagger, \qquad \left(\chi_{1,2}^\dagger\right)_t = i \left[\left(\alpha_1^{[1]}\right)^\dagger\right]_x \chi_{1,2}^\dagger - i \left(\alpha_1^{[1]}\right)^\dagger \left[\psi_{1,2}^\dagger\right]_x,$$

$$\left(\rho_{1,2}^\dagger\right)_x = -\left(\alpha_2^{[1]}\right)^\dagger \psi_{1,2}^\dagger \qquad \left(\rho_{1,2}^\dagger\right)_t = i \left[\left(\alpha_2^{[1]}\right)^\dagger\right]_x \rho_{1,2}^\dagger - i \left(\alpha_1^{[1]}\right)^\dagger \left[\psi_{1,2}^\dagger\right]_x.$$

$$(4.21)$$

The introduction of these eigenfunctions allows us to straightforwardly define the corresponding singular manifold $\phi_{1,2}$, by the exact derivative

$$d\phi_{1,2} = \psi_{1,2} \psi_{1,2}^\dagger \, dx + \left(\chi_{1,2} \chi_{1,2}^\dagger + \rho_{1,2} \rho_{1,2}^\dagger\right) dy + i \left[\psi_{1,2} \left(\psi_{1,2}^\dagger\right)_x - \psi_{1,2}^\dagger \left(\psi_{1,2}\right)_x\right] dt.$$

$$(4.22)$$

The spectral problem given in (4.16) and (4.17) can be interpreted [83, 141, 248]

as a system of nonlinear equations for the fields $\{\boldsymbol{\alpha}^{[1]}, m^{[1]}\}$ and the eigenfunctions $\{\psi_{1,2}, \chi_{1,2}, \rho_{1,2}\}$. Hence, it is expected that these eigenfunctions may follow a similar Painlevé expansion, constructed in a neighbourhood of the singular manifold $\phi_1 = 0$, of the form

$$\psi_{1,2} = \psi_2 - \psi_1 \frac{\Delta_{1,2}}{\phi_1}, \qquad \chi_{1,2} = \chi_2 - \chi_1 \frac{\Delta_{1,2}}{\phi_1}, \qquad \rho_{1,2} = \rho_2 - \rho_1 \frac{\Delta_{1,2}}{\phi_1}, \qquad \text{(4.23a)}$$

where their complex conjugates are given by

$$\psi_{1,2}^\dagger = \psi_2^\dagger - \psi_1^\dagger \frac{\Delta_{1,2}^\dagger}{\phi_1}, \qquad \chi_{1,2}^\dagger = \chi_2^\dagger - \chi_1^\dagger \frac{\Delta_{1,2}^\dagger}{\phi_1}, \qquad \rho_{1,2}^\dagger = \rho_2^\dagger - \rho_1^\dagger \frac{\Delta_{1,2}^\dagger}{\phi_1}. \qquad \text{(4.23b)}$$

Substitution of (4.23) in their respective spectral problems provides the expressions for the coefficients $\{\Delta_{1,2}, \Delta_{1,2}^\dagger\}$ as exact derivatives. Those results are subsumed in the 2×2 Δ-matrix, defined as

$$d\Delta_{i,j} = \psi_i^\dagger \psi_j \, dx + \left(\chi_i^\dagger \chi_j + \rho_i^\dagger \rho_j \right) dy + i \left[\psi_j \left(\psi_i^\dagger \right)_x - \psi_i^\dagger \left(\psi_j \right)_x \right] dt, \qquad \text{(4.24)}$$

for all $i, j = 1, 2$. If $i = j$, equation (4.24) reduces to expression (4.18), which implies that $\Delta_{i,i} = \phi_i$, $i = 1, 2$. If $i \neq j$, the identity $\Delta_{j,i} = \Delta_{i,j}^\dagger$ holds.

The singular manifold $\phi_{1,2}$ can be expanded with a similar Painlevé series, of the form

$$\phi_{1,2} = \phi_2 - \frac{\Delta_{1,2}\Delta_{2,1}}{\phi_1}, \qquad \text{(4.25)}$$

such that $\phi_{1,2}$ satisfies (4.22) and the coefficients $\{\Delta_{1,2}, \Delta_{2,1}\}$ are given in (4.24).

Second iteration and τ-function

Considering the fact that $\phi_{1,2}$ is the associated singular manifold for the iterated solution $\boldsymbol{\alpha}^{[1]}$, we shall now construct the solution for the fields $\boldsymbol{\alpha}^{[2]}$ by iterating expressions (4.19), obtaining

$$\alpha_1^{[2]} = \alpha_1^{[1]} + \frac{\chi_{1,2}\, \psi_{1,2}^\dagger}{\phi_{1,2}}, \qquad \left(\alpha_1^{[2]} \right)^\dagger = \left(\alpha_1^{[1]} \right)^\dagger + \frac{\chi_{1,2}^\dagger\, \psi_{1,2}}{\phi_{1,2}},$$

$$\alpha_2^{[2]} = \alpha_2^{[1]} + \frac{\rho_{1,2}\, \psi_{1,2}^\dagger}{\phi_{1,2}}, \qquad \left(\alpha_2^{[2]} \right)^\dagger = \left(\alpha_2^{[1]} \right)^\dagger + \frac{\rho_{1,2}^\dagger\, \psi_{1,2}}{\phi_{1,2}}, \qquad \text{(4.26)}$$

$$m^{[2]} = m^{[1]} + \frac{(\phi_{1,2})_x}{\phi_{1,2}}.$$

The second iteration can be easily expressed in terms of the seed solution $\{\boldsymbol{\alpha}^{[0]}, m^{[0]}\}$ by means of relations (4.19), giving rise to the following results

$$
\alpha_1^{[2]} = \alpha_1^{[0]} + \frac{\psi_2^\dagger \Delta_{1,1}\, \chi_2 + \psi_1^\dagger \Delta_{2,2}\, \chi_1 - \psi_2^\dagger \Delta_{1,2}\, \chi_1 - \psi_1^\dagger \Delta_{2,1}\, \chi_2}{\tau_{1,2}},
$$

$$
\alpha_2^{[2]} = \alpha_2^{[0]} + \frac{\psi_2^\dagger \Delta_{1,1}\, \rho_2 + \psi_1^\dagger \Delta_{2,2}\, \rho_1 - \psi_2^\dagger \Delta_{1,2}\, \rho_1 - \psi_1^\dagger \Delta_{2,1}\, \rho_2}{\tau_{1,2}},
$$

$$
\left(\alpha_1^{[2]}\right)^\dagger = \left(\alpha_1^{[0]}\right)^\dagger + \frac{\chi_2^\dagger \Delta_{1,1}\, \psi_2 + \chi_1^\dagger \Delta_{2,2}\, \psi_1 - \chi_1^\dagger \Delta_{2,1}\, \psi_2 - \chi_2^\dagger \Delta_{1,2}\, \psi_1}{\tau_{1,2}}, \qquad (4.27)
$$

$$
\left(\alpha_2^{[2]}\right)^\dagger = \left(\alpha_2^{[0]}\right)^\dagger + \frac{\rho_2^\dagger \Delta_{1,1}\, \psi_2 + \rho_1^\dagger \Delta_{2,2}\, \psi_1 - \rho_1^\dagger \Delta_{2,1}\, \psi_2 - \rho_2^\dagger \Delta_{1,2}\, \psi_1}{\tau_{1,2}},
$$

$$
m^{[2]} = m^{[0]} + \frac{\left(\tau_{1,2}\right)_x}{\tau_{1,2}},
$$

where the τ-function $\tau_{1,2}$ has been introduced as

$$
\tau_{1,2} = \phi_1 \phi_{1,2} = \phi_1 \phi_2 - \Delta_{1,2} \Delta_{2,1}, \qquad (4.28)
$$

with coincides with $\tau_{1,2} = \det \Delta$ regarding definition (4.24).

1.5. Lump solutions

This Subsection is aimed at the characterization of lump soliton solutions for the multi-component NLS system defined in (4.1), whose spectral problem is given in (4.13) and (4.14). Solutions of this kind may be straightforwardly constructed in a similar fashion of the standard procedure considered in the previous Chapters. The major difference rests on the fact that lump solutions decay regularly and rationally in all spatial dimensions. Therefore, the τ-function should present a polynomial dependence in its constituent coordinates (x, y, t) in order to reproduce this behaviour in the associated solutions.

1. *Seed solution*

Let us propose the following seed solution

$$
\boldsymbol{\alpha}^{[0]} = \begin{pmatrix} \beta_1 \\ \beta_2 \end{pmatrix}, \qquad (4.29)
$$

where β_1, β_2 are two arbitrary complex constants. Henceforth, we will denote the modulus of the seed solution $\left|\alpha^{[0]}\right|^2$ as

$$\mu = \left|\alpha^{[0]}\right|^2 = \left(\boldsymbol{\alpha}^{[0]}\right)^\dagger \cdot \left(\boldsymbol{\alpha}^{[0]}\right) = \beta_1^\dagger \beta_1 + \beta_2^\dagger \beta_2. \tag{4.30}$$

Then, the seed solution for the probability density $m_y^{[0]}$ reads

$$m_y^{[0]} = -\left(\boldsymbol{\alpha}^{[0]}\right)^\dagger \cdot \left(\boldsymbol{\alpha}^{[0]}\right) = -\mu. \tag{4.31}$$

2. *Eigenfunctions*

This seed solution implies that the spectral problem defined in (4.16)-(4.17) may be written as

$$\begin{aligned}
\left(\psi_j\right)_y &= -\beta_1^\dagger \chi_j - \beta_2^\dagger \rho_j, & \left(\psi_j\right)_t &= -i\left(\psi_j\right)_{xx}, \\
\left(\chi_j\right)_x &= -\beta_1 \psi_j, & \left(\chi_j\right)_t &= i\beta_1\left(\psi_j\right)_x, \\
\left(\rho_j\right)_x &= -\beta_2 \psi_j, & \left(\rho_j\right)_t &= i\beta_2\left(\psi_j\right)_x,
\end{aligned} \tag{4.32}$$

and its complex conjugate

$$\begin{aligned}
\left(\psi_j^\dagger\right)_y &= -\beta_1 \chi_j^\dagger - \beta_2 \rho_j^\dagger, & \left(\psi_j^\dagger\right)_t &= i\left(\psi_j^\dagger\right)_{xx}, \\
\left(\chi_j^\dagger\right)_x &= -\beta_1^\dagger \psi_j^\dagger, & \left(\chi_j^\dagger\right)_t &= -i\beta_1^\dagger\left(\psi_j^\dagger\right)_x, \\
\left(\rho_j^\dagger\right)_x &= -\beta_2^\dagger \psi_j^\dagger, & \left(\rho_j^\dagger\right)_t &= -i\beta_2^\dagger\left(\psi_j^\dagger\right)_x,
\end{aligned} \tag{4.33}$$

where $j = 1, 2$.

The key observation to construct the solutions for this spectral problem lies in the fact that the linear system of PDEs constituting the Lax pair is fully decoupled in the triads $\{\psi_j, \chi_j, \rho_j\}$ and $\{\psi_j^\dagger, \chi_j^\dagger, \rho_j^\dagger\}$, for $j = 1, 2$. Therefore, the eigenfunctions for the spectral problem (4.32)-(4.33) may be written as

$$\begin{aligned}
\psi_j(k_j) &= e^{k_j R(x,y,t,k_j)}\, P^{[N]}(x, y, t, k_j), \\
\chi_j(k_j) &= -\frac{\beta_1}{k_j}\, e^{k_j R(x,y,t,k_j)}\, Q^{[N]}(x, y, t, k_j), \\
\rho_j(k_j) &= -\frac{\beta_2}{k_j}\, e^{k_j R(x,y,t,k_j)}\, Q^{[N]}(x, y, t, k_j),
\end{aligned} \tag{4.34}$$

and

$$\psi_j^\dagger(h_j) = e^{-h_j R(x,y,t,h_j)} \, \tilde{P}^{[M]}(x,y,t,h_j),$$

$$\chi_j^\dagger(h_j) = \frac{\beta_1^\dagger}{h_j} \, e^{-h_j R(x,y,t,h_j)} \, \tilde{Q}^{[M]}(x,y,t,h_j),$$

$$\rho_j^\dagger(h_j) = \frac{\beta_2^\dagger}{h_j} \, e^{-h_j R(x,y,t,h_j)} \, \tilde{Q}^{[M]}(x,y,t,h_j),$$

(4.35)

for $j = 1, 2$, where $\{k_j, h_j\}$ are arbitrary complex parameters, $N, M \in \mathbb{N}$ and $R(x, y, t, k_j)$ is the linear polynomial

$$R(x, y, t, k_j) \equiv R(k_j) = x + \frac{\mu}{k_j^2} y - i k_j t. \tag{4.36}$$

$P^{[N]}(x, y, t, k_j)$ and $Q^{[N]}(x, y, t, k_j)$ are polynomials of degree N in x that can be expressed in the form

$$P^{[N]}(x, y, t, k_j) = \sum_{l=0}^{N} a_l(y, t, k_j) x^l, \quad Q^{[N]}(x, y, t, k_j) = \sum_{l=0}^{N} b_l(y, t, k_j) x^l, \tag{4.37}$$

defined for $N \geq 0$ and where the leading coefficients in this representation can be taken as $a_N = 1$, $b_N = 1$ without loss of generality.

Substitution of the eigenfunctions (4.34) into the spectral problem (4.32) yields recursion relations for the coefficients $a_l(k_j), b_{l'}(k_j), l, l' = 0, \ldots, N-1$ of the form

$$\frac{\partial a_l}{\partial t} = -i \, (l + 1) \left[2 k_j a_{l+1} + (l + 2) \, a_{l+2} \right],$$

$$\frac{\partial a_l}{\partial y} = \frac{\mu}{k_j} \, (b_l - a_l),$$

$$\frac{\partial b_l}{\partial t} = -i \, (l + 1) \left[2 k_j b_{l+1} + (l + 2) \, b_{l+2} \right],$$

$$\frac{\partial b_l}{\partial y} = -\frac{l + 1}{k_j} \left[\frac{\partial b_{l+1}}{\partial y} + \frac{\mu}{k_j} b_{l+1} \right],$$

(4.38)

where the following identity holds

$$a_l = \frac{l + 1}{k_j} b_{l+1} + b_l, \qquad l = 0, \ldots, N - 1, \tag{4.39}$$

and we have the freedom to set $a_N = 1$, $b_N = 1$, $a_{N+1} = 0$, $b_{N+1} = 0$ for a fixed N, and consider $P^{[N]}(0, 0, 0, k_j) = 0$ for any $N \geq 0$. For example, the first

polynomials up to order $N = 2$ are given by

$$P^{[0]}(x, y, t, k_j) = 1, \tag{4.40a}$$
$$Q^{[0]}(x, y, t, k_j) = 1,$$

$$P^{[1]}(x, y, t, k_j) = x - \left(2ik_j t + \frac{\mu y}{k_j^2} \right), \tag{4.40b}$$

$$Q^{[1]}(x, y, t, k_j) = x - \left(2ik_j t + \frac{\mu y}{k_j^2} + \frac{1}{k_j} \right),$$

$$P^{[2]}(x, y, t, k_j) = \left[x - \left(2ik_j t + \frac{\mu y}{k_j^2} \right) \right]^2 - 2 \left(it - \frac{\mu y}{k_j^3} \right), \tag{4.40c}$$

$$Q^{[2]}(x, y, t, k_j) = \left[x - \left(2ik_j t + \frac{\mu y}{k_j^2} + \frac{1}{k_j} \right) \right]^2 - 2 \left(it - \frac{\mu y}{k_j^3} - \frac{1}{2k_j^2} \right).$$

Moreover, the introduction of (4.35) in (4.33), combined with the subsequent comparison with (4.38), allows us to establish the following relations for the polynomials $\tilde{P}^{[M]}(x, y, t, h_j)$, $\tilde{Q}^{[M]}(x, y, t, h_j)$ with the former ones

$$\tilde{P}^{[M]}(x, y, t, h_j) = P^{[M]}(x, y, -t, -h_j),$$
$$\tilde{Q}^{[M]}(x, y, t, h_j) = Q^{[M]}(x, y, -t, -h_j), \tag{4.41}$$

where $P^{[M]}(x, y, t, h_j)$, $Q^{[M]}(x, y, t, h_j)$ defined in terms of the wavenumber h_j are the corresponding polynomials for eigenfunctions (4.34), satisfying (4.38).

Hence, the proposed solutions for the spectral problem, together with the recursion relations defined above, allow us to obtain an infinite number of possible eigenfunctions in terms of the integers N, M and the parameters $\{k_j, h_j\}$, $j = 1, 2$, which can be easily interpreted as wavenumbers.

The sole additional constraint to be fulfilled by the eigenfunctions (4.34)-(4.35) is that the associated singular manifolds ϕ_j, $j = 1, 2$ in (4.18), the product $\Delta_{1,2}\Delta_{2,1}$ in (4.24) and, consequently, the τ-function in (4.28), should be reduced to polynomial expressions in the independent variables (x, y, t). In order to achieve this goal, equation (4.18) suggests the following relation between the wavenumbers k_j, h_j for the eigenfunctions defined in (4.34) and (4.35)

$$k_j = h_j, \qquad j = 1, 2. \tag{4.42}$$

Besides, from (4.36) it is easy to see that $\left[R(x,t,y,k_j)\right]^\dagger = R\left(x,y,t,-k_j^\dagger\right)$, whilst equations (4.38)-(4.40) provide $\left[P^{[N]}(x,t,y,k_j)\right]^\dagger = P^{[N]}\left(x,y,-t,k_j^\dagger\right)$, $\left[Q^{[N]}(x,t,y,k_j)\right]^\dagger = Q^{[N]}\left(x,y,-t,k_j^\dagger\right)$. Regarding these identities and in view of relations (4.18), (4.24) and (4.28), the following reasonable choice arises

$$k_2 = -k_1^\dagger. \tag{4.43}$$

Gathering these previous results, the eigenfunctions for the spectral problem (4.32)-(4.33) associated to the seed solution (4.29) may be finally written as

$$\psi_1 = e^{k_1 R(k_1)}\, P^{[N]}(x,y,t,k_1), \qquad\qquad \psi_1^\dagger = e^{-k_1 R(k_1)}\, P^{[M]}(x,y,-t,-k_1),$$

$$\chi_1 = -\frac{\beta_1}{k_1}\, e^{k_1 R(k_1)}\, Q^{[N]}(x,y,t,k_1), \qquad \chi_1^\dagger = \frac{\beta_1^\dagger}{k_1}\, e^{-k_1 R(k_1)}\, Q^{[M]}(x,y,-t,-k_1),$$

$$\rho_1 = -\frac{\beta_2}{k_1}\, e^{k_1 R(k_1)}\, Q^{[N]}(x,y,t,k_1), \qquad \rho_1^\dagger = \frac{\beta_2^\dagger}{k_1}\, e^{-k_1 R(k_1)}\, Q^{[M]}(x,y,-t,-k_1),$$

$$\tag{4.44}$$

and

$$\psi_2 = e^{-k_1^\dagger R^\dagger(k_1)}\, P^{[M]}(x,y,t,-k_1^\dagger), \qquad \psi_2^\dagger = e^{k_1^\dagger R^\dagger(k_1)}\, P^{[N]}(x,y,-t,k_1^\dagger),$$

$$\chi_2 = \frac{\beta_1}{k_1^\dagger}\, e^{-k_1^\dagger R^\dagger(k_1)}\, Q^{[M]}(x,y,t,-k_1^\dagger), \qquad \chi_2^\dagger = -\frac{\beta_1^\dagger}{k_1^\dagger}\, e^{k_1^\dagger R^\dagger(k_1)}\, Q^{[N]}(x,y,-t,k_1^\dagger),$$

$$\rho_2 = \frac{\beta_2}{k_1^\dagger}\, e^{-k_1^\dagger R^\dagger(k_1)}\, Q^{[M]}(x,y,t,-k_1^\dagger), \qquad \rho_2^\dagger = -\frac{\beta_2^\dagger}{k_1^\dagger}\, e^{k_1^\dagger R^\dagger(k_1)}\, Q^{[N]}(x,y,-t,k_1^\dagger),$$

$$\tag{4.45}$$

which depends on the complex parameter k_1 and the integers $N, M \geq 0$, where $R(k_1) \equiv R(x,y,t,k_1)$ is given in (4.36) and $R^\dagger(k_1)$ stands for its complex conjugate.

It should be worth noticing that this ansatz for the eigenfunctions is exclusively written in terms of a unique wavenumber k_1. This fact indicates that, despite performing the second iteration for the fields and computing the τ-function $\tau_{1,2}$, the arising solution cannot be properly considered as a "two-lump solution", since it is not a two-soliton solution. Actually, the associated $\tau_{1,2}$ will provide a *one-soliton solution* displaying the *interaction of a nontrivial number of travelling lumps of the same amplitude* which depends on the choice of the integers N, M. The obtention of the proper two-lump solution would require the presence of two sets of eigenfunctions with two different (and non related) wavenumbers k_1, k_2. This will yield the computation of the fourth iteration for the fields, with associated

τ-function $\tau_{1,2,3,4}$. An example regarding the analysis of the two-soliton solution of lump kind for a nonlinear PDE in $2+1$ dimensions is addressed in Section 2 of the present Chapter.

The ensuing Subsections are devoted to the characterization of the one-soliton solution for the multi-component NLS system (4.1). Therefore, distinct lump solutions will arise for different values of the polynomial degrees N, M. Hereafter, the simplest three cases will be presented, corresponding to all possible combinations for the choices $N, M = 0, 1$, respectively.

1.5.1.   One-soliton solution of type $0+0$ (One lump)

This case of study corresponds to the elementary choice $N = 0$, $M = 0$. Then, according to (4.36) and (4.40a), we have

$$P^{[0]}(x,y,\pm t,\kappa) = 1, \qquad Q^{[0]}(x,y,\pm t,\kappa) = 1, \qquad R(\kappa) = x + \frac{\mu}{\kappa^2}y - i\kappa t, \quad (4.46)$$

where κ can take any of the following values $\kappa = \{\pm k_1, \pm k_1^\dagger\}$. The eigenfunctions in (4.44)-(4.45) possess the expressions

$$\psi_j = e^{k_j R_j}, \qquad \chi_j = -\frac{\beta_1}{k_j} e^{k_j R_j}, \qquad \rho_j = -\frac{\beta_2}{k_j} e^{k_j R_j},$$

$$\psi_j^\dagger = e^{-k_j R_j}, \qquad \chi_j^\dagger = \frac{\beta_1^\dagger}{k_j} e^{-k_j R_j}, \qquad \rho_j^\dagger = \frac{\beta_2^\dagger}{k_j} e^{-k_j R_j}, \quad (4.47)$$

where $j = 1, 2$, $R_j = R(k_j)$ and $k_2 = -k_1^\dagger$, such that $R_2 = R\left(-k_1^\dagger\right) = R_1^\dagger$.

The integration of equations (4.18) provides the following polynomial expressions for the associated singular manifolds

$$\phi_1 = x - 2ik_1 t - \frac{\mu y}{k_1^2}, \qquad \phi_2 = x + 2ik_1^\dagger t - \frac{\mu y}{\left(k_1^\dagger\right)^2}, \quad (4.48)$$

where $\mu = \beta_1^\dagger \beta_1 + \beta_2^\dagger \beta_2$. We may thereupon compute the remaining terms of the Δ-matrix following (4.24), whose integration provides

$$\Delta_{1,2} = -\frac{e^{-k_1 R_1} e^{-k_1^\dagger R_1^\dagger}}{k_1 + k_1^\dagger}, \qquad \Delta_{2,1} = \frac{e^{k_1 R_1} e^{k_1^\dagger R_1^\dagger}}{k_1 + k_1^\dagger}. \quad (4.49)$$

If we set $k_1 = a_1 + ib_1$, $k_1^\dagger = a_1 - ib_1$, with $a_1, b_1 \in \mathbb{R}$, expressions (4.48) and (4.49)

reduce to

$$\phi_1 = \left[x - 2b_1 t - \frac{(a_1^2 - b_1^2)\mu y}{\left(a_1^2 + b_1^2\right)^2} \right] + i \left[\frac{2a_1 b_1 \mu y}{\left(a_1^2 + b_1^2\right)^2} - 2a_1 t \right], \qquad \phi_2 = \phi_1^\dagger,$$

$$\Delta_{1,2} = -\frac{e^{-k_1 R_1} e^{-k_1^\dagger R_1^\dagger}}{2a_1}, \qquad \Delta_{2,1} = \frac{e^{k_1 R_1} e^{k_1^\dagger R_1^\dagger}}{2a_1},$$

(4.50)

where ϕ_2 may be computed as the complex conjugate of ϕ_1. Then, the corresponding τ-function defined in (4.28) is therefore given by

$$\tau_{1,2} = \left[x - 2b_1 t - \frac{(a_1^2 - b_1^2)\mu y}{\left(a_1^2 + b_1^2\right)^2} \right]^2 + \left[\frac{2a_1 b_1 \mu y}{\left(a_1^2 + b_1^2\right)^2} - 2a_1 t \right]^2 + \frac{1}{4a_1^2},$$

(4.51)

which is a positive-definite polynomial expression of second degree in the independent variables (x, y, t) with no singularities.

The second iteration for each component of the vector $\boldsymbol{\alpha}^{[2]}$ can be easily computed via expressions (4.27). The lump profile may emerge by analyzing the probability density of such second iteration, encoded in the scalar field $m_y^{[2]} = -\left(\boldsymbol{\alpha}^{[2]}\right)^\dagger \cdot \left(\boldsymbol{\alpha}^{[2]}\right)$ by means of the third equation in the initial NLS system (4.1). According to (4.27), the field $m_y^{[2]}$ takes the form

$$m_y^{[2]} = m_y^{[0]} + \left[\frac{(\tau_{1,2})_x}{\tau_{1,2}} \right]_y = -\mu + \left[\frac{(\tau_{1,2})_x}{\tau_{1,2}} \right]_y,$$

(4.52)

where $\tau_{1,2}$ is given in (4.51). Figure 4.1 displays the lump soliton solution $m_y^{[2]}$ defined in (4.52), which represents a localized travelling rational structure propagating along the direction $x - \frac{(a_1^2 - 3b_1^2)\mu y}{\left(a_1^2 + b_1^2\right)^2} = 0$, as it will be shown later.

Asymptotic behaviour

It is possible to perform the following Galilean transformations

$$x = X + v_x t, \qquad\qquad v_x = \frac{a_1^2 - 3b_1^2}{b_1},$$

$$y = \frac{\left(a_1^2 + b_1^2\right)^2}{\mu} \left(Y + v_y t\right), \qquad v_y = \frac{1}{b_1},$$

(4.53)

such that the τ-function in (4.51) is expressed as

$$\tau_{1,2} = \left[X - \left(a_1^2 - b_1^2\right)Y\right]^2 + \left[2a_1b_1Y\right]^2 + \frac{1}{4a_1^2}. \tag{4.54}$$

This τ-function retrieves a lump configuration of the form

$$\left[\frac{(\tau_{1,2})_X}{\tau_{1,2}}\right]_Y \sim \frac{\left[X - \frac{\left(a_1^2+b_1^2\right)^2 Y}{a_1^2-b_1^2}\right]^2 - \left[\frac{2a_1b_1\left(a_1^2+b_1^2\right)Y}{a_1^2-b_1^2}\right]^2 - \frac{1}{4a_1}}{\left[\left(X - \left(a_1^2 - b_1^2\right)Y\right)^2 + \left(2a_1b_1Y\right)^2 + \frac{1}{4a_1^2}\right]^2}, \tag{4.55}$$

which corresponds to a static lump, with similar profile to the one displayed in Figure 4.1, centered at the origin of the XY-plane, with constant amplitude $\mathcal{A}_{00} = -\frac{\left(3a_1^2-b_1^2\right)^2 \mu}{\left(a_1^2+b_1^2\right)^2}$.
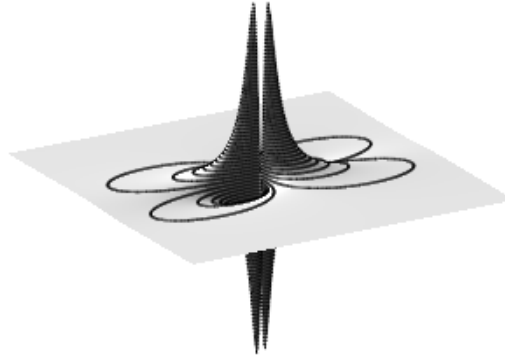
Figure 4.1: *One-soliton solution of type* $0{+}0$*,* $m_y^{[2]}$*, with* $\beta_1 = \beta_2 = 1$*,* $a_1 = \frac{1}{2}$*,* $b_1 = \frac{1}{2}$*.*

The direction of propagation of the travelling lump associated to the τ-function (4.51) can be easily obtained by balance of the dominant terms depending on t. According to (4.53), the spatial coordinates in the asymptotic regime behave as $x \sim \frac{a_1^2-3b_1^2}{b_1}t$, $y \sim \frac{\left(a_1^2+b_1^2\right)^2}{\mu b_1}t$, which retrieves the relation $x - \frac{\left(a_1^2-3b_1^2\right)\mu y}{\left(a_1^2+b_1^2\right)^2} = 0$ in the xy-plane as the direction of propagation of the corresponding solution.

1.5.2. One-soliton solution of type $0 + 1$ (Two lumps)

A new class of lump solutions can be derived by setting $N = 0$ and $M = 1$. In accordance with (4.38) and (4.40), the associated polynomials up to first degree are

$$P^{[0]} = 1, \qquad\qquad\qquad Q^{[0]} = 1, \qquad\qquad (4.56\text{a})$$

$$P^{[1]} = x - \left(2ik_1 t + \frac{\mu y}{k_1^2}\right), \qquad Q^{[1]} = P^{[1]} - \frac{1}{k_1}, \qquad (4.56\text{b})$$

where we have omitted the explicit dependence of the polynomials in the coordinates and the wavenumber k_1 for simplicity. Then, the associated eigenfunctions in (4.44)-(4.45) read

$$\psi_1 = e^{k_1 R_1}, \qquad\qquad \chi_1 = -\frac{\beta_1}{k_1} e^{k_1 R_1}, \qquad\qquad \rho_1 = -\frac{\beta_2}{k_1} e^{k_1 R_1},$$

$$\psi_1^\dagger = P^{[1]} e^{-k_1 R_1}, \qquad \chi_1^\dagger = \frac{\beta_1^\dagger}{k_1}\left(P^{[1]} + \frac{1}{k_1}\right) e^{-k_1 R_1}, \quad \rho_1^\dagger = \frac{\beta_2^\dagger}{k_1}\left(P^{[1]} + \frac{1}{k_1}\right) e^{-k_1 R_1},$$

$$\psi_2 = \left(P^{[1]}\right)^\dagger e^{-k_1^\dagger R_1^\dagger}, \quad \chi_2 = \frac{\beta_1}{k_1^\dagger}\left(P^{[1]} + \frac{1}{k_1}\right)^\dagger e^{-k_1^\dagger R_1^\dagger}, \quad \rho_2 = \frac{\beta_2}{k_1^\dagger}\left(P^{[1]} + \frac{1}{k_1}\right)^\dagger e^{-k_1^\dagger R_1^\dagger},$$

$$\psi_2^\dagger = e^{k_1^\dagger R_1^\dagger}, \qquad\qquad \chi_2^\dagger = -\frac{\beta_1^\dagger}{k_1^\dagger} e^{k_1^\dagger R_1^\dagger}, \qquad\qquad \rho_2^\dagger = -\frac{\beta_2^\dagger}{k_1^\dagger} e^{k_1^\dagger R_1^\dagger},$$

$$(4.57)$$

where $R_1 \equiv R(x,y,t,k_1)$ is given in (4.36) as $R_1 = x + \frac{\mu y}{k_1^2} - ik_1 t$.

The different elements of the Δ-matrix have been defined in differential form in (4.24), whose integration yields

$$\phi_1 = \frac{1}{2}\left[\left(X - \left(a_1^2 - b_1^2\right)Y\right)^2 - 4a_1^2 b_1^2 Y^2 - \frac{2a_1\left(a_1^2 - 3b_1^2\right)}{a_1^2 + b_1^2}\left(Y + \frac{t}{b_1}\right)\right]$$

$$\quad + i\left[2a_1 b_1\left(X - \left(a_1^2 - b_1^2\right)Y\right)Y + t + \frac{\left(3a_1^2 - b_1^2\right)\left(b_1 Y + t\right)}{a_1^2 + b_1^2}\right],$$

$$\phi_2 = \phi_1^\dagger, \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad (4.58)$$

$$\Delta_{1,2} = -\frac{1}{2a_1}\left[\left(X - \left(a_1^2 - b_1^2\right)Y + \frac{1}{2a_1}\right)^2 + 4a_1^2 b_1^2 Y^2 + \frac{1}{4a_1^2}\right] e^{-k_1 R_1} e^{-k_1^\dagger R_1^\dagger},$$

$$\Delta_{2,1} = \frac{e^{k_1 R_1} e^{k_1^\dagger R_1^\dagger}}{2a_1},$$

where we have selected $k_1 = a_1 + ib_1$ and the expressions have been written in terms of the coordinates (X, Y) introduced in (4.53). The τ-function now is given by

$$\tau_{1,2} = \frac{1}{4}\left[\left(X - \left(a_1^2 - b_1^2\right)Y\right)^2 - 4a_1^2 b_1^2 Y^2 - \frac{2a_1\left(a_1^2 - 3b_1^2\right)}{a_1^2 + b_1^2}\left(Y + \frac{t}{b_1}\right)\right]^2$$

$$+ \left[2a_1 b_1 \left(X - \left(a_1^2 - b_1^2 \right) Y \right) Y + t + \frac{\left(3a_1^2 - b_1^2 \right) \left(b_1 Y + t \right)}{a_1^2 + b_1^2} \right]^2 \tag{4.59}$$

$$+ \frac{1}{4a_1^2} \left[\left(X - \left(a_1^2 - b_1^2 \right) Y + \frac{1}{2a_1} \right)^2 + 4a_1^2 b_1^2 Y^2 + \frac{1}{4a_1^2} \right],$$

obtained by direct substitution of (4.58) in (4.28). It is worth stressing that $\tau_{1,2}$ is a positive-definite polynomial expression of fourth degree in (X, Y) and second degree in t with no zeros for any value of the parameters. The lump soliton solutions therefore arise from equation (4.52) by the computation of $m_y^{[2]} = -\mu + \left[\frac{(\tau_{1,2})_x}{\tau_{1,2}} \right]_y$, where now $\tau_{1,2}$ is given by (4.59).

Asymptotic behaviour

With the aim of better characterizing the resulting lump solution, we shall analyze the asymptotic behaviour of such solution. In order to achieve this, we propose the following transformation

$$X = \hat{X} + (ct)^{\frac{1}{2}}, \qquad Y = \hat{Y} + z\,(ct)^{\frac{1}{2}}, \tag{4.60}$$

such that $\tau_{1,2}$ in (4.59) is a polynomial expression of second degree in all the independent variables $(\hat{X}, \hat{Y}, t)$ and where c, z are arbitrary parameters to be determined.

From expression (4.59), it is immediate to note that $\tau_{1,2} \sim \mathcal{O}(t^2)$, whilst $(\tau_{1,2})_X \sim \mathcal{O}(t)$ in the limit $t \to \pm \infty$. This trivially retrieves $\lim\limits_{t \to \pm \infty} \left[\frac{(\tau_{1,2})_X}{\tau_{1,2}} \right]_Y = 0$, unless the highest powers in t of (4.59) vanish, and then this limit may become nonzero. The imposition of such condition allows us to obtain the following relations for the coefficients c and z,

$$c = \frac{2a_1}{b_1 \left(a_1^2 + b_1^2 \right) \left[\left(a_1^2 - b_1^2 \right) z - 1 \right] z}, \qquad z^2 + \frac{z}{2b_1} - \frac{1}{2b_1^2 \left(a_1^2 + b_1^2 \right)} = 0,$$

whose solutions are

$$c = -\frac{1}{4a_1 b_1^3} \left[a_1^4 + 6a_1^2 b_1^2 - 3b_1^4 + s \left(a_1^2 + b_1^2 \right)^{\frac{3}{2}} \left(a_1^2 + 9b_1^2 \right)^{\frac{1}{2}} \right], \tag{4.61}$$

$$z = \frac{1}{4b_1^2 \left(a_1^2 + b_1^2 \right)} \left[- \left(a_1^2 + b_1^2 \right) + s \left(a_1^4 + 10a_1^2 b_1^2 + 9b_1^4 \right)^{\frac{1}{2}} \right], \tag{4.62}$$

where $s = \pm 1$. This asymptotic analysis retrieves two possible solutions for the set

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of parameters $\{c, z\}$, which indicates the existence of two privileged directions for transformation (4.60) in the asymptotic limit.

In this case, the dominant terms in $\tau_{1,2}$ behave as

$$
\tau_{1,2} \sim \left[\left(1 - z\left(a_1^2 - b_1^2\right) \right) \hat{X} + \left(\left(a_1^4 - 6a_1^2 b_1^2 + b_1^4\right) z - a_1^2 + b_1^2 \right) \hat{Y} - \frac{a_1 \left(a_1^2 - 3b_1^2\right) z}{a_1^2 + b_1^2} \right]^2
$$

$$
+ \left[2a_1 b_1 z \hat{X} + 2a_1 b_1 \left(1 - 2z\left(a_1^2 - b_1^2\right) \right) \hat{Y} + \frac{b_1 \left(3a_1^2 - b_1^2\right) z}{a_1^2 + b_1^2} \right]^2
$$

$$
+ \frac{1}{4a_1^2} \left[\left(a_1^2 + b_1^2\right)^2 z^2 - 2\left(a_1^2 - b_1^2\right) z + 1 \right],
$$

$$(4.63)$$

which corresponds to a single static lump, presenting a similar profile of solution (4.51). The derivation of this solution, in combination with results (4.61), allows us to conclude that the one-soliton solution obtained in (4.59) will display the interaction of two nonlinear waves of lump-type with the same amplitude travelling along the lines $Y = \hat{Y} + z\left(X - \hat{X}\right)$, where the values of z are about to be specified thereupon. The illustration of such behaviour may be found in Figure 4.2.

The explicit asymptotic behaviour of the solution under consideration can be characterized as follows. The corresponding analysis depends on the sign of the constant c, which is given by the sign of the product $a_1 b_1$ and the value of s. Assuming $a_1 b_1 > 0$[1], the asymptotic analysis reads:

- At $t \to -\infty$, transformation (4.60) is well defined for $c < 0$, which is equivalent to consider $s = 1$ in (4.61). The corresponding value of z results in

$$
z_- = \frac{1}{4b_1^2 \left(a_1^2 + b_1^2\right)} \left[-\left(a_1^2 + b_1^2\right) + \left(a_1^4 + 10a_1^2 b_1^2 + 9b_1^4\right)^{\frac{1}{2}} \right] > 0. \tag{4.64}
$$

Therefore, there exists two lump-like waves that approach along the direction defined by $Y = \hat{Y} + z_-\left(X - \hat{X}\right)$.

- By an analogous analysis, at $t \to \infty$, $c > 0$ which implies $s = -1$, yielding

$$
z_+ = -\frac{1}{4b_1^2 \left(a_1^2 + b_1^2\right)} \left[\left(a_1^2 + b_1^2\right) + \left(a_1^4 + 10a_1^2 b_1^2 + 9b_1^4\right)^{\frac{1}{2}} \right] < 0, \tag{4.65}
$$

---

[1]In case $a_1 b_1 < 0$, the criteria regarding the choice of s should be the opposite, and consequently, the asymptotic directions are switched and the motion of the lump solution is reversed.

such that $Y = \hat{Y} + z_+ \left(X - \hat{X} \right)$ defines the asymptotic direction followed by the two lumps when moving away.

The *scattering angle* between the lumps, understood as the angle formed by the two asymptotic directions, is given by

$$\tan \theta_{\mathrm{S}} = \frac{\left(a_1^4 + 10 a_1^2 b_1^2 + 9 b_1^4 \right)^{\frac{1}{2}}}{1 - 2 b_1^2 \left(a_1^2 + b_1^2 \right)}. \tag{4.66}$$

Figure 4.2 displays the one-soliton solution obtained from $m_y^{[2]}$ with τ-function (4.59) at different times. It is worth stressing that, despite the fact that this solution has been computed as a one-solution solution, it presents two components of lump-type with *equal* conformation.

<center>(a)   (b)   (c)</center>
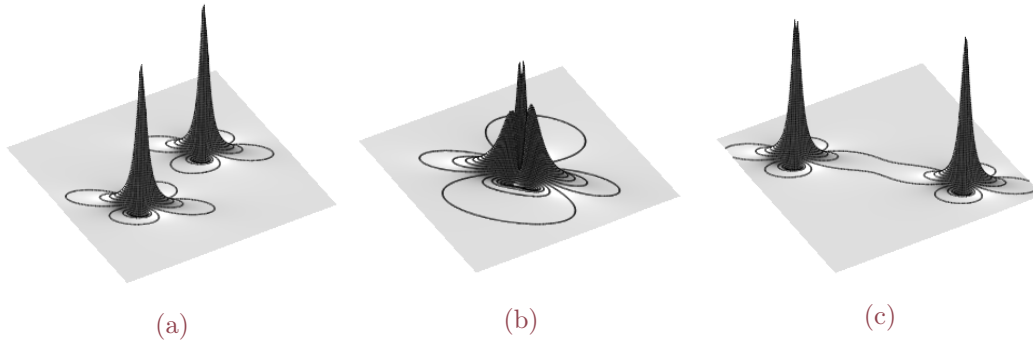
Figure 4.2: *One-soliton solution of type $0 + 1$, $m_y^{[2]}$, with parameters $\beta_1 = 1, \beta_2 = 1$, $a_1 = 1$, $b_1 = 2$, at different times:* (a) $t < 0$, (b) $t = 0$ *and* (c) $t > 0$.

The pictures reflect the interaction of those *lump components*, of asymptotic profile (4.63), such that they gradually approach to each other (Fig. 4.2a), converge at the origin in $t = 0$ (Fig. 4.2b) and then drift apart (Fig. 4.2c). The sole effect of the interaction over the travelling lumps lies in a shift on their propagation directions given by (4.66).

1.5.3. One-soliton solution of type $1 + 1$ (Three lumps)

The last class of lump solutions to be analyzed for this example corresponds to the choice $N = 1$ and $M = 1$. The associated polynomials $P^{[1]}(x, y, t, k_1)$, $Q^{[1]}(x, y, t, k_1)$ for this case are given in (4.56b), which clearly satisfy (4.38). Expressions (4.44)-(4.45) therefore give rise to the following eigenfunctions

$$\psi_1 = P^{[1]} e^{k_1 R_1}, \qquad \chi_1 = -\frac{\beta_1}{k_1}\left(P^{[1]} - \frac{1}{k_1}\right) e^{k_1 R_1}, \quad \rho_1 = -\frac{\beta_2}{k_1}\left(P^{[1]} - \frac{1}{k_1}\right) e^{k_1 R_1},$$

$$\psi_1^\dagger = P^{[1]} e^{-k_1 R_1}, \qquad \chi_1^\dagger = \frac{\beta_1^\dagger}{k_1}\left(P^{[1]} + \frac{1}{k_1}\right) e^{-k_1 R_1}, \quad \rho_1^\dagger = \frac{\beta_2^\dagger}{k_1}\left(P^{[1]} + \frac{1}{k_1}\right) e^{-k_1 R_1},$$

$$\psi_2 = \left(P^{[1]}\right)^\dagger e^{-k_1^\dagger R_1^\dagger}, \quad \chi_2 = \frac{\beta_1}{k_1^\dagger}\left(P^{[1]} + \frac{1}{k_1}\right)^\dagger e^{-k_1^\dagger R_1^\dagger}, \quad \rho_2 = \frac{\beta_2}{k_1^\dagger}\left(P^{[1]} + \frac{1}{k_1}\right)^\dagger e^{-k_1^\dagger R_1^\dagger},$$

$$\psi_2^\dagger = \left(P^{[1]}\right)^\dagger e^{k_1^\dagger R_1^\dagger}, \quad \chi_2^\dagger = -\frac{\beta_1^\dagger}{k_1^\dagger}\left(P^{[1]} - \frac{1}{k_1}\right)^\dagger e^{k_1^\dagger R_1^\dagger}, \quad \rho_2^\dagger = -\frac{\beta_2^\dagger}{k_1^\dagger}\left(P^{[1]} - \frac{1}{k_1}\right)^\dagger e^{k_1^\dagger R_1^\dagger},$$

$$\text{(4.67)}$$

where $R_1 = x + \dfrac{\mu y}{k_1^2} - ik_1 t$, $R_1^\dagger = x + \dfrac{\mu y}{\left(k_1^\dagger\right)^2} + ik_1^\dagger t$, from (4.36).

Following (4.24), the associated singular manifolds ϕ_1, ϕ_2 and the elements $\Delta_{1,2}$, $\Delta_{2,1}$ are given by

$$\phi_1 = \frac{\left[X - \left(a_1^2 - b_1^2\right) Y\right]^3}{3} - 4a_1^2 b_1^2 \left[X - \left(a_1^2 - b_1^2\right) Y\right] Y^2 - \frac{a_1^4 - 6a_1^2 b_1^2 + b_1^4}{\left(a_1^2 + b_1^2\right)^2}\left(Y + \frac{t}{b_1}\right)$$

$$+ i\left[-\frac{8a_1^3 b_1^3 Y^3}{3} + 2a_1 b_1\left(X - \left(a_1^2 - b_1^2\right) Y\right)^2 Y - \frac{4a_1\left(a_1^2 - b_1^2\right)\left(b_1 Y + t\right)}{\left(a_1^2 + b_1^2\right)^2}\right],$$

$$\phi_2 = \phi_1^\dagger,$$

$$\Delta_{1,2} = -\frac{1}{2a_1}\left[\left(X - \left(a_1^2 - b_1^2\right) Y + \frac{1}{2a_1}\right)^2 + 4a_1^2 b_1^2 Y^2 + \frac{1}{4a_1^2}\right] e^{-k_1 R_1} e^{-k_1^\dagger R_1^\dagger},$$

$$\Delta_{2,1} = \frac{1}{2a_1}\left[\left(X - \left(a_1^2 - b_1^2\right) Y - \frac{1}{2a_1}\right)^2 + 4a_1^2 b_1^2 Y^2 + \frac{1}{4a_1^2}\right] e^{k_1 R_1} e^{k_1^\dagger R_1^\dagger},$$

$$\text{(4.68)}$$

where $k_1 = a_1 + ib_1$ and the coordinates (X, Y) can be found in (4.53).

Hence, the τ-function (4.28) reads

$$\tau_{1,2} = \left[-\frac{8a_1^3 b_1^3 Y^3}{3} + 2a_1 b_1\left(X - \left(a_1^2 - b_1^2\right) Y\right)^2 Y - \frac{4a_1\left(a_1^2 - b_1^2\right)\left(b_1 Y + t\right)}{\left(a_1^2 + b_1^2\right)^2}\right]^2$$

$$+ \left[\frac{\left(X - \left(a_1^2 - b_1^2\right) Y\right)^3}{3} - 4a_1^2 b_1^2\left(X - \left(a_1^2 - b_1^2\right) Y\right) Y^2 - \frac{a_1^4 - 6a_1^2 b_1^2 + b_1^4}{\left(a_1^2 + b_1^2\right)^2}\left(Y + \frac{t}{b_1}\right)\right]^2$$

$$+ \frac{1}{4a_1^2} \left[\left\{ \left(X - \left(a_1^2 - b_1^2 \right) Y \right)^2 + 4a_1^2 b_1^2 Y^2 \right\}^2 + 4b_1^2 Y^2 + \frac{1}{4a_1^4} \right], \tag{4.69}$$

which is a positive-definite polynomial of sixth degree in (X,Y) and third degree in t. Moreover, it possesses no zeros.

Asymptotic behaviour

The asymptotic analysis should follow in complete analogy to the previous one. In order to study the behaviour at $t \to \pm\infty$ of the solution arising from (4.69), we shall introduce a transformation of the form

$$X = \hat{X} + (ct)^{\frac{1}{3}}, \qquad Y = \hat{Y} + z\,(ct)^{\frac{1}{3}}, \tag{4.70}$$

where c, z are arbitrary parameters to be determined such that the highest powers in t of (4.69) identically hold. This condition immediately retrieves the following equations for c and z,

$$c = \frac{6\left(a_1^2 - b_1^2\right)}{b_1 \left(a_1^2 + b_1^2\right) \left[\left(a_1^2 - 3b_1^2\right)\left(3a_1^2 - b_1^2\right) z^2 - 6\left(a_1^2 - b_1^2\right) z + 3\right] z}, \tag{4.71a}$$

$$z^3 - \frac{3z}{\left(a_1^2 + b_1^2\right)^2} + \frac{2\left(a_1^2 - b_1^2\right)}{\left(a_1^2 + b_1^2\right)^4} = 0. \tag{4.71b}$$

Expression (4.71b) defines a depressed cubic equation for z, which is proved to have three real roots of the form [376]

$$z_l = \frac{2}{a_1^2 + b_1^2} \cos\left(\frac{\theta}{3} + \frac{2\pi l}{3}\right), \quad \text{with} \quad \theta = \arccos\left(-\frac{a_1^2 - b_1^2}{a_1^2 + b_1^2}\right), \quad l = 0, 1, 2. \tag{4.72}$$

The substitution of any of these three values for z provides a unique value of c by means of (4.71a). This analysis results in three solutions for the set of parameters $\{c, z\}$, which implies the existence of three asymptotic directions given by $Y = \hat{Y} + z_l\left(X - \hat{X}\right)$, $l = 0, 1, 2$.

Then, the asymptotic behaviour of $\tau_{1,2}$ in (4.69) is

$$\tau_{1,2} \sim \left[\left(\left(a_1^4 - 6a_1^2 b_1^2 + b_1^4\right) z^2 - 2\left(a_1^2 - b_1^2\right) z + 1 \right)\hat{X} \right.$$
$$\left. - \left(\left(a_1^2 - b_1^2\right)\left(a_1^4 - 14a_1^2 b_1^2 + b_1^4\right) z^2 - 2\left(a_1^4 - 6a_1^2 b_1^2 + b_1^4\right) z + a_1^2 - b_1^2 \right)\hat{Y} \right]^2$$
$$+ 4a_1^2 b_1^2 \left[2\left(1 - \left(a_1^2 - b_1^2\right) z \right) z\hat{X} + \left(\left(a_1^2 - 3b_1^2\right)\left(3a_1^2 - b_1^2\right) z^2 - 4\left(a_1^2 - b_1^2\right) z + 1 \right)\hat{Y} \right]^2$$

$$+ \frac{1}{4a_1^2} \left[\left(9a_1^4 + 2a_1^2 b_1^2 + 9b_1^2\right) z^2 - 18 \left(a_1^2 - b_1^2\right) z + \frac{9a_1^4 - 14a_1^2 b_1^2 + 9b_1^2}{\left(a_1^2 + b_1^2\right)^2} \right], \qquad (4.73)$$

which represents a static lump. Thus, the τ-function (4.69) gives rise to three equal components of lump-type, which travel along the lines $Y = \hat{Y} + z_l \left(X - \hat{X}\right)$, with z_l, $l = 0, 1, 2$ given in (4.72), and possess the ability to interact among them. The behaviour of this solution may be appreciated in Figure 4.3, where the lump waves come closer to each other (Fig. 4.3a), coalesce at the origin in $t = 0$ (Fig. 4.3b) and then they move away (Fig 4.3c). It is also worth mentioning that in this case, the lump solitons display no scattering upon interaction.

(a)   (b)   (c)
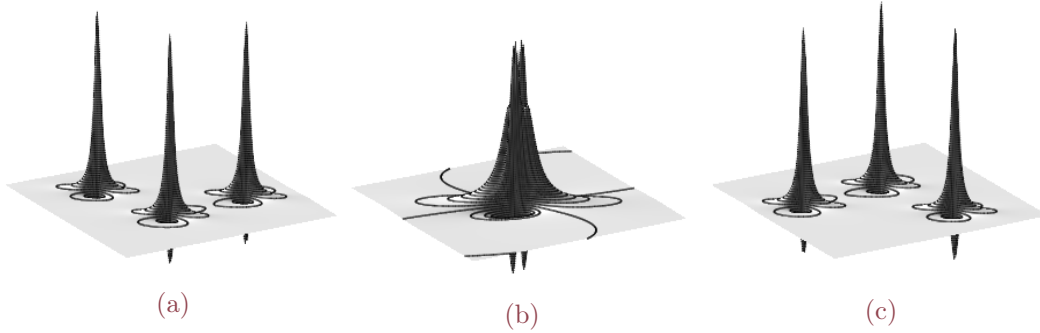
Figure 4.3: *One-soliton solution of type* $1 + 1$, $m_y^{[2]}$, *with parameters* $\beta_1 = 1, \beta_2 = 1$, $a_1 = 1$, $b_1 = 2$, *at different times:* (a) $t < 0$, (b) $t = 0$ *and* (c) $t > 0$.

2. Generalized Nizhnik-Novikov-Veselov equation

Korteweg-de Vries (KdV) equation [249] is one of the most famous $(1+1)$-integrable models in the realm of soliton theory and Mathematical Physics, widely studied in literature [3, 130, 183, 263, 303, 304, 373, 426]. Among the several generalizations of KdV, we are interested in its integrable extensions to higher spatial dimensions, particularly to $2 + 1$ dimensions. One of the first integrable generalizations of KdV is the ubiquitous Kadomtsev–Petviashvili (KP) equation [231], which is known to share several of the truly remarkable properties with its one-dimensional counterpart [3, 131, 168, 213, 230, 283, 287, 301, 307, 370].

Another renowed integrable generalization of KdV to $2 + 1$ dimensions is the so-called Nizhnik-Novikov-Veselov (NNV) equation, first introduced by the authors in [317, 319, 391, 392]. Boiti *et al.* solved this equation by the IST method and obtained a weak Lax pair in the subspace of coordinate space [53, 54]. Furthermore, a plethora

of soliton-like solutions [360, 384, 407] and other kinds of localized structures [255, 280, 336, 347, 436] have been found for this system via different procedures. Besides, the boundary value problem for NNV equation has been studied in [360], Moutard transformations [44] and auto-Bäcklund transformations [403] have been used to find new exacts solutions, and a superposition rule has been given in [219].

This Section addresses the analysis of a generalized Nizhnik-Novikov-Veselov equation, understood as a symmetric generalization of the KdV equation to $2 + 1$ dimensions, given by

$$
\begin{aligned}
&u_t + au_{xxx} + bu_{yyy} + cu_x + du_y - 3a(uv)_x - 3b(uw)_y = 0, \\
&u_x = v_y, \\
&u_y = w_x,
\end{aligned}
\tag{4.74}
$$

where a, b, c and d are arbitrary parameters. This equation, known to be completely integrable, has been investigated in [255, 347], where exponentially localized solutions have been generated and their dynamics has been analyzed.

By introducing the change of variables,

$$
u = 2m_{xy}, \qquad v = \frac{c}{3a} - 2m_{xx}, \qquad w = \frac{d}{3b} - 2m_{yy},
\tag{4.75}
$$

the system of PDEs given in (4.74) is transformed into the following nonlinear differential equation

$$
m_{xyt} + a \left(m_{xxxy} + 6m_{xx}m_{xy} \right)_x + b \left(m_{yyyx} + 6m_{yy}m_{xy} \right)_y = 0,
\tag{4.76}
$$

where $m = m(x, y, t)$ is a scalar field.

2.1. Painlevé test and integrability

An integrability analysis for (4.76) can be easily performed by means of the Painlevé test [417]. The field m is then required to possess a generalized Laurent expansion of the form

$$
m(x, y, t) = \sum_{j=0}^{\infty} a_j \phi^{j-\alpha},
\tag{4.77}
$$

in a neighbourhood of the singularity manifold $\phi(x, y, t) = 0$. $a_j(x, y, t)$, $\forall j$ are the coefficients of the expansion, about to be determined, and α is the leading index of the expansion. A preliminary inspection of the dominant terms at leading-order immediately provides $\alpha = 0$. According to the procedure illustrated in Subsection 2.1 of the previous Chapter, expansion (4.77) should be slightly modified as

$$m = \tilde{m}_0 \log(\phi) + \sum_{j=0}^{\infty} a_j \phi^j, \tag{4.78}$$

in order to incorporate an additional logarithmic term in the series such that the WTC method is prospectively applicable to equation (4.76).

1. A leading-order analysis results in

$$\tilde{m}_0(x, y, t) = 1, \tag{4.79}$$

   which is uniquely determined, and therefore, (4.78) has one branch of expansion.

2. The associated resonance condition is given by

$$j(j+1)(j-1)(j-2)(j-4)(j-6) = 0, \tag{4.80}$$

   which yields five resonances, among which the resonance in $j = -1$ accounts for the arbitrariness of the function ϕ.

3. The resonance conditions for $j = 0, 1, 4, 6$ identically hold, successfully retrieving the coefficients a_0, a_1, a_4, a_6 as arbitrary.

Thus, we shall conclude that equation (4.76), and consequently the initial system (4.74), passes the Painlevé test and then, it is precise to consider both of them as integrable systems.

2.2. Singular manifold method and spectral problem

According to the SMM [410], the truncated expansion arising from Painlevé analysis for the field m should be

$$m^{[1]} = m^{[0]} + \log(\phi), \tag{4.81}$$

which can be regarded as an auto-Bäcklund transformation between two solutions for the initial equation (4.76), the seed solution $m^{[0]}$ and the iterated one $m^{[1]}$.

Substitution of relation (4.81) into the starting problem (4.76) yields an expression in negatives powers of ϕ. The procedure to adopt in this Subsection has been subtly modified in order to properly grapple with the arising singular manifold equations.

Since (4.76) is symmetric under the interchange of $(x, a) \leftrightarrow (y, b)$, it is therefore reasonable to suggest the ansatz

$$\phi_t = aG_a(x, y, t) + bG_b(x, y, t), \tag{4.82}$$

such that the terms in a and b cancel independently. If we substitute now relation (4.82) into the singular manifold equations, we obtain two polynomial expressions in powers of ϕ, with separated contributions of terms in a and b, respectively. If we require all the coefficients of these polynomials to be zero, after some algebraic manipulations handled with MAPLE, the following results arise

$$G_a = -\phi_{xxx} - 6\phi_x m_{xx}^{[0]}, \qquad G_b = -\phi_{yyy} - 6\phi_y m_{yy}^{[0]}, \tag{4.83}$$

where the remaining contributions to the singular manifold equations can be independently integrated as

$$\frac{\phi_{xy} + 2\phi m_{xy}^{[0]}}{\phi_x} + K_2(y) + K_1(y) \int \left(\frac{\phi}{\phi_x}\right)^2 dx = 0, \tag{4.84}$$

$$\frac{\phi_{xy} + 2\phi m_{xy}^{[0]}}{\phi_y} + H_2(x) + H_1(x) \int \left(\frac{\phi}{\phi_y}\right)^2 dy = 0, \tag{4.85}$$

being $H_j(x)$, $K_j(y)$ for $j = 1, 2$ arbitrary functions.

The compatibility of equations (4.84) and (4.85) yields

$$H_1(x) = 0, \qquad H_2(x) = 0, \qquad K_1(y) = 0, \qquad K_2(y) = 0, \tag{4.86}$$

and therefore

$$\phi_{xy} + 2\phi m_{xy}^{[0]} = 0. \tag{4.87}$$

This latter equation, when combined with expressions (4.82) and (4.83), finally provides

$$\phi_t + a\left(\phi_{xxx} + 6\phi_x m_{xx}^{[0]}\right) + b\left(\phi_{yyy} + 6\phi_y m_{yy}^{[0]}\right) = 0. \tag{4.88}$$

It is immediate to verify that equations (4.87) and (4.88) constitute a Lax pair for the generalized NNV equation defined in (4.76). The above spectral problem is in sharp contrast to the notion of weak Lax pair postulated by Boiti *et al.* for this equation [53, 54].

As it has been illustrated, the associated linear problem in this case straightfor-

wardly arises from the direct integration of the singular manifold equations. The introduction of additional variables as an intermediate step to linearize these equations has not been needed, and hence, the singular manifold ϕ will play the role of the eigenfunction in this spectral problem. It is also worth mentioning that this Lax pair does not explicitly possess a spectral parameter. This novel scenario will require to consequentially adapt the Darboux transformation approach[2] in order to derive solutions for (4.76), as it is described in the following paragraphs.

2.3. Darboux transformations

Let us consider two singular manifolds ϕ_i, $i = 1, 2$ that behave in turn as eigenfunctions for the spectral problem (4.87)-(4.88) with seed solution $m^{[0]}$, *i.e.*

$$
\begin{aligned}
&(\phi_i)_{xy} + 2\phi_i m^{[0]}_{xy} = 0, \\
&(\phi_i)_t + a\left[(\phi_i)_{xxx} + 6(\phi_i)_x m^{[0]}_{xx}\right] + b\left[(\phi_i)_{yyy} + 6(\phi_i)_y m^{[0]}_{yy}\right] = 0,
\end{aligned}
\tag{4.89}
$$

for $i = 1, 2$.

First iteration

The truncated expansion given by equation (4.81) can be regarded as an iterative method to construct new solutions for the initial equation (4.76), such that an iterated solution $m^{[1]}$ can be obtained from the seed solution $m^{[0]}$, as

$$
m^{[1]} = m^{[0]} + \log(\phi_1).
\tag{4.90}
$$

We should now denote $\phi_{1,2}$ as the corresponding eigenfunction for the iterated solution $m^{[1]}$, which satisfies the spectral problem

$$
\begin{aligned}
&(\phi_{1,2})_{xy} + 2\phi_{1,2} m^{[1]}_{xy} = 0, \\
&(\phi_{1,2})_t + a\left[(\phi_{1,2})_{xxx} + 6(\phi_{1,2})_x m^{[1]}_{xx}\right] + b\left[(\phi_{1,2})_{yyy} + 6(\phi_{1,2})_y m^{[1]}_{yy}\right] = 0.
\end{aligned}
\tag{4.91}
$$

This Lax pair (4.91) can be considered as a nonlinear system in the fields and eigenfunction together. It means that the truncated Painlevé expansion given by (4.90) should be combined with a similar expansion for the eigenfunction $\phi_{1,2}$, of the form

---

[2]Actually, the computation associated to the Darboux transformation procedure will be simplified, since the intermediate eigenfunctions no longer appear in the process, and hence, the successive iterations are directly performed over the singular manifold itself.

$$\phi_{1,2} = \phi_2 - \frac{\Delta_{1,2}}{\phi_1}, \tag{4.92}$$

performed in a neighbourhood of the singular manifold $\phi_1 = 0$ and where $\Delta_{1,2}(x, y, t)$ is a coefficient to be determined. Substitution of equations (4.90) and (4.92) into equations (4.91) yields the coefficients $\Delta_{i,j}$, $i, j = 1, 2$ as the exact derivative

$$
\begin{aligned}
d\Delta_{i,j} = d\Delta(\phi_i, \phi_j) &= 2\phi_j(\phi_i)_x dx + 2(\phi_j)_y \phi_i \, dy \\
&+ 2a \left[(\phi_j)_x (\phi_i)_{xx} - (\phi_i)_x (\phi_j)_{xx} - \phi_j(\phi_i)_{xxx} - 6m^{[0]}_{xx} \phi_j (\phi_i)_x \right] dt \\
&+ 2b \left[(\phi_i)_y (\phi_j)_{yy} - (\phi_j)_y (\phi_i)_{yy} - \phi_i(\phi_j)_{yyy} - 6m^{[0]}_{yy} \phi_i (\phi_j)_y \right] dt,
\end{aligned}
\tag{4.93}
$$

where the following identity holds

$$\Delta_{i,j} = 2\phi_i \phi_j - \Delta_{j,i}. \tag{4.94}$$

The Painlevé expansion given by equation (4.90) and equation (4.92) may also be considered as a binary Darboux transformation that relates the Lax pairs given by (4.89) and (4.91).

Second iteration and τ-function

The singular manifold $\phi_{1,2}$ allows us to iterate equation (4.90) again in the following form

$$m^{[2]} = m^{[1]} + \log(\phi_{1,2}), \tag{4.95}$$

which in terms of the seed solution $m^{[0]}$ reads

$$m^{[2]} = m^{[0]} + \log(\tau_{1,2}), \tag{4.96}$$

where the τ-function is defined as

$$\tau_{1,2} = \phi_{1,2} \, \phi_1 = \phi_1 \phi_2 - \Delta_{1,2}. \tag{4.97}$$

In view of equation (4.94), it is trivial to obtain $\Delta_{1,1} = \phi_1^2$, $\Delta_{2,2} = \phi_2^2$, $\Delta_{2,1} = 2\phi_1\phi_2 - \Delta_{1,2}$, such that $\tau_{1,2}$ therefore satisfies

$$\tau_{12}^2 = \det \Delta, \tag{4.98}$$

where Δ is the 2×2 matrix of entries $\Delta_{i,j}$, with $i, j = 1, 2$, defined in (4.93). Thus, it is possible to construct the solution $m^{[2]}$ for the second iteration with the exclusive knowledge of two eigenfunctions ϕ_1 and ϕ_2 for the seed solution $m^{[0]}$.

2.4. Lump solutions

This Subsection is devoted to the obtention of lump soliton solutions for the generalized NNV equation (4.76). This kind of solutions can be straighforwardly derived by employing a similar procedure to the one described in the previous Section for the multi-component NLS system.

Seed solution and eigenfunctions

Let us consider the seed solution

$$m^{[0]} = q_0 xy, \tag{4.99}$$

where q_0 is an arbitrary constant. Then, solutions of the associated spectral problem (4.89) can be obtained through the following form

$$\phi_i(x, y, t, k_i) = e^{k_i[x+J(y,t,k_i)]} P^{[n]}(x, y, t, k_i), \qquad i = 1, 2, \tag{4.100}$$

depending on two complex parameters k_i, $i = 1, 2$, where $J(y, t, k_i)$ is the linear polynomial

$$J(k_i) \equiv J(y, t, k_i) = -2\frac{q_0}{k_i^2}y + \left(-ak_i^2 + \frac{8bq_0^3}{k_i^4}\right)t \tag{4.101}$$

and $P^{[n]}(k_i) \equiv P^{[n]}(x, y, t, k_i)$ is a polynomial of degree n in x of the form

$$P^{[n]}(k_i) = \sum_{l=0}^{n} a_j(y, t, k_i)\psi(x, y, t, k_i)^l, \tag{4.102}$$

with

$$\psi(k_i) \equiv \psi(x, y, t, k_i) = k_i^2\left(x + \frac{2q_0}{k_i^2}y - 3\left(ak_i^2 + \frac{8bq_0^3}{k_i^4}\right)t\right). \tag{4.103}$$

The coefficients $a_l(y, t, k_i)$ for any l in (4.102) can be easily computed by substituting equations (4.100)-(4.103) into equations (4.89), obtaining after some algebraic manipulations

$$\frac{\partial a_l}{\partial y} = -k_i(l+1)\frac{\partial a_{l+1}}{\partial y} - 2q_0k_i(l+1)(l+2)a_{l+2},$$

$$\frac{\partial a_l}{\partial t} = (l+1)\left(bk_i\frac{\partial^3 a_{l+1}}{\partial y^3} - 12bq_0\frac{\partial^2 a_{l+1}}{\partial y^2} + \frac{36bq_0^2}{k_i}\frac{\partial a_{l+1}}{\partial y}\right)$$

$$+ (l+1)(l+2)\left(2bq_0k_i\frac{\partial^2 a_{l+2}}{\partial y^2} - 24bq_0^2\frac{\partial a_{l+2}}{\partial y} - \frac{3}{k_i}\left(ak_i^6 - 16bq_0^3\right)a_{l+2}\right)$$

$$- (l+1)(l+2)(l+3)\left(ak_i^6 + 8bq_0^3\right)a_{l+3}, \tag{4.104}$$

where we can set $a_n = 1$, $a_{n-1} = 0$. From the above, it is obvious that there is an infinite number of possible eigenfunctions characterized by an integer n and a wavenumber k_i, $i = 1, 2$.

2.4.1. One-lump solution for $n = 1$ (One lump)

The simplest case corresponds to the choice $n = 1$, such that the eigenfunctions in (4.100) possess the following form

$$\phi_i(k_i) = e^{k_i[x+J(k_i)]}\psi(k_i), \qquad i = 1, 2, \tag{4.105}$$

where $J(k_i)$ and $\psi(k_i)$ are given in (4.101) and (4.103), respectively.

According to equation (4.93), the coefficients $\Delta_{i,j}$, $i = 1, 2$ can be calculated as

$$\begin{aligned}
\Delta_{i,j} &= \frac{2k_i}{k_i + k_j}\left[\left(\psi(k_i) + \frac{k_i k_j}{k_i + k_j}\right)\left(\psi(k_j) - \frac{k_j^2}{k_i + k_j}\right) + \frac{k_i^2 k_j^2}{(k_i + k_j)^2}\right] \\
&\times e^{k_i[x+J(k_i)]+k_j[x+J(k_j)]}.
\end{aligned} \tag{4.106}$$

Moreover, it is important to note that the following relation also vanishes $2\phi_i\phi_j = \Delta_{i,j} + \Delta_{j,i}$.

A second iteration for the field $m^{[2]}$ now provides

$$m^{[2]} = q_0 xy + \log(\tau_{1,2}), \tag{4.107}$$

such that the second iteration for the initial field $u^{[2]}$ in (4.74) can be easily obtained from (4.75), as

$$u^{[2]} = -2m_{xy}^{[2]} = -2\left[q_0 + \left(\frac{(\tau_{1,2})_x}{\tau_{1,2}}\right)_y\right]. \tag{4.108}$$

The τ-function $\tau_{1,2}$ has been introduced in (4.97), which combined with (4.94), yields

$$\tau_{1,2} = \frac{1}{2}\left(\Delta_{2,1} - \Delta_{1,2}\right). \tag{4.109}$$

Substitution of the proper expressions for the coefficients $\Delta_{1,2}$ and $\Delta_{2,1}$ in (4.106) allows to simplify the τ-function in (4.109) as

$$\tau_{1,2} = -\frac{k_1 - k_2}{k_1 + k_2}e^{k_1[x+J(k_1)]+k_2[x+J(k_2)]}\Omega_{1,2}, \tag{4.110}$$

where the coefficient $\Omega_{1,2}$ is given by

$$\Omega_{1,2} = \Big[\psi(k_1) + g(k_1, k_2)\Big]\Big[\psi(k_2) + g(k_2, k_1)\Big] + d(k_1, k_2) \tag{4.111}$$

and the functions $g(k_i, k_j)$, $d(k_i, k_j)$ are

$$g(k_i, k_j) = \frac{2k_j k_i^2}{k_i^2 - k_j^2}, \qquad d(k_i, k_j) = \frac{2k_i^2 k_j^2(k_i^2 + k_j^2)}{(k_i^2 - k_j^2)^2}, \qquad i, j = 1, 2. \tag{4.112}$$

Therefore, equation (4.108) reads

$$u^{[2]} = -2\left[q_0 + \left(\frac{(\Omega_{1,2})_x}{\Omega_{1,2}}\right)_y\right]. \tag{4.113}$$

In order to have real expressions, we should set k_2 as the complex conjugate of k_1, by identifying

$$k_1 = A + iB, \qquad k_2 = A - iB, \qquad A, B \in \mathbb{R}. \tag{4.114}$$

By using the ansatz (4.114) in (4.111), we finally obtain

$$\Omega_{1,2} = \left[2ABx + 12AB\left(-a(A^2 - B^2) + \frac{4bq_0^3}{(A^2 + B^2)^2}\right)t - \frac{A^2 + B^2}{2B}\right]^2$$

$$+ \left[(A^2 - B^2)x + 2q_0 y + \left(3a(6A^2B^2 - A^4 - B^4) - 24q_0^3\frac{A^2 - B^2}{(A^2 + B^2)^2}\right)t + \frac{A^2 + B^2}{2A}\right]^2$$

$$+ (B^2 - A^2)\left[\frac{A^2 + B^2}{2AB}\right]^2,$$

which has no zeros for $B^2 > A^2$. This means that equation (4.113) will not have singularities for those values of the parameters.

Asymptotic behaviour

Actually, it is possible to define a Galilean transformation of the form

$$x = X + X_0 + v_x t, \qquad y = Y + Y_0 + v_y t, \tag{4.115}$$

where

$$X_0 = \frac{A^2 + B^2}{4AB^2}, \qquad\qquad v_x = \left[6a(A^2 - B^2) - \frac{24bq_0^3}{(A^2 + B^2)^2}\right], \tag{4.116}$$

$$Y_0 = -\frac{(A^2+B^2)^2}{8q_0 AB^2}, \qquad v_y = \frac{1}{q_0}\left[-\frac{3a}{2}(A^2+B^2)^2 + 24bq_0^3\frac{(A^2-B^2)}{(A^2+B^2)^2}\right].$$

The function $\Omega_{1,2}$ is expressed in the new coordinates as the static solution

$$\Omega_{1,2} = \left[(A^2-B^2)X + 2q_0Y\right]^2 + [2ABX]^2 + (B^2-A^2)\left[\frac{A^2+B^2}{2AB}\right]^2. \quad (4.117)$$

The second iteration for $u^{[2]}$ is given in (4.113), and similarly, one can define the iterations for $v^{[2]}$ and $w^{[2]}$ through (4.75), of the form

$$v^{[2]} = \frac{c}{3a} - 2\left[q_0 + \left(\frac{(\Omega_{1,2})_x}{\Omega_{1,2}}\right)_x\right], \qquad w^{[2]} = \frac{c}{3b} - 2\left[q_0 + \left(\frac{(\Omega_{1,2})_x}{\Omega_{1,2}}\right)_x\right]. \quad (4.118)$$

The lump solution for $u^{[2]}$ via (4.113) is represented in Figure 4.4. It is interesting to note that one gets a similar lump profile for $v^{[2]}$ and $w^{[2]}$.
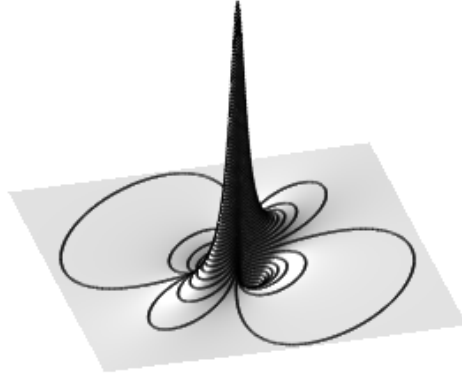
Figure 4.4: *One-soliton solution $u^{[2]}$ for $n=1$ when $q_0 = 0.3$, $A = 0.5$, $B = 1$.*

2.4.2. One-soliton solution for $n=2$ (Two lumps)

As already depicted in the analysis of lump solutions for the previous example, displayed in Subsection 1.5 of this Chapter, the polynomial of lowest degree for the eigenfunctions provides a travelling lump soliton that can be transformed in a static solution after a Galilean transformation. Then, it is expected that when polynomials of higher degrees are considered, the dynamics of the solutions become more intricate. In particular, the following level of complexity is achieved by setting $n=2$.

Substituting $n=2$ into equation (4.100) straightforwardly provides

$$\phi_i(k_i) = \Big[a_0(k_i) + \psi(k_i)^2\Big]e^{k_i[x+J(k_i)]}, \qquad i = 1, 2, \tag{4.119}$$

where $J(k_i)$ and $\psi(k_i)$ are given in (4.101) and (4.103), respectively, and

$$a_0(k_i) = -4q_0 k_i y - \frac{6}{k_i}\left(ak_i^6 - 16bq_0^3\right)t, \tag{4.120}$$

which has been easily obtained from equations (4.104).

The different elements of the Δ-matrix can be computed through the integration of equation (4.93), yielding

$$
\begin{aligned}
\Delta_{i,j}&\left(\frac{k_1 + k_2}{2k_1}\right)e^{-k_i[x+J(k_i)]-k_j[x+J(k_j)]} = -\frac{2k_2^2\psi(k_1)^2\psi(k_2)}{k_1 + k_2} + \frac{2k_1 k_2\psi(k_1)\psi(k_2)^2}{k_1 + k_2}\\
&+ \psi(k_1)^2\psi(k_2)^2 + \left[a_0(k_2) + \frac{2k_2^4}{(k_1 + k_2)^2}\right]\psi(k_1)^2 + \left[a_0(k_1) - \frac{2k_2 k_1^3}{(k_1 + k_2)^2}\right]\psi(k_2)^2\\
&+ \frac{2k_1 k_2}{k_1 + k_2}\left[a_0(k_2) + \frac{2k_2^3(k_2 - 2k_1)}{(k_1 + k_2)^2}\right]\psi(k_1) - \frac{2k_2^2}{k_1 + k_2}\left[a_0(k_1) + \frac{2k_1^3(k_1 - 2k_2)}{(k_1 + k_2)^2}\right]\psi(k_2)\\
&+ a_0(k_1)a_0(k_2) + \frac{2k_2}{(k_1 + k_2)^2}\left[k_2^3 a_0(k_1) - k_1^3 a_0(k_2)\right] + \frac{4k_2^2 k_1(k_1 - k_2)}{(k_1 + k_2)^2}\psi(k_1)\psi(k_2)\\
&+ \frac{12k_1^3 k_2^4(k_1 - k_2)}{(k_1 + k_2)^4}.
\end{aligned}
$$
$$\tag{4.121}$$

Once the elements $\Delta_{i,j}$, $i, j = 1, 2$ have been computed, the τ-function immediately arises from expression (4.109). Taking the ansatz (4.110) for $\tau_{1,2}$, the coefficient $\Omega_{1,2}$ can be written as

$$
\begin{aligned}
\Omega_{1,2} = &\left[\Big(\psi(k_1) + g(k_1, k_2)\Big)^2 + a_0(k_1) - \frac{k_1}{k_2}g(k_1, k_2)^2\right]\\
&\times \left[\Big(\psi(k_2) + g(k_2, k_1)\Big)^2 + a_0(k_2) - \frac{k_2}{k_1}g(k_2, k_1)^2\right]\\
&+ 4d(k_1, k_2)\left[\Big(\psi(k_1) - c(k_1, k_2)\Big)\Big(\psi(k_2) - c(k_2, k_1)\Big)\right] + p(k_1, k_2),
\end{aligned}
$$
$$\tag{4.122}$$

where $g(k_i, k_j)$, $d(k_i, k_j)$ are defined in equation (4.112) and

$$c(k_i, k_j) = k_i^2\frac{k_i^2 - k_i k_j + 2k_j^2}{(k_i + k_j)(k_i^2 + k_j^2)}, \qquad p(k_i, k_j) = \frac{8k_i^4 k_j^4(k_i^2 + k_j^2 + k_i k_j)}{(k_i^2 + k_j^2)(k_i + k_j)^4}.$$

If we select $k_1 = A + iB$ and $k_2 = A - iB$, $A, B \in \mathbb{R}$, $\Omega_{1,2}$ takes the form

$$\Omega_{1,2} = \left[\left((A^2 - B^2)X + 2q_0Y\right)^2 - 4A^2B^2X^2 - 4Aq_0Y + 8A^2B^2h_1t\right.$$

$$+\frac{(3A^2 - B^2)(A^4 - B^4)}{4A^2B^2}\Bigg]^2 + \Bigg[4(A^2 - B^2)ABX^2 + 8q_0ABXY - 4q_0BY$$

$$-8A^2B^2h_2t + \frac{(3A^2 - B^2)(A^2 + B^2)}{2AB}\Bigg]^2 + \frac{B^2 - 3A^2}{B^2 - A^2}\left[\frac{(A^2 + B^2)^2}{2A^2}\right]^2$$

$$+ (B^2 - A^2)\left[\frac{A^2 + B^2}{2AB}\right]^2\left[(A^2 - B^2)X + 2q_0Y - \frac{2A^4 - A^2B^2 - B^4}{2A(A^2 - B^2)}\right]^2$$

$$+ (B^2 - A^2)\left[\frac{A^2 + B^2}{2AB}\right]^2\left[2ABX + \frac{A^4 - A^2B^2 + 2B^4}{2B(A^2 - B^2)}\right]^2, \tag{4.123}$$

where h_1 and h_2 are constants defined by

$$h_1 = \frac{3}{A(A^2 + B^2)^2}\left[8bq_0^3 + a\left(3A^6 - B^6 + 5A^4B^2 + A^2B^4\right)\right],$$

$$h_2 = \frac{3}{B(A^2 + B^2)}\left[8bq_0^3 - a\left(3B^6 - A^6 + 5B^4A^2 + B^2A^4\right)\right], \tag{4.124}$$

and (X, Y) are the coordinates introduced in equation (4.115). By analyzing equation (4.123), it is easy to see that $\Omega_{1,2}$ does not have zeros when $B^2 > 3A^2$.

Asymptotic behaviour

In the following, we wish to study the asymptotic behaviour at $t \to \pm\infty$ of the lump solution (4.113) generated by the functions $\Omega_{1,2}$ obtained in (4.123). Therefore, we need to perform a transformation of the form

$$X = \hat{X} + (ct)^{\frac{1}{2}}, \qquad Y = \hat{Y} + z\,(ct)^{\frac{1}{2}}, \tag{4.125}$$

such that parameters c and z should be fixed so as to cancel the higher powers in t of expression (4.123). This requirement results in

$$c^2 - 2h_1c - h_2^2 = 0, \tag{4.126a}$$

$$z = \frac{B^2 - A^2}{2q_0} + \frac{ABh_2}{q_0c}, \tag{4.126b}$$

where h_1, h_2 are given in (4.124). Equation (4.126a) can be trivially solved, providing $c = h_1 \pm \sqrt{h_1^2 + h_2^2}$, which yields two possible solutions for z when substituted in (4.126b). These two solutions indicate the existence of two privileged directions at $t \to \pm\infty$ given by $Y = \hat{Y} + z\left(X - \hat{X}\right)$. In the asymptotic limit, the function $\Omega_{1,2}$

behaves as

$$
\begin{aligned}
\Omega_{1,2} \sim & \left[\left(2h_2(A^2 - B^2) - 4ABc \right) \hat{X} + 4q_0 h_2 \hat{Y} - 2Ah_2 + \left(\frac{A^2 - B^2}{B} \right) c \right]^2 \\
& \times \left[\left(4ABh_2 + 2(A^2 - B^2)c \right) \hat{X} + 4q_0 c \hat{Y} - 2Bh_2 + \left(\frac{A^2 - B^2}{A} \right) c \right]^2 \\
& + (h_2^2 + c^2)(B^2 - A^2) \left[\frac{A^2 + B^2}{2AB} \right]^2 ,
\end{aligned}
\tag{4.127}
$$

which corresponds to a static lump. Then, this one-soliton solution is expected to exhibit two interacting components of lump-type with equal conformation, analogously to the results found in the previous example for the lump solution of type $0 + 1$ (cf. Subsection 1.5.2 of this Chapter).

Let us consider the two possible solutions of equation (4.127) separately.

- At $t \longrightarrow -\infty$, transformation (4.125) is well defined if $c < 0$, which provides

$$
c_- = -\sqrt{h_1^2 + h_2^2} + h_1 < 0, \qquad z_- = \frac{B^2 - A^2}{2q_0} - \frac{ABh_2}{q_0 (\sqrt{h_1^2 + h_2^2} + h_1)} .
\tag{4.128}
$$

Hence, there are two equal lumps approaching along the line $X = \hat{X} + (c_- t)^{\frac{1}{2}}$, $Y = \hat{Y} + z_- (c_- t)^{\frac{1}{2}}$, which yields the direction $Y = \hat{Y} + z_-(X - \hat{X})$ in the XY-plane.

- At $t \longrightarrow \infty$, c should be selected as a positive parameter, *i.e.*

$$
c_+ = \sqrt{h_1^2 + h_2^2} + h_1 > 0, \qquad z_+ = \frac{B^2 - A^2}{2q_0} - \frac{ABh_2}{q_0 \left(\sqrt{h_1^2 + h_2^2} + h_1 \right)} ,
$$

such that there are again two equal lumps moving away along the line $X = \hat{X} + (c_+ t)^{\frac{1}{2}}$, $Y = \hat{Y} + z_- (c_+ t)^{\frac{1}{2}}$, providing $Y = \hat{Y} + z_+(X - \hat{X})$.

The scattering angle between the two lump components is given by

$$
\tan \theta_S = \frac{8q_0 AB \sqrt{h_1^2 + h_2^2}}{4q_0^2 h_2 + 4ABh_1(A^2 - B^2) + h_2(A^4 - 6A^2 B^2 + B^4)} .
\tag{4.129}
$$

The one-soliton solution for $u^{[2]}$ is shown in Figure 4.5. It is interesting to note that the same lump profile is obtained for $v^{[2]}$ and $w^{[2]}$. Figure 4.5 represents the interaction of the two aforementioned lump components, equally conformed. It is immediate to observe that the effect of such interaction is a mere rotation in the

propagation direction of the lumps, with no exchange of energy. Figure 4.5b shows the coalesced state of two components, wherein the two lumps just pass through each other.

(a)   (b)   (c)
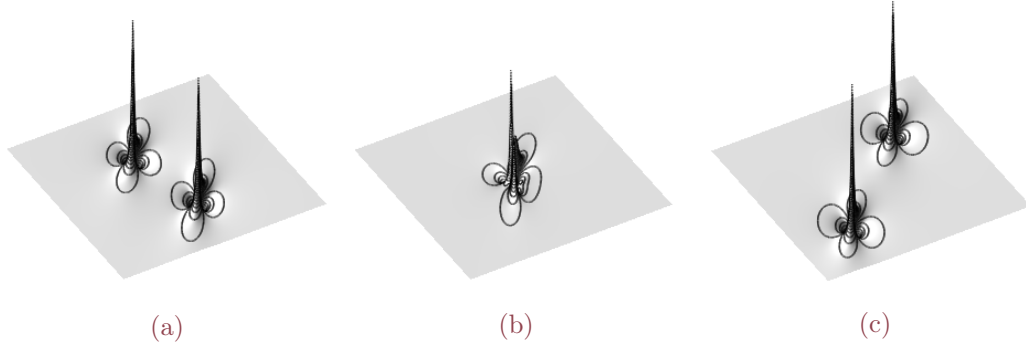
Figure 4.5: *One-soliton solution $u^{[2]}$ for $n = 2$, when $q_0 = 0.5$, $a = 1$, $b = 66$, $A = 0.5$, $B = 1$, at different times:* (a) $t < 0$, (b) $t = 0$ *and* (c) $t > 0$.

2.4.3. Two-soliton solution (Two different lumps)

In a similar manner to what was obtained in the previous Section, solution (4.100) for the eigenfunctions retrieves the one-soliton solution through the second iteration of the fields, which depends on a single wavenumber k_1 and its complex conjugate $k_2 = k_1^\dagger$. Conversely, the proper two-soliton solution would require the presence of two different wavenumbers, giving rise to two independent lump configurations, of different amplitude. It could be therefore inferred that, in order to introduce an extra wavenumber, we should now consider *four* singular manifolds ϕ_i, $i = 1, \ldots, 4$ for the spectral problem associated to the seed solution (4.89), instead of selecting exclusively two of them. This fact obviously allows us to construct up to the fourth iteration of the field $m^{[4]}$, with the consequent derivation of a τ-function $\tau_{1,2,3,4}$. The first and the third iteration introduce two different wavenumbers k_1, k_3, whilst the second and the fourth iteration provide the respective complex conjugates $k_2 = k_1^\dagger$ and $k_4 = k_3^\dagger$, [161].

Generalized Darboux transformations

The Darboux transformation approach conducted in Subsection 2.3 should be slightly modified in order to cope with this novel situation. This procedure has been constructed following the prescription developed in [161].

Let us therefore consider four eigenfunctions ϕ_i, $i = 1, \ldots, 4$ for the spectral problem

(4.87)-(4.88) with seed solution $m^{[0]}$, given by

$$
\begin{aligned}
&(\phi_i)_{xy} + 2\phi_i m^{[0]}_{xy} = 0, \\
&(\phi_i)_t + a\left[(\phi_i)_{xxx} + 6(\phi_i)_x m^{[0]}_{xx}\right] + b\left[(\phi_i)_{yyy} + 6(\phi_i)_y m^{[0]}_{yy}\right] = 0.
\end{aligned}
\tag{4.130}
$$

Let $\phi_{1,2}$ be the eigenfunction associated to the first iterated solution $m^{[1]}$ by means of the Lax pair (4.91). Then, as it has been shown from (4.95), the second iteration for the field $m^{[2]}$ can be obtained as

$$
m^{[2]} = m^{[1]} + \log(\phi_{1,2}).
\tag{4.131}
$$

We may now recursively iterate expression (4.131) so as to derive the third and the fourth iteration for the field $m^{[3]}$ and $m^{[4]}$, of the form

$$
m^{[3]} = m^{[2]} + \log(\phi_{1,2,3}), \qquad m^{[4]} = m^{[3]} + \log(\phi_{1,2,3,4}),
\tag{4.132}
$$

such that $\phi_{1,2,3}$ and $\phi_{1,2,3,4}$ are the eigenfunctions associated to the second $m^{[2]}$ and third $m^{[3]}$ iterations, respectively, satisfying the corresponding Lax pairs (4.87)-(4.88)

$$
\begin{aligned}
&(\phi_{1,2,3})_{xy} + 2\phi_{1,2,3} m^{[2]}_{xy} = 0, \\
&(\phi_{1,2,3})_t + a\left[(\phi_{1,2,3})_{xxx} + 6(\phi_{1,2,3})_x m^{[2]}_{xx}\right] + b\left[(\phi_{1,2,3})_{yyy} + 6(\phi_{1,2,3})_y m^{[2]}_{yy}\right] = 0,
\end{aligned}
\tag{4.133}
$$

and

$$
\begin{aligned}
&(\phi_{1,2,3,4})_{xy} + 2\phi_{1,2,3,4} m^{[3]}_{xy} = 0, \\
&(\phi_{1,2,3,4})_t + a\left[(\phi_{1,2,3,4})_{xxx} + 6(\phi_{1,2,3,4})_x m^{[3]}_{xx}\right] + b\left[(\phi_{1,2,3,4})_{yyy} + 6(\phi_{1,2,3,4})_y m^{[3]}_{yy}\right] = 0.
\end{aligned}
\tag{4.134}
$$

The truncated Painlevé expansion for $\phi_{1,2}$ given in (4.92) may be generalized as

$$
\phi_{1,j} = \phi_j - \frac{\Delta_{1,j}}{\phi_1},
\tag{4.135}
$$

for $j = 2, \ldots, 4$, such that any $\phi_{1,j}$ satisfies the spectral problem (4.91) for the first iteration $m^{[1]}$. Then, the associated coefficients $\Delta_{1,j}$ correspond to the elements in first row of the restated 4×4 Δ-matrix,

$$
\Delta = \begin{pmatrix}
\Delta_{1,1} & \Delta_{1,2} & \Delta_{1,3} & \Delta_{1,4} \\
\Delta_{2,1} & \Delta_{2,2} & \Delta_{2,3} & \Delta_{2,4} \\
\Delta_{3,1} & \Delta_{3,2} & \Delta_{3,3} & \Delta_{3,4} \\
\Delta_{4,1} & \Delta_{4,2} & \Delta_{4,3} & \Delta_{4,4}
\end{pmatrix},
\tag{4.136}
$$

whose generic entries can be easily computed by setting

$$\Delta_{i,j} = \Delta(\phi_i, \phi_j), \qquad\qquad i,j = 1,\ldots,4, \qquad\qquad (4.137)$$

according to their definition given in (4.93)

$$\begin{aligned}
d\Delta_{i,j} &= d\Delta(\phi_i, \phi_j) = 2\phi_j(\phi_i)_x dx + 2(\phi_j)_y\, \phi_i\, dy \\
&+ 2a\left[(\phi_j)_x\,(\phi_i)_{xx} - (\phi_i)_x\,(\phi_j)_{xx} - \phi_j(\phi_i)_{xxx} - 6m^{[0]}_{xx}\,\phi_j\,(\phi_i)_x\right] dt \\
&+ 2b\left[(\phi_i)_y\,(\phi_j)_{yy} - (\phi_j)_y\,(\phi_i)_{yy} - \phi_i(\phi_j)_{yyy} - 6m^{[0]}_{yy}\,\phi_i\,(\phi_j)_y\right] dt,
\end{aligned} \qquad (4.138)$$

where identity (4.94) holds

$$\Delta_{j,i} = 2\phi_i\phi_j - \Delta_{i,j}, \qquad\qquad (4.139)$$

and whose direct application provides $\Delta_{i,i} = \phi_i^2$, $i = 1,\ldots,4$, for the diagonal terms. A third iteration for the eigenfunctions immediately yields

$$\phi_{1,i,j} = \phi_{1,j} - \frac{\Delta_{1,i,j}}{\phi_{1,i}} \qquad\qquad (4.140)$$

for $i,j = 2,\ldots,4$, where $\phi_{1,j}$ is given in (4.135) and the coefficients of the expansion $\Delta_{1,i,j}$ are found to satisfy

$$\Delta_{1,i,j} = \Delta(\phi_{1,i}, \phi_{1,j}) = \Delta_{i,j} - \frac{\Delta_{i,1}\Delta_{1,j}}{\phi_1^2}, \qquad\qquad (4.141)$$

by means of the definitions (4.138) and (4.139).

Finally, the fourth iteration provides

$$\begin{aligned}
\phi_{1,i,j,k} &= \phi_{1,i,k} - \frac{\Delta_{1,i,j,k}}{\phi_{1,i,j}}, \\
\Delta_{1,i,j,k} &= \Delta(\phi_{1,i,j}, \phi_{1,i,k}) = \Delta_{i,j,k} - \frac{\Delta_{1,j,i}\Delta_{1,i,k}}{\phi_{1,i}^2},
\end{aligned} \qquad\qquad (4.142)$$

for $i,j,k = 2,3,4$ and where the different elements $\{\phi_{1,i,k}, \Delta_{i,j,k}\}$ can be recursively obtained from expressions (4.140) and (4.141).

The fourth iteration for the field $m^{[4]}$, expressed in terms of the seed solution $m^{[0]}$, then becomes

$$m^{[4]} = m^{[3]} + \log(\phi_{1,2,3,4}) = m^{[2]} + \log(\phi_{1,2,3}) + \log(\phi_{1,2,3,4})$$

$$= m^{[1]} + \log(\phi_{1,2}) + \log(\phi_{1,2,3}) + \log(\phi_{1,2,3,4}) \tag{4.143}$$
$$= m^{[0]} + \log(\phi_1) + \log(\phi_{1,2}) + \log(\phi_{1,2,3}) + \log(\phi_{1,2,3,4}),$$

which can be summarized as

$$m^{[4]} = m^{[0]} + \log(\tau_{1,2,3,4}), \tag{4.144}$$

where the τ-function $\tau_{1,2,3,4}$ has been defined as

$$\tau_{1,2,3,4} = \phi_{1,2,3,4}\, \phi_{1,2,3}\, \phi_{1,2}\, \phi_1. \tag{4.145}$$

With the previous definitions, we can construct the τ-function for the fourth iteration $m^{[4]}$ from the eigenfunctions of the seed solution $m^{[0]}$ in the following form

$$\tau_{1,2,3,4} = \frac{1}{4}\left(\Delta_{2,1} - \Delta_{1,2}\right)\left(\Delta_{4,3} - \Delta_{3,4}\right) - \frac{1}{4}\left(\Delta_{4,2} - \Delta_{2,4}\right)\left(\Delta_{3,1} - \Delta_{1,3}\right)$$
$$+ \frac{1}{4}\left(\Delta_{4,1} - \Delta_{1,4}\right)\left(\Delta_{3,2} - \Delta_{2,3}\right), \tag{4.146}$$

where every coefficient can be computed through the exact derivative (4.138) and we have used identity (4.139) $\phi_i\phi_j = \frac{1}{2}(\Delta_{j,i} + \Delta_{i,j})$, $i,j = 1,\ldots,4$.

Analogously to (4.98), it is possible to express $\tau_{1,2,3,4}$ in a more compact form, by means of the equation

$$\tau_{1,2,3,4}^2 = \det\Delta, \tag{4.147}$$

where Δ is given in (4.136).

Hence, the fourth iteration $m^{[4]}$ can be straightforwardly constructed starting from four solutions $\phi_1, \phi_2, \phi_3, \phi_4$ of the spectral problem (4.130) associated to the chosen seed solution $m^{[0]}$, selected as (4.99) in this case.

Two-soliton solution for $n = 1$ (Two lumps)

In the following, we shall consider the simplest possible case to characterize the two-lump solution. In consequence, this restriction implies the choice $n = 1$. Then, the eigenfunctions given by (4.100) again take the form established in equation (4.105),

$$\phi_i(k_i) = e^{k_i[x+J(k_i)]}\psi(k_i), \qquad i = 1,\ldots,4, \tag{4.148}$$

where $J(k_i)$ and $\psi(k_i)$ are given in (4.101) and (4.103), respectively. In this case, and regarding expression (4.106), the coefficients of the Δ-matrix $\Delta_{i,j}$, $i = 1,\ldots,4$

now read

$$\Delta_{i,j} = \frac{2k_i}{k_i + k_j} \left[\left(\psi(k_i) + \frac{k_i k_j}{k_i + k_j} \right) \left(\psi(k_j) - \frac{k_j^2}{k_i + k_j} \right) + \frac{k_i^2 k_j^2}{(k_i + k_j)^2} \right]$$
$$\times e^{k_i[x+J(k_i)]+k_j[x+J(k_j)]}. \tag{4.149}$$

Then, the fourth iteration for the field $u^{[2]}$ may be analogously computed as

$$u^{[2]} = -2m_{xy}^{[2]} = -2 \left[q_0 + \left(\frac{(\tau_{1,2,3,4})_x}{\tau_{1,2,3,4}} \right)_y \right], \tag{4.150}$$

where $\tau_{1,2,3,4}$ is given in (4.146).

Regarding the ansatz (4.114), the wavenumbers k_1, k_2, k_3, k_4 should be chosen as

$$\begin{aligned} k_1 &= A_1 + iB_1, & k_2 &= k_1^\dagger = A_1 - iB_1, \\ k_3 &= A_2 + iB_2, & k_4 &= k_3^\dagger = A_2 - iB_2, \end{aligned} \tag{4.151}$$

with $A_1, B_1, A_2, B_2 \in \mathbb{R}$.

In view of the ensuing calculations, it is convenient to define a center-of-mass coordinate system as

$$x = X_{cm} + \frac{1}{2} \left(v_x^1 + v_x^2 \right) t \qquad y = Y_{cm} + \frac{1}{2} \left(v_y^1 + v_y^2 \right) t, \tag{4.152}$$

where $\left(v_x^i, v_y^i \right)$, $i = 1, 2$ are the individual velocities of each soliton given in (4.116)

$$v_x^i = \left(6a(A_i^2 - B_i^2) - \frac{24bq_0^3}{(A_i^2 + B_i^2)^2} \right), \tag{4.153}$$
$$v_y^i = \frac{1}{q_0} \left(\frac{-3a}{2} \left(A_i^2 + B_i^2 \right)^2 + 24bq_0^3 \frac{(A_i^2 - B_i^2)}{(A_i^2 + B_i^2)^2} \right).$$

Using the change of variables given in equations (4.152)-(4.153), the expressions of the linear polynomials $\psi(k_i)$, $i = 1, \dots, 4$ defined in (4.103) reduce to

$$\begin{aligned} \psi(k_1) &= k_1^2 (X_{cm} - V_x t) + 2q_0 (Y_{cm} - V_y t), \\ \psi(k_2) &= k_2^2 (X_{cm} - V_x t) + 2q_0 (Y_{cm} - V_y t), \\ \psi(k_3) &= k_3^2 (X_{cm} + V_x t) + 2q_0 (Y_{cm} + V_y t), \\ \psi(k_4) &= k_4^2 (X_{cm} + V_x t) + 2q_0 (Y_{cm} + V_y t), \end{aligned} \tag{4.154}$$

such that the relative velocities in each spatial coordinate are defined as

$$V_x = \frac{1}{2}\left(v_x^1 - v_x^2\right), \qquad V_y = \frac{1}{2}\left(v_y^1 - v_y^2\right). \tag{4.155}$$

In the center of mass system, the solution asymptotically yields two different and completely independent lumps, of distinct conformation and amplitude, which move with equal and opposite velocities. In order to clarify this point, let us proceed with the study concerning the asymptotic behaviour of each lump.

• If we define the asymptotic coordinates

$$X_{cm} = X_1 - X_0^1 + V_x t, \qquad Y_{cm} = Y_1 - Y_0^1 + V_y t, \tag{4.156}$$

it can be proved, after cumbersome intermediate calculations handled with MAPLE, that the limit of the τ-function when $t \to \pm\infty$ can be written as the static lump

$$\tau_{1,2,3,4} \sim \left[(A_1^2 - B_1^2)X_1 + 2q_0 Y_1\right]^2 + \left[2A_1 B_1 X_1\right]^2 + \left(B_1^2 - A_1^2\right)\left[\frac{(A_1^2 + B_1^2)}{2A_1 B_1}\right]^2, \tag{4.157}$$

where

$$X_0^1 = -\frac{A_1^2 + B_1^2}{4A_1 B_1^2}$$
$$- \frac{4A_2\left[(A_2^2 + B_2^2)^3 + (A_2^2 - 3B_2^2)(A_1^2 + B_1^2)^2 + 2(A_2^2 + B_2^2)^2(B_1^2 - A_1^2)\right]}{\gamma(A_1, B_1, A_2, B_2)},$$
$$q_0 Y_0^1 = \frac{A_1^2 + B_1^2}{8A_1 B_1^2}$$
$$- \frac{2A_2(A_1^2 + B_1^2)^2[(A_1^2 + B_1^2)^2 + (A_2^2 - 3B_2^2)(A_2^2 + B_2^2) + 2(A_2^2 + B_2^2)(B_1^2 - A_1^2)]}{\gamma(A_1, B_1, A_2, B_2)},$$

and the quantity $\gamma(A_1, B_1, A_2, B_2)$ is defined as

$$\gamma(A_1, B_1, A_2, B_2) = \left[(A_1 + A_2)^2 + (B_1 - B_2)^2\right]\left[(A_1 - A_2)^2 + (B_1 - B_2)^2\right]$$
$$\times \left[(A_1 + A_2)^2 + (B_1 + B_2)^2\right]\left[(A_1 - A_2)^2 + (B_1 + B_2)^2\right]. \tag{4.158}$$

• Analogously, if we now define

$$X_{cm} = X_2 - X_0^2 - V_x t, \qquad Y_{cm} = Y_2 - Y_0^2 - V_y t, \tag{4.159}$$

the limit of the τ-function when $t \to \pm\infty$ becomes the static lump

$$\tau_{1,2,3,4} \sim \left[(A_2^2 - B_2^2)X_2 + 2q_0Y_2\right]^2 + \left[2A_2B_2X_2\right]^2 + (B_2^2 - A_2^2)\left[\frac{(A_2^2 + B_2^2)}{2A_2B_2}\right]^2, \quad (4.160)$$

where

$$X_0^2 = -\frac{A_2^2 + B_2^2}{4A_2B_2^2}$$
$$-\frac{4A_1\left[(A_1^2 + B_1^2)^3 + (A_1^2 - 3B_1^2)(A_2^2 + B_2^2)^2 + 2(A_1^2 + B_1^2)^2(B_2^2 - A_2^2)\right]}{\gamma(A_1, B_1, A_2, B_2)},$$

$$q_0Y_0^2 = \frac{A_2^2 + B_2^2}{8A_2B_2^2}$$
$$-\frac{2A_1(A_2^2 + B_2^2)^2[(A_2^2 + B_2^2)^2 + (A_1^2 - 3B_1^2)(A_1^2 + B_1^2) + 2(A_1^2 + B_1^2)(B_2^2 - A_2^2)]}{\gamma(A_1, B_1, A_2, B_2)},$$

with $\gamma(A_1, B_1, A_2, B_2)$ given by (4.158).

In this system of reference, the asymptotic behaviour of the solution for $t \to \pm\infty$ corresponds to two different lumps moving with equal and opposite velocities along parallel lines as shown in Figures 4.6a and 4.6c. Figure 4.6b represents the coalesced state of two lump solution where again the lumps seem to merge and move away in opposite directions later. Similarly, it is possible to define analogous lump profiles for $v^{[2]}$ and $w^{[2]}$.

(a)   (b)   (c)
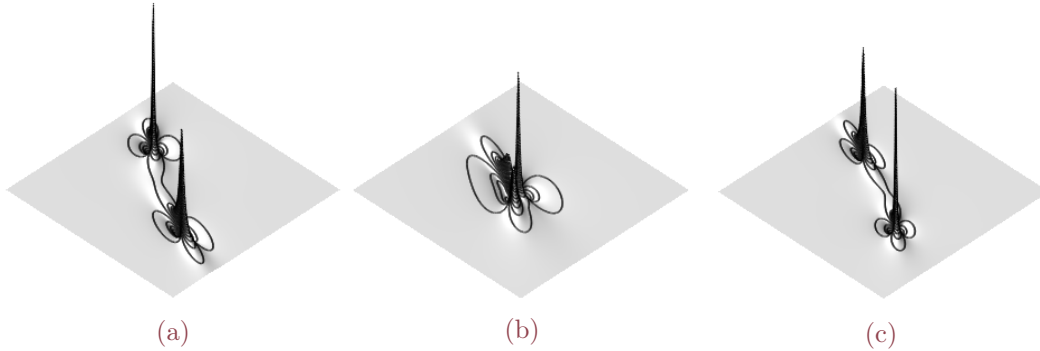
Figure 4.6: *Two-lump solution $u^{[2]}$ for $n = 1$, with parameters $a = 1$, $b = 0.2$, $q_0 = 0.5$, $A_1 = 0.5$, $B_1 = 1$, $A_2 = 0.5$, $B_2 = \frac{4}{3}$, at different times:* (a) $t < 0$, (b) $t = 0$ *and* (c) $t > 0$.

This Chapter marks the end of the study of integrable systems from the perspective of the SMM and the characterization of soliton solutions. The ensuing Chapters address a different approach to nonlinear integrable models with a known Lax pair, focusing

on the analysis of Lie symmetries and their associated similarity reductions. In particular, this machinery provides valuable information about the geometric structure of such systems. On the one hand, the reduction process allows us to explore the limits of integrability by establishing relations among integrable systems defined in diverse spatial dimensions. On the other hand, the analysis of Lax pairs from the symmetry point of view presents a new framework to characterize and delve more deeply into the role of the spectral parameter, where its isospectral or nonisospectral nature turns out to be crucial.

Chapter 5

Lie symmetries for differential equations

One of the most celebrated methods to address the issue of finding symmetries for differential equations is the so-called *Lie's method*. This approach is framed within the groundbreaking ideas of S. Lie in the XIX century [273–275], which would give rise to the modern theory of Lie groups. Lie's classical method can be straightforwardly extended to the study of other kinds of symmetries for differential equations, such us contact symmetries, Lie-Bäcklund or generalized symmetries, potential symmetries, etc.

Lie group analysis has been proved to be a fundamental technique to study any sort of systems involving differential equations. The study of symmetries represents a crucial aspect of the analysis of integrability of such equations, since this invariance property may be used to achieve partial or complete integration of the aforementioned systems. For example, invariance under a given transformation implies the possibility of reducing the number of independent variables.

There exist several generalizations of the concept of symmetry developed in Lie's theory, and its subsequent applications concerning the construction of group invariant solutions for partial differential equations. One of the renowned approaches to this matter lies in the so-called *nonclassical symmetries*, also known as *conditional symmetries*. The *nonclassical method* was first introduced by Bluman and Cole in [50] as an attempt to obtain new generalized self-similar solutions for the linear heat equation. This procedure, later generalized by Olver and Rosneau [326, 327] throughout the notion of *weak symmetries*, is based on the additional constraint imposed by the invariance of certain submanifolds, the *invariant surface conditions*. This method has proved to be extremely fruitful when calculating new similarity solutions for nonlinear PDEs [40, 93, 101, 154, 156, 272, 320, 321] and further theoretical remarks may be found in [31, 49, 99, 100, 420].

A second generalization to analyze similarity solutions of PDEs was due to Clarkson and Kruskal [98], leading to the so-called *direct method*. This algorithmic procedure

requires the choice of a particular ansatz for the solution of the PDE directly in terms of the reduced variable and the reduced field, and it has been successfully applied to numerous significant PDEs providing new symmetry reductions and exact solutions for them [91, 98, 101, 278, 281, 322]. The major distinction of this method over the ones described above is that the direct method is not based in group theory. The controversial problem of correspondence between the nonclassical method and the direct method have been addressed by several authors [94, 98, 101, 136, 272, 322, 346] and the conditions for the equivalence of both procedures have been widely studied in literature [41, 94, 147, 324]. It is also worthwhile to highlight the connection between these generalizations of Lie's classical theory with the singular manifold method and their role in the analysis of integrability for nonlinear PDEs. The link between the SMM and the direct method was first notice by Cariello and Tabor [72] and in [136] a generalization of the direct method in combination with the SMM is proposed as an alternative procedure to compute nonclassical symmetries. In a series of publications [137, 145–147], Estévez and Gordoa studied the synergy between the SMM, nonclassical symmetries and the direct method for several integrable models.

This Chapter is devoted to the study and characterization of the Lie symmetry method and its applications to differential equations. We will review basic concepts related to the general theory of Lie groups and the analysis of symmetries for PDEs, providing a solid methodology for further practical applications. There exist different approaches to describe the theoretical framework of Lie symmetries, and in this thesis we will focus on the geometric point of view [323, 329]. A more algebraic approach can be found in [51, 52, 378].

The first Section gives a basic outline of the general concepts concerning Lie groups, Lie algebras and differential geometry, presenting a unified picture to set a solid theoretical framework for the ensuing Sections. Sections 2 and 3 provide the geometric description of the theory of Lie symmetries for differential equations. Lie's method for classical symmetries is examined in depth, explicitly stating the quasi-algorithmic procedure to compute the classical symmetries. The nonclassical method is also presented, and its comparison with the classical procedure, both at a theoretical and computational level, is established. Section 4 is dedicated to the description of the symmetry reduction method, which yields the associated similarity reductions. Finally, we close the Chapter with Section 5, which specifically illustrates every aspect of the whole procedure described above for the toy example considered in this thesis, the NLS equation in $1 + 1$ dimensions.

Further applications of Lie's classical method to a plethora of integrable models in Physics and related disciplines described by systems of PDEs are extensively treated in Chapter 6.

1. Preliminaries on Lie groups, Lie algebras and differential geometry

In this Section we will be interested in the theory of Lie groups as a tool to study Lie symmetries for differential equations within a geometric framework. A brief overview of the main concepts regarding Lie groups and Lie algebras, and their connection with group actions and local groups of transformations on manifolds, is presented. The basic notions treated in this thesis are examined briefly and without proofs. Advanced topics on Lie groups and Lie algebras may be found in [245, 406], whilst notions on differential geometry and manifolds are widely treated in [14, 267, 372]. For further discussion on applications of Lie groups, we refer to [371], and particularly to [52, 323, 329, 378] concerning the issue of symmetries for differential equations.

1.1. Lie groups of transformations

Local Lie groups

A *Lie group* is a group G that also carries the structure of a differentiable manifold, such that the group operation of multiplication

$$
\begin{aligned}
\mu: \quad G \times G \quad &\to \quad G, \\
(g, h) \quad &\mapsto \quad \mu(g, h) = gh
\end{aligned}
\tag{5.1}
$$

and inversion

$$
\begin{aligned}
\iota: \quad G \quad &\to \quad G, \\
g \quad &\mapsto \quad \iota(g) = g^{-1}
\end{aligned}
\tag{5.2}
$$

are smooth maps of manifolds.

Example 5.6. The simplest example of a Lie group is the additive group $G = (\mathbb{R}, +)$, which represents the set of real numbers under the group operation of addition $\mu: \mathbb{R} \times \mathbb{R} \to \mathbb{R}$, $\mu(x, y) = x + y$, for every $x, y \in \mathbb{R}$. The inverse operation is defined as $\iota: \mathbb{R} \to \mathbb{R}$, $\iota(x) = -x$ and the identity element is $e_G = 0$. The group $(\mathbb{R}, +)$ can be easily generalized to the n-dimensional case, giving rise to the abelian Lie group $(\mathbb{R}^n, +)$, where the group operation is the vector addition in $\mathbb{R}^n$.

We may not be interested in the consideration of the full Lie group, but only in the group elements close to the identity. In this sense, the notion of local Lie group arises.

Let us consider an n-dimensional smooth manifold N with a distinguished element $e \in N$, and the open subsets $\mathcal{V}$, $\mathcal{W}$ defined such that $(\{e\} \times N) \cup (N \times \{e\}) \subset \mathcal{V} \subset$

$N \times N$ and $e \in \mathcal{W} \subset N$. A *local Lie group* G is a smooth manifold N, equipped with an identity element $e_G \equiv e \in N$, the smooth multiplication map $\mu : \mathcal{V} \to N$ and the smooth inversion map $\iota : \mathcal{W} \to N$, such that $\mathcal{W} \times \iota(\mathcal{W}) \subset \mathcal{V}$ and $\iota(\mathcal{W}) \times \mathcal{W} \subset \mathcal{V}^1$, which obeys the following properties:

(i) (Local associativity): If (x, y), (y, z), $(\mu(x, y), z)$, $(x, \mu(y, z))$ are all well defined and belong to $\mathcal{V}$, then $\mu(x, \mu(y, z)) = \mu(\mu(x, y), z)$.

(ii) (Identity): $\mu(e, x) = x = \mu(x, e)$, for all $x \in N$.

(iii) (Local inverse): $\mu(\iota(x), x) = e = \mu(x, \iota(x))$, for all $x \in \mathcal{W}$.

It is worthwhile to remark that the usual axiom of closure for global groups is omitted. The manifold N is frequently assumed to be an open subset of the Euclidean space, $N \subseteq \mathbb{R}^n$, such that the product and inversion maps may be expressed in terms of a local coordinate system.

Lie group action on manifolds

One of the most important applications of Lie groups to manifold theory involves actions of Lie groups on manifolds. Lie groups naturally arise as groups of transformations acting smoothly on manifolds, which are closely related to symmetries, as it will be shown later. For example, the groups $SO(2, \mathbb{R})$ and $SO(3, \mathbb{R})$ emerge as the groups of rotation in the plane $\mathbb{R}^2$ or the Euclidean space, $\mathbb{R}^3$, respectively. Generally, $SO(n, \mathbb{R})$ accounts for the rotation group in $\mathbb{R}^n$. The connection between Lie groups and groups of transformations on manifold may be straightforwardly constructed.

Let N be a smooth manifold and G a (local) Lie group. Let $\mathcal{U}$ be an open neighbourhood of $\{e_G\} \times N$, being e_G the identity element in G, such that $\{e_G\} \times N \subset \mathcal{U} \subset G \times N$. A *local group of transformations* acting on N is determined by a (local) Lie group G and a differentiable map

$$
\begin{aligned}
\Phi : \quad \mathcal{U} &\to N, \\
(g, x) &\mapsto \Phi(g, x)
\end{aligned}
\tag{5.3}
$$

for some $g \in G$ and $x \in N$ such that $(g, x) \in \mathcal{U}$, with the following properties:

(i) $\Phi(g, \Phi(h, x)) = \Phi((gh), x)$ for $g, h \in G$, $x \in N$ such that (h, x), $(g, \Phi(h, x))$, $(gh, x) \in \mathcal{U}$.

---

[1]This condition is imposed in order to ensure that both left and right inverses of a group element are defined. Regarding this matter, the definition of a local Lie group may vary from author to author, the one presented here is based on [325].

(ii) $\Phi(e_G, x) = x$ for all $x \in N$.

(iii) $\Phi(g^{-1}, \Phi(g, x)) = x$ with $g \in G$, $x \in N$ such that (g, x) and $(g^{-1}, \Phi(g, x)) \in \mathcal{U}$.

The map $\Phi : \mathcal{U} \to N$ defines the *local left-action*[2] of G on the manifold N and, in the following, we will use the shorthand notation $\Phi(g, x) = g \cdot x$. It can be seen from (iii) that every local group of transformations is a diffeomorphism in its domain of definition.

When $\mathcal{U} = G \times N$, the local action is said to be a *global action*, or simply an action, *i.e.* $g \cdot x$ is defined for every $g \in G$ and every $x \in N$, where properties (i)-(ii) read

(i) $\Phi(g, \Phi(h, x)) = \Phi((gh), x),$

(ii) $\Phi(e_G, x) = x,$ $\forall g, h \in G, \ x \in N,$ (5.4)

and property (iii) may be easily derived from (i) and (ii). Generally, a coordinate neighbourhood of the identity in any Lie group is a local Lie group. This identification, translated to actions of groups, implies that any global action of a Lie group on a smooth manifold restricts to a local action on any sufficiently small coordinate neighbourhood.

Orbit of a Lie group

Let G be a (local) group of transformations acting on a manifold N. A subset $V \subset N$ is called *G-invariant* or *invariant under G* if, for every $g \in G$, $x \in V$ such that $g \cdot x$ is defined, $g \cdot x \in V$.

An *orbit* of a local group of transformations is defined as the minimal (nonempty) G-invariant subset of the manifold N. A rigorous definition reads as follows. The subset $\mathcal{O} \subset N$ is an orbit if it satisfies the following properties:

(i) If $x \in \mathcal{O}$, $g \in G$ such that $g \cdot x$ is defined, then $g \cdot x \in \mathcal{O}$.

(ii) If $\tilde{\mathcal{O}} \subset \mathcal{O}$ and $\tilde{\mathcal{O}}$ satisfies (i), then either $\tilde{\mathcal{O}} = \mathcal{O}$, or $\tilde{\mathcal{O}}$ is empty.

Example 5.7. Let us consider the translation of a vector $x \in \mathbb{R}^n$. Let $y \neq 0$ be a fixed vector in $\mathbb{R}^n$, and $G = (\mathbb{R}, +)$. The map

$$\Phi(\epsilon, x) = x + \epsilon y, \tag{5.5}$$

---

[2]It is possible to define a completely equivalent right-action and derive analogous properties and results to the ones here presented. For further remarks on this formulation, we refer the reader to any textbook on the subject of Lie groups, for example [406].

where $\epsilon \in \mathbb{R}$ is a real parameter, defines a global group action on $N = \mathbb{R}^n$. Besides, the orbits are straight lines parallel to the vector y.

1.2. Lie algebras

In the following theoretical description, only $\mathbb{R}$-algebras will be considered, but the present formulation may be easily generalized to $\mathbb{K}$-algebras for a given field $\mathbb{K}$. A $\mathbb{R}$-*Lie algebra* $\mathfrak{g}$ is a $\mathbb{R}$-vector space equipped with a binary operation, called *Lie bracket*,

$$[\cdot,\cdot]: \begin{array}{rcl} \mathfrak{g} \times \mathfrak{g} & \to & \mathfrak{g}, \\ (u,v) & \mapsto & [u,v] \end{array} \tag{5.6}$$

which satisfies the following properties:

(i)   ($\mathbb{R}$-bilinearity) :

$$\begin{aligned} [au + bv, w] &= a[u,w] + b[v,w], \\ [u, av + bw] &= a[u,v] + b[u,w], \end{aligned} \tag{5.7}$$

(ii)   (Anticommutativity) : $[u,v] = -[v,u]$,

(iii)   (Jacobi identity) : $[u,[v,w]] + [w,[u,v]] + [v,[w,u]] = 0$,

for scalars $a, b \in \mathbb{R}$ and elements of the algebra $u, v, w \in \mathfrak{g}$.

The dimension of the Lie algebra is its dimension as a vector space over $\mathbb{R}$.

Given a finite-dimensional Lie algebra $\mathfrak{g}$ with basis $\mathcal{B} = \{e_1, e_2, \ldots, e_n\}$ (for the underlying vector space), it follows that the Lie bracket of all elements of the Lie algebra is uniquely determined in the form

$$[e_i, e_j] = \sum_{k=1}^{n} C_{ij}^k e_k, \tag{5.8}$$

for each $i, j = 1, \ldots, n$, and where the scalars $C_{ij}^k \in \mathbb{R}$ are called the *structure constants* of $\mathfrak{g}$ (with respect to the chosen basis $\mathcal{B}$). It can be checked by direct computation that the structure constants satisfy the following properties, inherited from the properties of the Lie bracket,

(i) (Antisymmetry) $C_{ij}^k = -C_{ji}^k$,

(ii) (Jacobi identity) $\sum_{l=1}^{n} \left(C_{ij}^l C_{lk}^r + C_{jk}^l C_{li}^r + C_{ki}^l C_{lj}^r \right) = 0$,

for all $i, j, k, r = 1, \ldots, n$.

The structure constants play a crucial role in Lie algebra representations, specially in the adjoint representation[3], and also in the classification problem.

A *derivation* on a Lie algebra $\mathfrak{g}$ is a linear map $D : \mathfrak{g} \to \mathfrak{g}$ that satisfies the Leibniz rule, *i.e.* for all $u, v \in \mathfrak{g}$,

$$D([u, v]) = [D(u), v] + [u, D(v)]. \tag{5.9}$$

1.3. Notions of differential geometry

Let us consider an n-dimensional differentiable manifold N and let p be a generic point of N. Let (V, φ) be a coordinate chart on N where $V \subset N$ is an open subset of N and the homomorphism $\varphi : V \to \mathbb{R}^n$ is the coordinate map, with local coordinates $x = (x_1, \ldots, x_n)$. Let us denote the set of real-valued differentiable functions on N as $C^\infty(N)$.

Vector fields

A tangent vector to N at a point p is denoted as v_p and the set of all tangent vectors at this given point define the tangent space to N at p, T_pN. T_pN is endowed with an n-dimensional vector space structure and the isomorphism $T_pN \simeq \mathbb{R}^n$ can be established. The collection (disjoint union) of all tangent spaces corresponding to all points $p \in N$ is called the *tangent bundle* of N, denoted by

$$TN = \bigsqcup_{p \in N} T_pN = \{(p, v) : p \in N, v \in T_pN\}. \tag{5.10}$$

There exists a canonical projection map $\pi : TN \to N, (p, v) \mapsto \pi(x, v) = p$, which sends each vector in T_pN to the point p at which it is tangent. Besides, the tangent bundle TN comes equipped with a natural topology and a smooth structure, giving rise to a $2n$-dimensional smooth manifold.

A *(smooth) vector field* on N is a section of the tangent bundle, *i.e.* a smooth map

$$\begin{aligned} X : \quad N \quad &\to \quad TN, \\ p \quad &\mapsto \quad X_p \equiv X(p) \end{aligned} \tag{5.11}$$

with the property $\pi \circ X = \mathrm{Id}_N$, being Id_N the identity on N. In other words, this map assigns to every point $p \in N$ to a tangent vector $X_p \in T_pN$. Given a smooth

---

[3]Given a finite-dimensional Lie algebra $\mathfrak{g}$ of a Lie group G, the adjoint representation is defined as $\mathrm{ad}(X)(Y) = [X, Y]$ for $X, Y \in \mathfrak{g}$.

coordinate chart (V, φ) for N, any vector field X may be expressed as

$$X = \sum_{i=1}^{n} X_i \frac{\partial}{\partial x_i} = \sum_{i=1}^{n} \xi_i(x) \frac{\partial}{\partial x_i}, \qquad (5.12)$$

where X_i is called the ith component of X in that chart, and it is given by the differentiable function $\xi_i : V \to \mathbb{R}$, for all $i = 1, \ldots, n$. Hence, X is actually a first order differential operator. The set of all vector fields on N is denoted by $\mathfrak{X}(N)$. It can be proved that $\mathfrak{X}(N)$ is a vector space under pointwise addition and scalar multiplication, and also a module over the ring $C^\infty(N)$ [267].

Vector fields may be regarded as derivations in the sense of definition (5.9) on the algebra of real-valued differentiable functions on N. If $X \in \mathfrak{X}(N)$, $f \in C^\infty(N)$, the vector field X defines a linear map

$$\begin{aligned} X: \quad C^\infty(N) &\rightarrow C^\infty(N), \\ f &\mapsto (Xf)(p) = X(f)|_p = X_p f \end{aligned} \qquad (5.13)$$

for all $p \in N$. $X_p f$ has the geometric interpretation of the directional derivative at p of the function f along any curve passing through p and having X_p as tangent at p. Since tangent vectors satisfy the product rule by definition[4], this property can be easily translated to vector fields, giving rise to the Leibniz rule

$$X(fg) = X(f)g + fX(g) \qquad (5.14)$$

for all $f, g \in C^\infty(N)$, which fulfills the definition of a derivation (5.9). Conversely, every derivation $D : C^\infty(N) \to C^\infty(N), f \mapsto Df$ defines a vector field $X \in \mathfrak{X}(N)$ as $Df = Xf$, and usually, both characterization are identically identified.

Let N and N' be two smooth manifolds and $F : N \to N'$ be a smooth map. Then, for each $p \in N$, the *differential of F at p* is the linear map $dF|_p : T_p N \to T_{F(p)} N'$ defined by

$$dF|_p(X)(f) = X(f \circ F), \qquad (5.15)$$

given $X \in T_p N$ and $f \in C^\infty(N')$. If F is a diffeomorphism, the differential induces the map of vector fields $F_* : \mathfrak{X}(N) \to \mathfrak{X}(N')$ defined by

$$(F_* X)_q = dF|_{F^{-1}(q)}(X_{F^{-1}(q)}). \qquad (5.16)$$

---

[4]Roughly speaking, and avoiding some subtleties, a tangent vector v_p to N at a point $p \in N$ is defined as a linear map $v_p : C^\infty(N) \to \mathbb{R}$ that satisfies the product rule at the point p of the form $v_p(fg) = f(p)v(g) + v(f)g(p)$ for any $f, g \in C^\infty(N)$. This definition allows us to interpret tangent vectors as derivations on the space of smooth functions defined near a point p.

Given two vector fields $X, Y \in \mathfrak{X}(N)$ (regarded as derivations), the *commutator* between them is given by the operation

$$[X, Y] = X \circ Y - Y \circ X, \tag{5.17}$$

which turns out to be a vector field on N. This statement may be easily proved by using the local description (5.12) for the vector fields X, Y. This operation is also called the *Lie bracket*, and it is immediate (but tedious) to prove that (5.17) satisfies the properties of $\mathbb{R}$-bilinearity, anticommutativity and the Jacobi identity given in definition (5.7). Thus, the vector space $\mathfrak{X}(N)$ together with the binary operation $[\cdot, \cdot]$ defined in (5.17) form an (infinite dimensional) Lie algebra.

Integral curves

A smooth curve on N can be parametrized by the map

$$\begin{aligned} \gamma: \quad I &\to \quad N, \\ \epsilon &\mapsto \quad p = \gamma(\epsilon) \end{aligned} \tag{5.18}$$

where $I \subseteq \mathbb{R}$ is an (open) interval of $\mathbb{R}$. The velocity vector associated to that curve at a point $\epsilon_0 \in I$ is given by the tangent vector

$$\dot{\gamma}(\epsilon_0) = d\gamma \left(\left. \frac{d}{d\epsilon} \right|_{\epsilon=\epsilon_0} \right) \equiv \left. \frac{d\gamma}{d\epsilon} \right|_{\epsilon=\epsilon_0} \in T_{\gamma(\epsilon_0)} N, \tag{5.19}$$

or, defined by its action on functions $f \in C^\infty(N)$, as

$$\dot{\gamma}(\epsilon_0) f = \left. \frac{d(f \circ \gamma)}{d\epsilon} \right|_{\epsilon=\epsilon_0}. \tag{5.20}$$

In local coordinates, for $\epsilon_0 \in I$, the coordinate representation of γ is given by $\gamma(\epsilon) = (\gamma_1(\epsilon), \ldots, \gamma_n(\epsilon))$ for ϵ sufficiently close to ϵ_0, and then

$$\dot{\gamma}(\epsilon_0) = \sum_{i=1}^n \left. \frac{d\gamma_i}{d\epsilon} \right|_{\epsilon=\epsilon_0} \left. \frac{\partial}{\partial x_i} \right|_{\gamma(\epsilon_0)}. \tag{5.21}$$

Given a vector field $X \in \mathfrak{X}(N)$, a smooth curve $\gamma : I \to N$ is said to be an *integral curve* of X at p if its velocity vector coincides with the value of the vector field X at that point,

$$\dot{\gamma}(\epsilon) = X_{\gamma(\epsilon)} = X(\gamma(\epsilon)), \qquad \forall \epsilon \in I. \tag{5.22}$$

This definition requires that the following autonomous system of ordinary differential equations holds

$$\begin{cases} \dot{\gamma}_1(\epsilon) = X_1(\gamma_1(\epsilon), \dots, \gamma_n(\epsilon)), \\ \quad \vdots \\ \dot{\gamma}_n(\epsilon) = X_n(\gamma_1(\epsilon), \dots, \gamma_n(\epsilon)). \end{cases} \tag{5.23}$$

The Cauchy problem defined by (5.23) and the initial data set $\gamma(0) = p_0 \in N$ is well-posed, *i.e.* the theorem of existence, uniqueness and smoothness for systems of ODEs (cf. Theorem D.1 in [267]) guarantees that the solution exists, is unique and varies smoothly on the initial conditions. In other words, for each point $p_0 \in N$, there exists a unique integral curve of X given by the smooth curve $\gamma : (-\delta, \delta) \to N$, with $\delta > 0$, such that $\gamma(0) = p_0$. Uniqueness means that if there is any other integral curve $\tilde{\gamma} : \tilde{I} \to N$ of X such that $\tilde{\gamma}(0) = p_0$, then $\tilde{\gamma}(\epsilon) = \gamma(\epsilon)$ for $\epsilon \in \tilde{I} \cap (-\delta, \delta)$.

A vector field $X \in \mathfrak{X}(N)$ is *complete* if for any $p \in N$ there is an integral curve $\gamma : \mathbb{R} \to N$ such that $\gamma(0) = p$. The set of all complete vector fields on N may be denoted by $\mathfrak{X}^\infty(N)$.

Let X be now a complete vector field on N. Hence, for any $p \in N$, there exists a unique integral curve $\gamma_p : \mathbb{R} \to N$ such that $\gamma_p(0) = p$.

Thus, for any fixed $\epsilon \in \mathbb{R}$, we define the differentiable map

$$\begin{aligned} \Phi_\epsilon : \quad N \quad &\to \quad N, \\ p \quad &\mapsto \quad \Phi_\epsilon(p) = \gamma_p(\epsilon) \end{aligned} \tag{5.24}$$

that possesses the following properties

$$\begin{aligned} &\text{(i)} \quad \Phi_\epsilon \circ \Phi_{\tilde{\epsilon}} = \Phi_{\epsilon+\tilde{\epsilon}}, \qquad \forall \epsilon, \tilde{\epsilon} \in \mathbb{R}, \\ &\text{(ii)} \quad \Phi_0 = \text{Id}_N, \text{ being } \text{Id}_N \text{ the identity on } N. \end{aligned} \tag{5.25}$$

Proof. Property (i) may be demonstrated as follows. For any $p \in N$ and any $\epsilon, \tilde{\epsilon} \in \mathbb{R}$, both sides of identity (i) define integral curves of the vector field X,

$$\begin{aligned} \gamma_1(\epsilon) &\equiv \Phi_\epsilon \circ \Phi_{\tilde{\epsilon}}(p) = \Phi_\epsilon(\gamma_p(\tilde{\epsilon})) = \gamma_{\gamma_p(\tilde{\epsilon})}(\epsilon), \\ \gamma_2(\epsilon) &\equiv \Phi_{\epsilon+\tilde{\epsilon}}(p) = \gamma_p(\epsilon + \tilde{\epsilon}). \end{aligned}$$

It is immediate to see that the initial conditions for both curves at $\epsilon = 0$ provide the same result,

$$\gamma_1(0) = \gamma_{\gamma_p(\tilde{\epsilon})}(0) = \gamma_p(\tilde{\epsilon}), \qquad \gamma_2(0) = \gamma_p(0 + \tilde{\epsilon}) = \gamma_p(\tilde{\epsilon}).$$

As both $\gamma_1(\epsilon)$ and $\gamma_2(\epsilon)$ are integral curves of X and $\gamma_1(0) = \gamma_2(0)$, hence, by

uniqueness of integral curves, it follows that $\gamma_1(\epsilon) = \gamma_2(\epsilon)$, and then, $\Phi_\epsilon \circ \Phi_{\tilde{\epsilon}}(p) = \Phi_{\epsilon+\tilde{\epsilon}}(p)$.

Property (ii) holds by definition, since $\Phi_0(p) = \gamma_p(0) = p$ for all $p \in N$. □

Besides, Φ_ϵ has an inverse, $\Phi_\epsilon^{-1} = \Phi_{-\epsilon}$ for $\epsilon \in \mathbb{R}$, since $\Phi_\epsilon \circ \Phi_{-\epsilon} = \Phi_0 = \mathrm{Id}_N$ and $\Phi_{-\epsilon} \circ \Phi_\epsilon = \Phi_0 = \mathrm{Id}_N$. Hence, we can conclude that $\Phi_\epsilon : N \to N$ is bijective and as is differentiable by construction, then the map $\Phi_\epsilon : N \to N$ is a diffeomorphism. The family of maps $\{\Phi_\epsilon\}_{\epsilon \in \mathbb{R}}$ defines a *one-parameter group of diffeomorphisms* on N, where the group laws are given by (5.25).

Flow generated by a vector field

This construction allows us to define the *flow generated by the vector field X* as the smooth map

$$\begin{aligned} \Phi : \quad \mathbb{R} \times N \quad &\to \quad N, \\ (\epsilon, p) \quad &\mapsto \quad \Phi(\epsilon, p) = \Phi_\epsilon(p) \end{aligned} \tag{5.26}$$

with the properties

$$\Phi(\epsilon, \Phi(\tilde{\epsilon}, p)) = \Phi(\epsilon + \tilde{\epsilon}, p), \qquad \Phi(0, p) = p, \tag{5.27}$$

for all $\epsilon, \tilde{\epsilon} \in \mathbb{R}$ and $p \in N$.

From those properties defined above and by virtue of (5.4), one may forthrightly see that the flow generated by the vector field X (5.26) corresponds to a global left-action of the additive group $G = (\mathbb{R}, +)$ on the n-dimensional manifold N. The flow (5.26) is usually referred as a *one-parameter group of transformations* of N. Then, a complete vector field X generates a one-parameter group of transformations where the integral curves $\gamma_p(\epsilon)$ are the orbits for the group.

Conversely, every one-parameter group of diffeomorphisms $\{\Phi_\epsilon\}_{\epsilon \in \mathbb{R}}$ allows us to define a vector field $\hat{X}$ on N of the form

$$\hat{X}_\Phi(p) = \left. \frac{d\Phi_\epsilon(p)}{d\epsilon} \right|_{\epsilon=0}, \tag{5.28}$$

or equivalently, by its action on functions,

$$\hat{X}_\Phi f = \left. \frac{d(f \circ \gamma_p)}{d\epsilon} \right|_{\epsilon=0}, \tag{5.29}$$

for all $p \in N$, $f \in C^\infty(N)$, and where the integral curves are given by $\Phi_\epsilon(p) = \gamma_p(\epsilon)$. Then, every vector field $X = \hat{X}_\Phi(p)$ of the form (5.28) is called the *the infinitesimal*

generator of the one-parameter group Φ_ϵ.

If X is not complete, one can also derive a similar theory for local flows generated by X, $\Phi : \{0\} \times N \subset \mathcal{U} \to N$, where the group laws (5.27) still hold for any $(\tilde{\epsilon}, p)$, $(\epsilon, \Phi(\tilde{\epsilon}, p))$, $(\epsilon + \tilde{\epsilon}, p) \in \mathcal{U}$, $p \in N$, with parameters sufficiently close to the identity. In this case, the set of maps $\{\Phi_\epsilon\}_{\epsilon \in I}$ with $I \subset \mathbb{R}$ such that $\{0\} \in I$, defines a local one-parameter group of diffemorphims.

Thus, every vector field is equivalent to a local one-parameter group of transformations and viceversa. When X is complete, the identification extends globally.

If the vector field is expressed as $X = \sum_{i=1}^{n} \xi_i(x)\partial_{x_i}$, then, the associated local one-parameter group of transformations is given by

$$\tilde{x}_i = \Phi(\epsilon, x_i) = x_i + \epsilon \xi_i(x) + \mathcal{O}\left(\epsilon^2\right), \qquad i = 1, \ldots, n, \tag{5.30}$$

with $\epsilon \in \mathbb{R}$ as the parameter of the group.

1.4. Lie algebra of a Lie group

The following lines are devoted to address the fundamental question regarding the correspondence between Lie groups and Lie algebras. Given a Lie group G, we can construct an associated Lie algebra $\mathfrak{g}$ (as it will be shown shortly) that completely captures the local structure of the group. This relation implies that all the information in the group G is contained in its Lie algebra, and it enables us to study Lie groups in terms of Lie algebras. In fact, many of applications of Lie groups to differential equations, including Lie symmetries, are based on this connection.

For every $g \in G$ we define the diffeomorphisms

$$
\begin{array}{llll}
L_g: & G & \to & G, \\
& h & \mapsto & \mu(g, h) = gh,
\end{array}
\qquad
\begin{array}{llll}
R_g: & G & \to & G, \\
& h & \mapsto & \mu(h, g) = hg,
\end{array}
\qquad \forall h \in G,
\tag{5.31}
$$

which are called *left* and *right translation* maps, respectively. Since $Lg \circ L_h = L_{gh}$, $R_g \circ R_h = R_{gh}$, their inverses are given by $(L_g)^{-1} = L_{g^{-1}}$, $(R_g)^{-1} = R_{g^{-1}}$. Besides, left and right translation commute, $L_g \circ R_h = R_h \circ L_g$.

A vector field X on G is called *left-invariant* if it is invariant under all left translations, *i.e.* for every $g, h \in G$,

$$dL_g|_h(X_h) = X_{L_g(h)} = X_{gh}. \tag{5.32}$$

Since L_g is a diffeomorphism and by virtue of (5.16), expression (5.32) can be ab-

breviated by $(L_g)_* X = X$. Analogously, X is *right-invariant* if $(R_g)_* X = X$.

Equivalently, any left-invariant vector field X is determined by its value at the identity X_e,

$$X_g = dL_g|_e(X_e), \qquad \forall g \in G, \tag{5.33}$$

where $e \equiv e_G$ is the identity of the group.

Let $\text{Lie}(G)$ be the set of all left-invariant vector fields on a Lie group G. It can be proved that $\text{Lie}(G)$ is a Lie subalgebra of $\mathfrak{X}(G)$ and it is therefore a Lie algebra. Then, $\text{Lie}(G)$ is said to be the Lie algebra associated to the Lie group G, and it will be denoted by $\mathfrak{g}$. This fact, together with proposition (5.33), allows us to establish an isomorphism between $\text{Lie}(G)$ and the tangent space of G at the identity, $T_e G$, through the so-called *evaluation map* $\varepsilon : \text{Lie}(G) \to T_e G$, $X \mapsto \varepsilon(X) = X_e$. Thus, $\text{Lie}(G)$ is a finite-dimensional $\mathbb{R}$-vector space of dimension $\dim(G)$, which can be fully identified with $\text{Lie}(G) \simeq T_e G$. For every $v \in T_e G$ we can define a left-invariant vector field on G by $v^L|_g = dL_g|_e(v)$ and the Lie bracket on $T_e G$ is simply given by $[v, w] = \left[v^L|_g, w^L|_g \right]_e$, for any $v, w \in T_e G$.

A *one-parameter subgroup of a Lie group* G is a Lie group homomorphism $\varphi : \mathbb{R} \to G$ from the additive group $(\mathbb{R}, +)$ to G such that

$$\varphi(\epsilon + \tilde{\epsilon}) = \varphi(\epsilon)\varphi(\tilde{\epsilon}) \tag{5.34}$$

for all $\epsilon, \tilde{\epsilon} \in \mathbb{R}$.

Given any tangent vector $v \in T_e G$ there is a unique one-parameter subgroup of G $\varphi_v : \mathbb{R} \to G$ such that $\dot{\varphi}_v(0) = v$. Furthermore, φ_v is characterized as the integral curve for the associated left-invariant vector field $X_g = dL_g|_e(v) \in \text{Lie}(G)$ with $\varphi_v(0) = e$. Moreover, left-invariant vector fields on Lie groups are complete. That means that X_g generates the corresponding flow $\Phi(\epsilon, g) = g\varphi_v(\epsilon)$, which is defined for all $\epsilon \in \mathbb{R}$.

This motivates the following definition. Given a Lie group G and its associated Lie algebra $\mathfrak{g}$, we define the *exponential map* of the Lie algebra $\mathfrak{g}$ into G by the map

$$\begin{array}{rcll} \exp : & \mathfrak{g} & \to & G, \\ & v & \mapsto & \exp(v) = \varphi_v(1), \end{array} \qquad \text{for any } v \in \mathfrak{g}. \tag{5.35}$$

By definition, its image lies in the connected component of the identity in G. The exponential map is a smooth map, it possesses the following properties

$$\begin{aligned} &\exp((\epsilon + \tilde{\epsilon})v) = \exp(\epsilon v)\exp(\tilde{\epsilon}v), \qquad \exp(0) = e, \\ &\exp(-v) = (\exp(v))^{-1}, \end{aligned} \tag{5.36}$$

for all $v \in \mathfrak{g}$, $\epsilon, \tilde{\epsilon} \in \mathbb{R}$ and the following relation holds

$$e^{\epsilon v} \equiv \exp(\epsilon v) = \varphi_v(\epsilon), \qquad \forall \epsilon \in \mathbb{R}, \tag{5.37}$$

where it can be checked that the unique one-parameter subgroup of G $\varphi_v : \mathbb{R} \to G$ is consistently defined by (5.37), since it satisfies the group laws (5.36).

Hence, this application maps the line ϵv in $\mathfrak{g}$ onto the one–parameter subgroup $\varphi_v(\epsilon)$, which is tangent to v at the identity e. In terms of the exponential map, the associated flow generated by X_g is given by $\Phi(\epsilon, g) = g \exp(\epsilon v)$, with $X_g = dL_g|_e(v)$.

Since $\exp(0) = e$, $\exp$ is locally a diffeomorphism from a neighbourhood of zero onto a neighbourhood of e in G, *i.e.* there exists an open subset $V \subset \mathfrak{g}$ containing 0 and a neighbourhood $W \subset G$ containing e such that $\exp|_V : V \to W$ is a diffeomorphism.

Thus, in summary, by virtue of the arguments presented above, we may conclude that there is a one-to-one correspondence between the following elements

1. One-parameter subgroups of G,

2. Left-invariant vector fields on G,

3. Tangent vectors at $e \in G$,

which allow us to deeply understand the inner structure of Lie groups, and exploit it in order to treat problems involving Lie groups and their action on manifolds by the corresponding problem for the Lie algebras. As every Lie algebra $\mathfrak{g}$ generates a corresponding Lie group G, given a finite-dimensional Lie algebra of vector fields on a manifold N, we can always reconstruct the (local) action of G via the exponentiation process.

For further remarks on the connection between Lie groups and Lie algebra beyond the work developed in the present manuscript, we refer the reader to [267]. This matter is known as the Lie group-Lie algebra correspondence, which has been addressed, on the local level, by the Fundamental Theorems of Lie (cf. Theorem 20.22 [267]).

2. Geometric structure of differential equations

Let us consider a system of partial differential equations of order p, involving n independent variables $x = (x_1, \ldots, x_n) \in \mathscr{X} \subseteq \mathbb{R}^n$ and m dependent variables $u = (u^1, \ldots, u^m) \in \mathscr{U} \subseteq \mathbb{R}^m$. The Euclidean space of independent and dependent variables will be denoted as $\mathcal{M} = \mathscr{X} \times \mathscr{U}$, and let us consider that our system of differential equations is defined in the open subset $M \subseteq \mathcal{M}$. The solution for this

system is locally given by the equations

$$u^j = f^j(x_1, \ldots, x_n), \qquad j = 1, \ldots, m, \tag{5.38}$$

or $u = f(x)$ for brevity, where $f : \mathscr{X} \to \mathscr{U}$ is a smooth function defined in the subdomain $\Omega \subset \mathscr{X}$.

Given a smooth function $f(x) = f(x_1, \ldots, x_n)$ depending on n independent variables, there exist $n_k \equiv \begin{pmatrix} n + k - 1 \\ k \end{pmatrix}$ different partial derivatives of f of order k. In the following, let us introduce the multi-index notation and consider a set of n nonnegative integers $\sigma = (i_1, \ldots, i_n)$ of order $|\sigma| = i_1 + \cdots + i_n > 0$. If we establish $|\sigma| = k$, then

$$\partial^\sigma f = \frac{\partial^{|\sigma|} f}{\partial x^\sigma} = \frac{\partial^k f}{\partial x_1^{i_1} \cdots \partial x_n^{i_n}} \tag{5.39}$$

represents the set of all kth derivatives of f.

Let us denote

$$u_\sigma^j = \partial^\sigma f^j(x) = \frac{\partial^k u^j}{\partial x_1^{i_1} \cdots \partial x_n^{i_n}}, \quad j = 1, \ldots, m, \quad i_1 + \cdots + i_n = k, \tag{5.40}$$

as the kth-order derivatives of the coordinate u^j, or u_σ for brevity. In the case with $|\sigma| = 0$, we identify $u_0(x) \equiv u(x)$ with the function itself. Complementarily, we may use the notation $u_{x_i}^j = \frac{\partial u^j}{\partial x_i}$ in reference to the derivative of u^j with respect to a particular coordinate x_i, for $i = 1, \ldots, n$, $j = 1, \ldots, m$.

We introduce the Euclidean space $\mathscr{U}^k \subseteq \mathbb{R}^{m \cdot n_k}$, of dimension $m \cdot n_k$, endowed with coordinates u_σ^j, $j = 1, \ldots, m$ and all multi-indices $\sigma = (i_1, \ldots, i_n)$ of order k. $\mathscr{U}^k$ constitutes the space of all the kth derivatives of $u = f(x)$.

Hence, we may construct the Euclidean space

$$\mathscr{U}_{(k)} = \mathscr{U} \times \mathscr{U}^1 \times \cdots \times \mathscr{U}^k \tag{5.41}$$

of dimension $m + m \cdot n_1 + \cdots + m \cdot n_k = m \begin{pmatrix} n + k \\ k \end{pmatrix}$, whose coordinates represent all the derivatives of the solution $u = f(x)$ up to order, and including, k. A typical point in $\mathscr{U}_{(k)}$ will be denoted by $u_{(k)}$, and it represents not only the kth-order derivative of u, but also all the derivatives up to order k.

Thus, the total space $\mathscr{M}_{(k)} = \mathscr{X} \times \mathscr{U}_{(k)}$ describes the space of independent and dependent variables, together with all the derivatives up to order k. This space $\mathscr{M}_{(k)}$ is called the kth-order jet space of $\mathscr{M}$. Alternatively, we may use the notation

$J^k(\mathcal{M})$ for the kth-order jet space and $j_x^k u$ to denote a general point of it. A local coordinate system in $\mathcal{M}_{(k)}$ is given by

$$\left(x_i, u_{(k)}^j\right) \equiv \left(x_i, u^j, u_1^j, \ldots, u_k^j\right), \tag{5.42}$$

where the indices range $i = 1, \ldots, n$, $j = 1, \ldots, m$ and u_σ^j runs such that $|\sigma| = 1, \ldots, k$.

Since the starting system of differential equations is defined over M, we may construct the space

$$J^k(M) = M_{(k)} = M \times \mathscr{U}^1 \times \cdots \times \mathscr{U}^k \tag{5.43}$$

as the kth-order jet space of M.

The main advantage of the geometric structure defined above is that it allows us to treat and consider all the variables involved in a system of differential equations as local coordinates in a higher space.

Example 5.8. Let us consider the case with two independent variables and two dependent variables, $n = 2$ and $m = 2$, with local coordinates $x = (x_1, x_2) \subseteq \mathbb{R}^2$ and $u = (u^1, u^2) \subseteq \mathbb{R}^2$, such that $u^1 = u^1(x_1, x_2)$, $u^2 = u^2(x_1, x_2)$.

The space $\mathscr{U}^1 \simeq \mathbb{R}^4$ constitutes the space of first derivatives of the dependent variables, with coordinates $(u_{x_1}^1, u_{x_2}^1, u_{x_1}^2, u_{x_2}^2)$. In a similar way, we construct the space of second derivatives $\mathscr{U}^2 \simeq \mathbb{R}^6$ with coordinates $(u_{x_1 x_1}^1, u_{x_1 x_2}^1, u_{x_2 x_2}^1, u_{x_1 x_1}^2, u_{x_1 x_2}^2, u_{x_2 x_2}^2)$. In general, the space $\mathscr{U}^k$ will be isomorphic to $\mathbb{R}^{2(k+1)}$, taking as local coordinates the $2(k+1)$ different partial derivatives of u^j, $u_k^j = \frac{\partial^k u^j}{\partial x_1^i \partial x_2^{k-i}}$, for $j = 1, 2$ and $i = 0, \ldots, k$.

Then, the space $\mathscr{U}_{(k)}$ can be constructed as the Cartesian product space $\mathscr{U}_{(k)} = \mathscr{U} \times \mathscr{U}^1 \times \cdots \times \mathscr{U}^k$ of dimension $\sum_{n=0}^{k} 2(n+1) = (k+1)(k+2)$, representing the space of all derivatives up to order k of $u = (u^1, u^2)$, with local coordinates $(u_{(k)}^1, u_{(k)}^2) \equiv (u^1, u^2, u_{x_1}^1, u_{x_2}^1, u_{x_1}^2, u_{x_2}^2, \ldots, u_k^1, u_k^2)$.

3. Lie symmetries

Let us now consider a system $\mathscr{S}$ of q partial differential equations of order p, depending on n independent variables $x = (x_1, \ldots, x_n) \in \mathscr{X}$ and m dependent variables $u = (u^1, \ldots, u^m) \in \mathscr{U}$, given by

$$\mathcal{E}_\nu(x, u_{(p)}) = 0, \qquad \nu = 1, \ldots, q, \tag{5.44}$$

where $u_{(p)}$ denotes the set of all the derivatives of u with respect to the variables x up to order p. We may use the notation $\mathcal{E}(x, u_{(p)}) = 0$ or $\mathcal{E} = 0$ to abbreviate and refer to the system (5.44). This system of PDEs can be understood as a vanishing collection of smooth functions $\mathcal{E}_\nu : \mathcal{M}_{(p)} \to \mathbb{R}$, $\nu = 1, \ldots, q$ defined on the pth-order jet space $\mathcal{M}_{(p)}$. Then, geometrically, the system (5.44) defines a submanifold in $\mathcal{M}_{(p)}$ of the form

$$\mathscr{S}_\mathcal{E} = \{(x, u_{(p)}) : \mathcal{E}_\nu(x, u_{(p)}) = 0\} \subset \mathcal{M}_{(p)}, \qquad \nu = 1, \ldots, q. \tag{5.45}$$

3.1. Notion of symmetry

We may say, as an intuitive approach, that a symmetry group of the system $\mathscr{S}$ is a group of transformations that maps solutions of $\mathscr{S}$ into other solutions of $\mathscr{S}$. Let $u = f(x)$ be a solution of $\mathscr{S}$ (5.44), where $f : \mathscr{X} \to \mathscr{U}$ is a smooth function defined in the subdomain $\Omega \subset \mathscr{X}$, and let $(x, u) \equiv (x, f(x)) \subset \mathcal{M}$ be its graph.

A *continuous group G of symmetries* for the system of partial differential equations $\mathscr{S}$ is a local group of diffeomorphisms, acting on an open subset $M \subseteq \mathcal{M}$, that transforms solutions (x, u) of $\mathscr{S}$ into new solutions $(\tilde{x}, \tilde{u}) = g \cdot (x, u)$ of the system $\mathscr{S}$, for all $g \in G$.

Let $X \in \mathfrak{g}$ be a generic element of the Lie algebra $\mathfrak{g}$ associated with the Lie group of transformations G. We consider vector fields of the form

$$X = \sum_{i=1}^{n} \xi_i(x, u) \frac{\partial}{\partial x_i} + \sum_{j=1}^{m} \eta^j(x, u) \frac{\partial}{\partial u^j}. \tag{5.46}$$

Then, the local action of the symmetry group G is given by the map $\Phi : \mathbb{R} \times \mathcal{M} \to \mathcal{M}$, which in coordinates yields

$$(\tilde{x}, \tilde{u}) = g \cdot (x, u) = (\Xi_g(x, u), \Psi_g(x, u)) \subset \mathcal{M} \tag{5.47}$$

for $g \equiv g(\epsilon) = \exp(\epsilon X) \in G$, $\epsilon \in \mathbb{R}$ and some smooth functions $\Xi_g : M \to \mathscr{X}$, $\Psi_g : M \to \mathscr{U}$. Hence, the following relations hold

$$\begin{aligned}
\xi_i(x, u) &= \left. \frac{d\tilde{x}_i}{d\epsilon} \right|_{\epsilon=0} = \left. \frac{d\Phi_{g(\epsilon)}(x, u)}{d\epsilon} \right|_{\epsilon=0}, \\
\eta^j(x, u) &= \left. \frac{d\tilde{u}^j}{d\epsilon} \right|_{\epsilon=0} = \left. \frac{d\Psi_{g(\epsilon)}(x, u)}{d\epsilon} \right|_{\epsilon=0},
\end{aligned} \tag{5.48}$$

with the initial conditions $\tilde{x}_i(x, u, \epsilon)|_{\epsilon=0} = x_i$, $\tilde{u}^j(x, u, \epsilon)|_{\epsilon=0} = u^j$, for $i = 1, \ldots, n$

and $j = 1, \ldots, m$.

This gives rise to a one-parametric group of infinitesimal transformations

$$\begin{cases} \tilde{x}_i = x_i + \epsilon \xi_i(x, u) + \mathcal{O}(\epsilon^2), \\ \tilde{u}^j = u^j + \epsilon \eta^j(x, u) + \mathcal{O}(\epsilon^2), \end{cases} \quad i = 1, \ldots, n, \quad j = 1, \ldots, m, \quad (5.49)$$

where $\epsilon \in \mathbb{R}$ is the parameter of the group, and the associated vector field for the group of transformations (5.49) is given by expression (5.46).

The infinitesimal transformation (5.49) is call a *Lie point transformation* if the associated infinitesimals $\xi_i(x, u)$ and $\eta^j(x, u)$ are functions of (x, u) for all $i = 1, \ldots, n$, $j = 1, \ldots, m$, *i.e* they do not depend on higher-order derivatives of u. Hence, the associated symmetry is said to be a *Lie point symmetry*. In the remainder of this thesis, only Lie point symmetries will be considered.

Nevertheless, in the most general case, coefficients $\xi_i(x, u, u_1, \ldots, u_k)$, $\eta^j(x, u, u_1, \ldots, u_k)$ may be allowed to depend on higher-order derivatives of u up to some finite order k. These kinds of transformations are usually called *generalized symmetries* or *Lie-Bäcklund symmetries*. The formalism of point symmetries can be generalized to construct an analogous geometric or algebraic framework to characterize those symmetries, as well as an algorithmic procedure to compute them [52, 323, 394, 395]. In the particular case of dependence on the first derivatives u_1, the infinitesimal transformations are said to be *contact transformations* and the associated symmetries are called *contact symmetries*. In fact, it can be proved that a contact transformation is a extended point transformation for systems of PDEs with a single dependent variable ($m = 1$) [52, 323].

3.2. Extended jet space and prolongations

Let us consider the local group of transformations G acting on $M \subseteq \mathcal{M}$ as a symmetry group for (5.44). There exists an induced local action of G on the pth jet space $\mathcal{M}_{(p)}$, called the *pth prolongation of G* and denoted by $\mathrm{pr}^{(p)}G$. This extension is defined such that the derivatives of the dependent variables u are naturally transformed in order to preserve the contact structure of the associated submanifold (5.45). The prolonged action of the group G amounts for the map $\tilde{\Phi} : \mathbb{R} \times \mathcal{M}_{(p)} \to \mathcal{M}_{(p)}$, $(x, u_{(p)}) \mapsto (\tilde{x}, \tilde{u}_{(p)}) = \tilde{\Phi}(\epsilon, x, u_{(p)}) = \mathrm{pr}^{(p)}g \cdot (x, u_{(p)})$. In this sense, we can reformulate the notion of symmetry as follows.

G is a *symmetry group of transformations* for the system $\mathscr{S}$ (5.44) if its prolongation leaves invariant the corresponding submanifold $\mathscr{S}_\mathcal{E} \subset \mathcal{M}_{(p)}$ (5.45). In terms of the prolongation of the group action, the previous proposition means that, if for every point $(x, u_{(p)}) \in \mathscr{S}_\mathcal{E}$, we have $\mathrm{pr}^{(p)}g \cdot (x, u_{(p)}) \in \mathscr{S}_\mathcal{E}$ for all $g \in G$ such that this is

defined, then G is a symmetry group for $\mathscr{S}$.

We will be concerned about the infinitesimal description of this problem, formulated in terms of the prolongation of the generating vector field of the group of transformations G. Let X be such vector field of the form (5.46), generator of the one-parameter group of transformations (5.49) on $M \subseteq \mathcal{M}$. The *pth prolongation of X*, denoted by $\mathrm{pr}^{(p)} X$ or $X^{(p)}$, is the vector field defined on the pth jet space $M_{(p)}$ that generates the prolonged one-parameter group of transformations $\mathrm{pr}^{(p)}[\exp(\epsilon X)]$. Thus, for any point $(x, u_{(p)}) \in \mathcal{M}_{(p)}$,

$$
\mathrm{pr}^{(p)} X \Big|_{(x,u_{(p)})} \equiv X^{(p)} \Big|_{(x,u_{(p)})} = \frac{d}{d\epsilon} \Big|_{\epsilon=0} \mathrm{pr}^{(p)}[\exp(\epsilon X)](x, u_{(p)}). \tag{5.50}
$$

It can be shown [323, 395] that the pth prolongation of a vector field X (5.46) may be displayed in local coordinates as

$$
X^{(p)} = \sum_{i=1}^{n} \xi_i(x, u) \frac{\partial}{\partial x_i} + \sum_{j=1}^{m} \eta^j(x, u) \frac{\partial}{\partial u^j} + \sum_{j=1}^{m} \sum_{0<|\sigma|\leq p} \eta_\sigma^j(x, u, \ldots, u_\sigma) \frac{\partial}{\partial u_\sigma^j}, \tag{5.51}
$$

where the derivatives u_σ^j, $j = 1, \ldots, m$ are given by (5.40) and $\sigma = (i_1, \ldots, i_n)$ stands for the usual multi-index such that $0 < |\sigma| = 1_1 + \cdots + i_n \leq p$ up to order p. The coefficients $\{\eta_\sigma^j(x, u, \ldots, u_\sigma)\}$, $j = 1, \ldots, m$ are analytic functions of their variables and denote the prolongations of $\eta^j(x, u)$ for a given multi-index σ. Those coefficients are usually called *extended infinitesimals*.

We may introduced the *total derivative operator* with respect to the coordinate x_l, defined in $\mathcal{M}_{(p)}$, as the differential operator

$$
\mathrm{D}_{x_l} = \frac{\partial}{\partial x_l} + \sum_{j=1}^{m} \sum_{0\leq|\sigma|\leq p} \frac{\partial u_\sigma^j}{\partial x_l} \frac{\partial}{\partial u_\sigma^j}, \tag{5.52}
$$

where the summation over the multi-index σ ranges such that $0 \leq |\sigma| \leq p$, with the ansatz $u_0^j \equiv u^j$ and $\frac{\partial u_\sigma^j}{\partial x_l} = \frac{\partial^{|\sigma|+1} u^j}{\partial x_1^{i_1} \cdots \partial x_l^{i_l+1} \cdots \partial x_n^{i_n}}$ for all $j = 1, \ldots, m$. Given the above multi-index notation, higher-order total derivatives are define by analogy with the higher-order partial derivative case, as

$$
\mathrm{D}^\sigma = \mathrm{D}_{x_1}^{i_1} \cdot \mathrm{D}_{x_2}^{i_2} \cdot \cdots \cdot \mathrm{D}_{x_n}^{i_n}. \tag{5.53}
$$

Thus, the explicit expression for the extended infinitesimals in (5.51) is given by the

so-called *general prolongation formula*,

$$\eta_\sigma^j(x, u, \ldots, u_\sigma) = \mathrm{D}^\sigma \left(\eta^j(x, u) - \sum_{i=1}^n \xi_i(x, u) \frac{\partial u^j}{\partial x_i} \right) + \sum_{i=1}^n \xi_i(x, u) \frac{\partial u_\sigma^j}{\partial x_i}, \qquad (5.54)$$

for $j = 1, \ldots, m$ and the multi-index $0 < |\sigma| \leq p$. This expression can be demonstrated by induction, the sketch of the proof is illustrated in Appendix A.1.

The prolonged vector field $X^{(p)}$ (5.51) yields a natural extension or *prolongation* for the infinitesimal transformations (5.49) up to derivatives of order p, giving rise to the following one-parameter Lie group of extended transformations acting on $\mathcal{M}_{(p)}$, of the form

$$\begin{cases} \tilde{x}_i = x_i + \epsilon \xi_i(x, u) + \mathcal{O}(\epsilon^2), \\ \tilde{u}^j = u^j + \epsilon \eta^j(x, u) + \mathcal{O}(\epsilon^2), \\ \quad \vdots \\ \tilde{u}_\sigma^j = u_\sigma^j + \epsilon \eta_\sigma^j(x, u, \ldots, u_\sigma) + \mathcal{O}(\epsilon^2), \end{cases} \qquad (5.55)$$

for all $i = 1, \ldots, n$, $j = 1, \ldots, m$ and $0 < |\sigma| \leq p$.

Finally, we introduce the notion of the *characteristic of a vector field X* of the form (5.46), which is usually referred to the collection of m functions defined by

$$Q^j(x, u_{(1)}) = \eta^j(x, u) - \sum_{i=1}^n \xi_i(x, u) \frac{\partial u^j}{\partial x_i}, \qquad j = 1, \ldots, m. \qquad (5.56)$$

With this definition, the general prolongation formula (5.54) may be alternatively written as

$$\eta_\sigma^j(x, u, \ldots, u_\sigma) = \mathrm{D}^\sigma Q^j + \sum_{i=1}^n \xi_i \frac{\partial u_\sigma^j}{\partial x_i}, \qquad j = 1, \ldots, m, \ 0 < |\sigma| \leq p. \qquad (5.57)$$

3.3. Classical Lie symmetries

In this context, the following fundamental result arises as a criterion of infinitesimal invariance for partial differential equations. Theorem 5.1 fully characterizes a symmetry group G for a given system of PDEs from the infinitesimal point of view, and provides a rigorous mechanism to performed the calculation of the aforementioned symmetries.

Theorem 5.1. Let

$$\mathcal{E}_\nu(x, u_{(p)}) = 0, \qquad \nu = 1, \ldots, q, \qquad (5.58)$$

be a system of q partial differential equations of order p, defined in the open subset $M \subset \mathcal{M}$, understood as a submanifold on the jet space $\mathcal{M}_{(p)}$ as

$$\mathscr{S}_{\mathcal{E}} = \{(x, u_{(p)}) : \mathcal{E}_\nu(x, u_{(p)}) = 0\} \subset \mathcal{M}_{(p)}, \qquad \nu = 1, \ldots, q. \tag{5.59}$$

A local group of transformations G acting on M is a symmetry group of (5.58) if

$$X^{(p)} \left[\mathcal{E}_\nu(x, u_{(p)}) \right] \Big|_{\mathcal{E}=0} = 0, \qquad \nu = 1, \ldots, q, \tag{5.60}$$

for every infinitesimal generator $X = \sum\limits_{i=1}^{n} \xi_i(x, u)\frac{\partial}{\partial x_i} + \sum\limits_{j=1}^{m} \eta^j(x, u)\frac{\partial}{\partial u^j}$ of G.

Proof. In order to simplify the notation, let us denote $j_x^p u \equiv (x, u_{(p)}) \in \mathcal{M}_{(p)}$ as a generic point of the pth-order jet space $\mathcal{M}_{(p)}$ and $j_{\tilde{x}}^p \tilde{u} = \tilde{\Phi}(\epsilon, j_x^p u) = (\tilde{x}, \tilde{u}_{(p)}) = \mathrm{pr}^{(p)} g \cdot (x, u_{(p)})$ as the transformed one under the action of the group G. By definition of symmetry, G is a symmetry group for $\mathscr{S}$ if for every point $j_x^p u \in \mathscr{S}_{\mathcal{E}}$, then $j_{\tilde{x}}^p \tilde{u} \in \mathscr{S}_{\mathcal{E}}$ for every $\epsilon \in \mathbb{R}$. This condition is equivalent to say that if $\mathcal{E}(j_x^p u) = 0$, then $\mathcal{E}(j_{\tilde{x}}^p \tilde{u}) = 0$. Hence, for every $j_{\tilde{x}}^p \tilde{u} \in \mathscr{S}_{\mathcal{E}}$, we have

$$\frac{d}{d\epsilon}\bigg|_{\epsilon=0} \mathcal{E}(j_{\tilde{x}}^p \tilde{u}) = \frac{d}{d\epsilon}\bigg|_{\epsilon=0} \mathcal{E}(\Phi(\epsilon, j_x^p u)) = \frac{d}{d\epsilon}\bigg|_{\epsilon=0} \mathcal{E}(j_x^p u) = 0. \tag{5.61}$$

On the other hand, by applying the chain rule and considering the result given in (5.28), we may write

$$\frac{d}{d\epsilon}\bigg|_{\epsilon=0} \mathcal{E}(j_{\tilde{x}}^p \tilde{u}) = \frac{d}{d\epsilon}\bigg|_{\epsilon=0} \mathcal{E}(\Phi(\epsilon, j_x^p u)) = d\mathcal{E}|_{j_x^p u} \left(\frac{d\Phi_\epsilon(j_x^p u)}{d\epsilon}\bigg|_{\epsilon=0} \right)$$
$$= d\mathcal{E}|_{j_x^p u} \left(X^{(p)}(j_x^p u) \right) = X^{(p)}(\mathcal{E})(j_x^p u). \tag{5.62}$$

Therefore, $X^{(p)}[\mathcal{E}(j_x^p u)] = 0$ whenever $\mathcal{E}(j_x^p u) = 0$ for every $j_x^p u \in \mathscr{S}_{\mathcal{E}}$, i.e. $X^{(p)}(\mathcal{E}) = 0$ on $\mathscr{S}_{\mathcal{E}}$. $\qquad\square$

Theorem 5.1 provides an effective algorithmic procedure to compute the symmetry group of a given system of PDEs. This process, often called *Lie's method*, is illustrated as follows:

> Given a system of PDEs of order p of the form (5.44), let us consider an ϵ-parametric group of transformations (5.49) generated by the vector field X (5.46).

1. In the first step, we should construct the pth-order jet space $M_{(p)} \subset \mathcal{M}_{(p)}$ associated to the starting system of PDEs, defined in (5.43). Then, it is possible to introduce the prolonged vector field $X^{(p)}$ as (5.51) and the extended group of infinitesimal transformations (5.54).

2. Now, Theorem 5.1 can be applied to (5.44) as a criterion of infinitesimal invariance, providing a set of q differential equations, given by (5.60), involving the variables (x, u), the derivatives of u up to order p, the coefficients ξ_i, η^j and their prolongations up to order p.

   Alternatively, one may introduce the extended Lie point transformations (5.54) in the starting system of PDEs (5.44), and perform a Taylor expansion in the group parameter ϵ, neglecting the second order and higher contributions $\mathcal{O}(\epsilon^2)$ of the formal series. The leading-order terms allow us to recover the original untransformed equations $\mathcal{E} = 0$, and the first-order terms should provide the q differential equations (5.60).

   The restriction $\mathcal{E} = 0$ in expression (5.60) can be implemented by isolating the higher-order derivatives u_σ^j from (5.44) (if possible) and properly substituting them in the condition $X^{(p)} \left[\mathcal{E}_\nu(x, u_{(p)}) \right] = 0$.

3. We can introduce the explicit expressions for the prolonged infinitesimals η_σ^j given by (5.54). At this point, we should remind that expression (5.60) is defined on the jet space $\mathcal{M}_{(p)}$ and only Lie point symmetries are being considered, *i.e.* ξ_i and η^j do not depend on derivatives of u. Therefore, we are able to select the coefficients of the remaining unconstrained derivatives of u, treated as independent coordinates in $\mathcal{M}_{(p)}$, and set those terms equal to zero. This procedure results in an overdetermined system of linear differential equations for the infinitesimals ξ_i, η^j, called the *determining equations*, whose solution provides the desired Lie symmetries.

As it is proved in [323], the resulting system of infinitesimal generators forms a Lie algebra of symmetries, which fully characterizes the symmetry group for the considered system of PDEs (5.44).

3.4. Nonclassical symmetries

Nonclassical symmetries may be understood as a generalization of the classical symmetries that not only leave invariant the system of PDEs under study but also certain submanifolds defined by the *invariant surface conditions*. The description of the theoretical framework of the so-called *nonclassical method* can be found in [50, 326, 327]. The set of nonclassical symmetries is potentially larger than the one obtained by

Lie's classical approach, since there are fewer determining equations, and hence, more symmetries. The additional symmetries provided by this method are inaccessible from the conventional Lie's method, but they can be recovered (not always) by other procedures, as the *direct method* [98]. Nevertheless, nonclassical symmetries no longer transform all the possible solutions of the PDE into new solutions, they leave invariant just a subset of them. In addition, nonclassical symmetries will not form a Lie algebra. The computation of nonclassical symmetries requires the resolution of a smaller system of PDEs for the infinitesimals, with the significant difference that the resulting overdetermined system is nonlinear. This fact entails a remarkably harder symmetry calculation.

Let us illustrate the nonclassical method to compute nonclassical symmetries hereafter. Let us consider a system of PDEs (5.44) and let $\mathscr{S}_{\mathcal{E}} = \{(x, u_{(p)}) : \mathcal{E}_\nu(x, u_{(p)}) = 0\} \subset \mathcal{M}_{(p)}$ be the associated submanifold containing all the solutions of this system. Let us consider now a one-parameter group of infinitesimal transformations for (5.44) of the form

$$\begin{cases} \tilde{x}_i = x_i + \epsilon \xi_i(x, u) + \mathcal{O}(\epsilon^2), \\ \tilde{u}^j = u^j + \epsilon \eta^j(x, u) + \mathcal{O}(\epsilon^2), \end{cases} \qquad i = 1, \dots, n, \quad j = 1, \dots, m, \qquad (5.63)$$

associated with the vector field $X = \sum_{i=1}^{n} \xi_i(x, u) \frac{\partial}{\partial x_i} + \sum_{j=1}^{m} \eta^j(x, u) \frac{\partial}{\partial u^j}$.

The so-called *invariant surface conditions* are closely related with the characteristic of this vector field (5.56) and are given by the m system of first-order PDEs

$$\Delta^j(x, u_{(1)}) \equiv \eta^j(x, u) - \sum_{i=1}^{n} \xi_i(x, u) \frac{\partial u^j}{\partial x_i} = 0, \qquad j = 1, \dots, m, \qquad (5.64)$$

or $\Delta = 0$ for brevity.

The nonclassical method requires that the symmetry group (5.63) leaves invariant not only the system (5.44) but also conditions (5.64). This implies that only the subset of $\mathscr{S}_{\mathcal{E}}$ defined by

$$\mathscr{S}_{\mathcal{E}\Delta} = \left\{ (x, u_{(p)}) : \mathcal{E}(x, u_{(p)}) = 0, \, \Delta(x, u_{(1)}) = 0 \right\} \subset \mathscr{S}_{\mathcal{E}} \subset \mathcal{M}_{(p)} \qquad (5.65)$$

must remain invariant under the infinitesimal transformations (5.63). Other solutions for the system (5.44) in $\mathscr{S}_{\mathcal{E}}$ that do not belong to the subset $\mathscr{S}_{\mathcal{E}\Delta}$ are not necessarily transformed back to the set $\mathscr{S}_{\mathcal{E}}$.

This procedure yields the following criterion of infinitesimal invariance (analogous to Theorem 5.1) for the nonclassical symmetries

$$X^{(p)}\left[\mathcal{E}_\nu(x, u_{(p)})\right]\Big|_{\mathcal{E}=0,\,\Delta=0} = 0, \qquad \nu = 1, \ldots, q, \tag{5.66a}$$

$$X^{(1)}\left[\Delta^j(x, u_{(1)})\right]\Big|_{\mathcal{E}=0,\,\Delta=0} = 0, \qquad j = 1, \ldots, m. \tag{5.66b}$$

It is worthwhile to notice that condition (5.66b) is identically satisfied, since it can be proved that (see Appendix A.2)

$$X^{(1)}\left[\Delta^j(x, u_{(1)})\right] = \sum_{l=1}^{m} \Delta^l \frac{\partial \Delta^j}{\partial u^l}, \qquad j = 1, \ldots, m, \tag{5.67}$$

which holds for every $\Delta^l = 0$, $l = 1, \ldots, m$. This result implies that equation (5.66b) imposes no additional restriction over the coefficients of X, and hence, every classical symmetry is also a nonclassical symmetry.

As (5.66b) does not provide further constraints in the infinitesimals, the invariant surface conditions can be interpreted as restrictions on the first derivatives of the dependent variables $\{u_{x_i}^j\}$, $i = 1, \ldots, n$, $j = 1, \ldots, m$, since due to (5.64) they are not longer independent. Then, we could assume that at least one infinitesimal for the independent variables is nonzero, $\xi_\alpha \neq 0$ for a fixed $1 \leq \alpha \leq n$, and eliminate the corresponding m first derivatives of $u_{x_\alpha}^j$ from (5.64), as

$$\frac{\partial u^j}{\partial x_\alpha} = \frac{\eta^j - \sum\limits_{i=1, i\neq\alpha}^{n} \xi_i \frac{\partial u^j}{\partial x_i}}{\xi_\alpha}, \qquad \forall j = 1, \ldots, m. \tag{5.68}$$

Besides equation (5.68), we should compute its differential consequences, *i.e.*

$$\mathrm{D}^{\hat{\sigma}}\left(\Delta^j\right) = 0 \tag{5.69}$$

for every multi-index $\hat{\sigma}$ such that $|\hat{\sigma}| \leq p - 1$, for all $j = 1, \ldots, m$, in order to obtain up to the pth higher-order derivatives associated to $u_{x_\alpha}^j$.

Direct substitution of these derivatives $\{u_{x_\alpha}^j, (u_{x_\alpha}^j)_{\hat{\sigma}}\}$, $\forall j = 1, \ldots, m$ in (5.66a) and setting the remaining derivatives equal to zero, will provide the overdetermined system of PDEs for the infinitesimals whose solution gives rise to the desired nonclassical symmetries. It is immediate to see that the resulting number of determining equations is smaller than the classical one, and consequently, the set of solutions will be larger. Furthermore, the arising system of PDEs is in general highly nonlinear.

Thus, condition (5.66) can be summarize in the following compact and sufficient [346] infinitesimal criterion of invariance associated to the nonclassical symmetries

$$X^{(p)}\left[\mathcal{E}_\nu(x, u_{(p)})\right]\Big|_{\mathcal{E}=0,\,\Delta=0,\mathrm{D}^{\hat\sigma}(\Delta)=0} = 0, \qquad \nu = 1, \ldots, q,\ |\hat\sigma| \le p - 1. \qquad (5.70)$$

Computationally speaking, the algorithm to determine the nonclassical symmetries is completely analogous to the classical one described in the previous Section, with the subtlety that now the criterion of invariance is given by (5.70). Moreover, a plethora of different cases arise depending on whether the coefficients ξ_α, $1 \le \alpha \le n$ are null or not. Each of these cases should be analyzed separately, giving rise to a richer spectrum of symmetries. Unfortunately, nonclassical symmetries do not form a vector space and neither a Lie algebra, since they do not necessarily satisfy the closure property [272].

4. Symmetry reduction and group invariant solutions

One of the central applications of Lie groups on differential equations is that symmetry group techniques provide methods to obtain solutions for those differential equations. In particular, Lie symmetries may be exploited to derive exact or special solutions of a given equation in terms of solutions of lower dimensional differential equations. If a partial differential equation is invariant under a one-parameter symmetry group, then it is possible to reduce the total number of independent variables by one by means of the *invariants* of the system. In the case of ordinary differential equations, this procedure implies the reduction of order by one, where the solutions to the original equation may be recover by quadratures from the solution of the reduced problem. For first-order ODEs, this method provides an explicit formula for the general solution. This procedure is know as the *symmetry reduction method* or *similarity reduction method*, the resulting equations after the reduction process are called *reduced equations* (or simply, *reductions*) and their corresponding solutions give rise to the *similarity solutions*. A detailed description of this technique is found in [51, 52, 323].

Let us consider a system of PDEs of order p of the form (5.44) and let (5.49) be a one-parameter group of infinitesimal transformations on M for this system generated by the vector field (5.46). A *(local) invariant* of the group of transformations (5.49) is a real-valued smooth function $F : M \to \mathbb{R}$ such that F is an invariant under the action of the group, *i.e.*

$$F(\tilde{x}, \tilde{u}) = F(g \cdot (x, u)) = F(x, u) \qquad (5.71)$$

for all $g \in G$. Then, $F(x, u)$ is solution of the following linear, homogeneous, first-order partial differential equation

$$X(F) = \sum_{i=1}^{n} \xi_i \frac{\partial F}{\partial x_i} + \sum_{j=1}^{m} \eta^j \frac{\partial F}{\partial u^j} = 0 \qquad (5.72)$$

defined on M. The general solution of equation (5.72) may be achieved by the so-called *method of characteristics* [163], which implies the integration of the *characteristic system* of ordinary differential equations

$$\frac{dx_1}{\xi_1(x,u)} = \cdots = \frac{dx_n}{\xi_n(x,u)} = \frac{du^1}{\eta^1(x,u)} = \cdots = \frac{du^m}{\eta^m(x,u)}. \qquad (5.73)$$

The solution of system (5.73) gives rise to a $n+m-1$ constants of integration, which allow to construct $n+m-1$ functionally independent invariants on M.

The standard procedure to solve the system of ODEs defined above is described in the following. We should fix one of the independent variables, x_n without loss of generality, and integrate the system (5.73) by pairs. The *reduced variables* are obtained by integrating the subsystem of (5.73) involving the independent variables

$$dz_1 = \frac{dx_1}{\xi_1} - \frac{dx_n}{\xi_n} = 0, \quad dz_2 = \frac{dx_2}{\xi_2} - \frac{dx_n}{\xi_n} = 0, \quad \dots, \quad dz_{n-1} = \frac{dx_{n_1}}{\xi_{n-1}} - \frac{dx_n}{\xi_n} = 0, \qquad (5.74)$$

giving rise to a set of $n-1$ new reduced variables $z = (z_1, \dots, z_{n-1})$, which enables us to reduce in one the number of independent variables of the associated reduced problem. The *reduced (scalar) fields* are given by the integration of the system

$$dU^1 = \frac{du^1}{\eta^1} - \frac{dx_n}{\xi_n} = 0, \quad dU^2 = \frac{du^2}{\eta^2} - \frac{dx_n}{\xi_n} = 0, \quad \dots, \quad dU^m = \frac{du^m}{\eta^m} - \frac{dx_n}{\xi_n} = 0, \qquad (5.75)$$

providing the remaining m invariants $U = (U^1, \dots, U^m)$ such that $U = U(z)$. It is worthwhile to remark that any smooth combination of functions involving the reduced variables and/or reduced fields (z, U) is also and invariant (of the same kind), since it is solution of (5.72).

Let us denote the following sets of coordinates as $\hat{x} = (x_1, \dots, x_{n-1})$. Hence, the general solution for the characteristic system (5.73) may be expressed as

$$z_k = z_k(\hat{x}, x_n, u), \qquad U^l = U^l(\hat{x}, x_n, u) \qquad \forall k = 1, \dots, n-1, \, l = 1, \dots, m. \qquad (5.76)$$

We can always invert relations (5.76) such that

$$x_i = x_i(x_n, z, U(z)), \qquad u^j = u^j(x_n, z, U(z)), \qquad \forall i = 1, \dots, n-1, \, j = 1, \dots, m. \qquad (5.77)$$

The above change of variables induces a transformation over the derivatives of u^j, $j =$

$1, \ldots, m$ up to order p, which can be easily computed by means of the chain rule. Hence, the desired reduced problem is obtained by introducing the change of variables (5.77) and their differential consequences over the derivatives of u into the starting problem (5.44), giving rise to a system of q partial differential equations of order p depending on $n-1$ independent variables $z = (z_1, \ldots, z_{n-1})$ and m dependent variables $U = (U^1, \ldots, U^m)$, of the form

$$\hat{\mathcal{E}}_\nu(z, U_{(p)}) = 0, \qquad \nu = 1, \ldots, q. \tag{5.78}$$

In particular, if $n = 2$, the starting problem (5.44) reduces to a system of q ordinary differential equations.

5. Toy example revisited: NLS equation in $1+1$ dimensions

Let us illustrate the main procedure of Lie's symmetry method described above for the toy example considered before, the famous NLS equation [3, 13, 130] defined in (2.58)

$$
\begin{aligned}
iu_t + u_{xx} - 2u^2 w &= 0, \\
-iw_t + w_{xx} - 2w^2 u &= 0.
\end{aligned}
\tag{5.79}
$$

This system of PDEs is defined in $1+1$ dimensions, it is expressed in terms of two independent variables, the spatial coordinate x and time t, which implies $n = 2$, and two complex conjugate scalar fields $\{u, w\}$, which provides $m = 2$. The associated jet space $\mathcal{M}_{(2)}$ may be constructed as in Example 5.8 for $p = 2$, with local coordinates

$$\left(x, t, u, w, u_x, u_t, w_x, w_t, u_{xx}, u_{xt}, u_{tt}, w_{xx}, w_{xt}, w_{tt}\right), \tag{5.80}$$

so that the system (5.79) can be considered as a submanifold in $\mathcal{M}_{(2)}$.

5.1. Classical Lie symmetries for NLS

Let us consider the following ϵ-parametric group of infinitesimal transformations

$$
\begin{cases}
\tilde{x} = x \ + \epsilon\xi_1(x, t, u, w) + \mathcal{O}(\epsilon^2), \\
\tilde{t} = t \ + \epsilon\xi_2(x, t, u, w) + \mathcal{O}(\epsilon^2), \\
\tilde{u} = u \ + \epsilon\eta_1(x, t, u, w) + \mathcal{O}(\epsilon^2), \\
\tilde{w} = w + \epsilon\eta_2(x, t, u, w) + \mathcal{O}(\epsilon^2),
\end{cases}
\tag{5.81}
$$

generated by the vector field

$$X = \xi_1 \frac{\partial}{\partial x} + \xi_2 \frac{\partial}{\partial t} + \eta_1 \frac{\partial}{\partial u} + \eta_2 \frac{\partial}{\partial w}. \tag{5.82}$$

It is possible to extend these infinitesimal transformations up to second-order derivatives,

$$\begin{cases}
\tilde{u}_{\tilde{x}} = u_x + \epsilon(\eta_1)_x(x, t, u_{(1)}, w_{(1)}) + \mathcal{O}(\epsilon^2), \\
\tilde{u}_{\tilde{t}} = u_t + \epsilon(\eta_1)_t(x, t, u_{(1)}, w_{(1)}) + \mathcal{O}(\epsilon^2), \\
\tilde{w}_{\tilde{x}} = w_x + \epsilon(\eta_2)_x(x, t, u_{(1)}, w_{(1)}) + \mathcal{O}(\epsilon^2), \\
\tilde{w}_{\tilde{t}} = w_t + \epsilon(\eta_2)_t(x, t, u_{(1)}, w_{(1)}) + \mathcal{O}(\epsilon^2), \\
\tilde{u}_{\tilde{x}\tilde{x}} = u_{xx} + \epsilon(\eta_1)_{xx}(x, t, u_{(2)}, w_{(2)}) + \mathcal{O}(\epsilon^2), \\
\tilde{u}_{\tilde{x}\tilde{t}} = u_{xt} + \epsilon(\eta_1)_{xt}(x, t, u_{(2)}, w_{(2)}) + \mathcal{O}(\epsilon^2), \\
\tilde{u}_{\tilde{t}\tilde{t}} = u_{tt} + \epsilon(\eta_1)_{tt}(x, t, u_{(2)}, w_{(2)}) + \mathcal{O}(\epsilon^2), \\
\tilde{w}_{\tilde{x}\tilde{x}} = w_{xx} + \epsilon(\eta_2)_{xx}(x, t, u_{(2)}, w_{(2)}) + \mathcal{O}(\epsilon^2), \\
\tilde{w}_{\tilde{x}\tilde{t}} = w_{xt} + \epsilon(\eta_2)_{xt}(x, t, u_{(2)}, w_{(2)}) + \mathcal{O}(\epsilon^2), \\
\tilde{w}_{\tilde{t}\tilde{t}} = w_{tt} + \epsilon(\eta_2)_{tt}(x, t, u_{(2)}, w_{(2)}) + \mathcal{O}(\epsilon^2),
\end{cases} \tag{5.83}$$

where the corresponding prolonged vector field reads

$$X^{(2)} = \xi_1 \frac{\partial}{\partial x} + \xi_2 \frac{\partial}{\partial t} + \eta_1 \frac{\partial}{\partial u} + \eta_2 \frac{\partial}{\partial w} + (\eta_1)_x \frac{\partial}{\partial u_x} + (\eta_1)_t \frac{\partial}{\partial u_t} + (\eta_2)_x \frac{\partial}{\partial w_x} + (\eta_2)_t \frac{\partial}{\partial w_t}$$

$$+ (\eta_1)_{xx} \frac{\partial}{\partial u_{xx}} + (\eta_1)_{xt} \frac{\partial}{\partial u_{xt}} + (\eta_1)_{tt} \frac{\partial}{\partial u_{tt}} + (\eta_2)_{xx} \frac{\partial}{\partial w_{xx}} + (\eta_2)_{xt} \frac{\partial}{\partial w_{xt}} + (\eta_2)_{tt} \frac{\partial}{\partial w_{tt}}. \tag{5.84}$$

By applying Theorem 5.1 to (5.79), we arrive at the following system of determining equations

$$i(\eta_1)_t + (\eta_1)_{xx} - 4uw\,\eta_1 - 2u^2\eta_2 = 0, \\
-i(\eta_2)_t + (\eta_2)_{xx} - 4uw\,\eta_2 - 2w^2\eta_1 = 0, \tag{5.85}$$

under the restriction imposed by the original NLS system $u_{xx} = -iu_t + 2u^2 w$, $w_{xx} = iw_t + 2uw^2$. The prolonged infinitesimals $(\eta_j)_t$, $(\eta_j)_{xx}$, $j = 1, 2$ can be computed by the general prolongation formula (5.54), and the substitution of the expressions for u_{xx} and w_{xx} in (5.85) yields

$$2\left(\xi_{2,u}u_x + \xi_{2,w}w_x + \xi_{2,x}\right)u_{xt} + \xi_{1,uu}u_x^3 + \left(-\frac{\partial^2\eta_1}{\partial u^2} + 2\xi_{1,uw}w_x + 2\xi_{1,xu} + \xi_{2,uu}u_t\right)u_x^2$$

$$+ \left[2\left(\xi_{1,xw} - \frac{\partial^2\eta_1}{\partial u\partial w} + \xi_{2,uw}u_t\right)w_x - 2\frac{\partial^2\eta_1}{\partial x\partial u} + 2\xi_{1,u}(3u^2w - iu_t) + 2\xi_{1,w}(uw^2 + iw_t)\right]$$

$$+ \xi_{1,xx} + i\xi_{1,t}\big]u_x + \left[i\xi_{2,t} - 2i\xi_{1,x} + 2\xi_{2,u}u^2w + 2\xi_{2,w}(w_t + uw^2) + \xi_{2,xx} + 2\xi_{2,xu}u_x\right]u_t$$

$$+ \left(\xi_{1,ww} u_x + \xi_{2,ww} u_t - \frac{\partial^2 \eta_1}{\partial w^2} \right) w_x^2 + 2 \left[(\xi_{2,xw} - i\xi_{1,w}) u_t - \frac{\partial^2 \eta_1}{\partial x \partial w} + 2\xi_{1,w} u^2 w \right] w_x$$

$$- 2i \frac{\partial \eta_1}{\partial w} w_t + \left[2 \left(2\xi_{1,x} - \frac{\partial \eta_1}{\partial u} \right) u^2 w - 2 \frac{\partial \eta_1}{\partial u} uw^2 - \frac{\partial^2 \eta_1}{\partial x^2} - i \frac{\partial \eta_1}{\partial t} + 4\eta_1 uw + 2\eta_2 u^2 \right] = 0,$$

$$(5.86)$$

$$2 \left(\xi_{2,u} u_x + \xi_{2,w} w_x + \xi_{2,x} \right) w_{xt} + \xi_{1,ww} w_x^3 + \left(-\frac{\partial^2 \eta_2}{\partial w^2} + 2\xi_{1,uw} u_x + 2\xi_{1,xw} + \xi_{2,ww} w_t \right) w_x^2$$

$$+ \left[2 \left(\xi_{1,xu} - \frac{\partial^2 \eta_2}{\partial u \partial w} + \xi_{2,uw} w_t \right) u_x - 2 \frac{\partial^2 \eta_2}{\partial x \partial w} + 2\xi_{1,w} (3uw^2 + iw_t) + 2\xi_{1,u} (u^2 w - iu_t) \right.$$

$$\left. + \xi_{1,xx} - i\xi_{1,t} \right] w_x + \left[-i\xi_{2,t} + 2i\xi_{1,x} + 2\xi_{2,w} uw^2 + 2\xi_{2,u} (u_t + u^2 w) + \xi_{2,xx} + 2\xi_{2,xw} w_x \right] w_t$$

$$+ \left(\xi_{1,uu} w_x + \xi_{2,uu} w_t - \frac{\partial^2 \eta_2}{\partial u^2} \right) u_x^2 + 2 \left[(\xi_{2,xu} + i\xi_{1,u}) w_t - \frac{\partial^2 \eta_2}{\partial x \partial u} + 2\xi_{1,u} uw^2 \right] u_x$$

$$+ 2i \frac{\partial \eta_2}{\partial u} u_t + \left[2 \left(2\xi_{1,x} - \frac{\partial \eta_2}{\partial w} \right) uw^2 - 2 \frac{\partial \eta_2}{\partial u} u^2 w - \frac{\partial^2 \eta_2}{\partial x^2} + i \frac{\partial \eta_2}{\partial t} + 4\eta_2 uw + 2\eta_1 w^2 \right] = 0,$$

$$(5.87)$$

where the subscripts $\{x, t, u, w\}$ denote partial derivation with respect the corresponding variables. We should now extract the coefficients in the different derivatives of u and w and equate them to zero, leading to an overdetermined system of linear PDEs for the infinitesimals $\{\xi_1, \xi_2, \eta_1, \eta_2\}$, whose solution provides

$$\xi_1(x, t, u, w) = a_1 + a_4 x + a_5 t,$$
$$\xi_2(x, t, u, w) = a_2 + 2a_4 t,$$
$$\eta_1(x, t, u, w) = \left(\frac{ia_5}{2} x - a_4 + ia_3 \right) u, \tag{5.88}$$
$$\eta_2(x, t, u, w) = \left(-\frac{ia_5}{2} x - a_4 - ia_3 \right) w,$$

where a_i, $i = 1, \ldots, 5$ are arbitrary real constants. Relations (5.88) constitute the classical Lie symmetries for the NLS equation [92, 165, 166]. The associated vector fields define the transformations

$$
\begin{array}{lll}
X_1 = \partial_x & \text{Space translation} & \\
X_2 = \partial_t & \text{Time translation} & \\
X_3 = i \left(u\partial_u - w\partial_w \right) & \text{Phase translation} & \\
X_4 = x\partial_x + 2t\partial_t - u\partial_u - w\partial_w & \text{Scaling} & (5.89) \\
X_5 = t\partial_x + \frac{ix}{2} \left(u\partial_u - w\partial_w \right) & \text{Galilean boost} &
\end{array}
$$

and constitutes the basis of a five-dimensional Lie algebra with nontrivial commutation relations

$$[X_1, X_4] = X_1, \qquad [X_1, X_5] = \frac{1}{2}X_3, \qquad [X_2, X_4] = 2X_2,$$
$$[X_2, X_5] = X_1, \qquad [X_4, X_5] = X_5. \tag{5.90}$$

This algebra is solvable and isomorphic to the one-dimensional extended Galilei-similitude algebra $\mathfrak{sg}(1)$, subalgebra of the so-called Schrödinger algebra $\mathfrak{sch}(1)$ [180, 356]. The matter of classification of symmetry groups associated to generalized versions of n-dimensional Schödinger-like equations has been a subject of keen interest during the past years [58, 59, 62, 177, 178, 315], where Schrödinger groups $\mathrm{Sch}(n)$ and extended Galilean groups $\mathrm{Gal}(n)$ play an important role. In particular, for the one-dimensional case, the group $\mathrm{Sch}(1) \simeq SL(2,\mathbb{R}) \rhd W_1$ naturally arises, expressed as the semidirect product of $SL(2,\mathbb{R})$ and the Weyl-Heisenberg group W_1, representing the symmetry group of the $(1+1)$-linear Schrödinger equation for the free particle [59, 315].

5.2. Nonclassical Lie symmetries for NLS

Nonclassical Lie symmetries for NLS equation (5.79) can be analogously computed by applying the procedure described in Subsection 3.4 of this Chapter.

Together with the extended group of infinitesimal transformations (5.84), we should impose the invariant surface conditions (5.64), which are given by

$$\Delta \equiv \begin{cases} \eta_1 - \xi_1 u_x - \xi_2 u_t = 0, \\ \eta_2 - \xi_1 w_x - \xi_2 w_t = 0. \end{cases} \tag{5.91}$$

In view of equations (5.91), the nonclassical method applied to (5.79) gives rise to two different cases of analysis:

1. $\xi_2 \neq 0$

   Without loss of generality, we may set $\xi_2 = 1$ in order to simplify the ensuing calculations. Then, solving (5.91) for u_t, w_t, we get

   $$u_t = \eta_1 - \xi_1 u_x, \qquad w_t = \eta_1 - \xi_1 w_x, \tag{5.92}$$

   whose differential consequences up to second order imply

   $$u_{xt} = \mathrm{D}_x(\eta_1) - \mathrm{D}_x(\xi_1)u_x - \xi_1 u_{xx}, \qquad u_{tt} = \mathrm{D}_t(\eta_1) - \mathrm{D}_t(\xi_1)u_x - \xi_1 u_{xt},$$
   $$w_{xt} = \mathrm{D}_x(\eta_2) - \mathrm{D}_x(\xi_1)w_x - \xi_1 w_{xx}, \qquad w_{tt} = \mathrm{D}_t(\eta_2) - \mathrm{D}_t(\xi_1)w_x - \xi_1 w_{xt}. \tag{5.93}$$

   Finally, by applying condition (5.70) to (5.79), computing the corresponding prolonged infinitesimal by the general prolongation formula, and properly substitut-

ing the restrictions imposed by (5.79) itself, (5.92) and (5.93), one arrives at the following system of nonlinear PDEs

$$\xi_{1,uu}u_x^3 + \left(2\xi_{1,xu} + 2i\xi_{1,u}\xi_1 - \frac{\partial^2\eta_1}{\partial u^2} + 2\xi_{1,uw}w_x\right)u_x^2 + \left(\xi_{1,ww}u_x - \frac{\partial^2\eta_1}{\partial w^2}\right)w_x^2$$

$$+ 2\left(\xi_{1,xw} - \frac{\partial^2\eta_1}{\partial u\partial w}\right)u_x w_x + 2\left[2\xi_{1,w}u^2 w - \frac{\partial^2\eta_1}{\partial x\partial w} + i(\frac{\partial\eta_1}{\partial w}\xi_1 - \xi_{1,w}\eta_1)\right]w_x$$

$$+ 2\left[(3\xi_{1,u}u + \xi_{1,w}w)uw - \frac{\partial^2\eta_1}{\partial x\partial u} + i(\xi_{1,w}\eta_2 - \xi_{1,u} + \xi_{1,x}\xi_1\eta_1) + \frac{1}{2}(i\xi_{1,t} + \xi_{1,xx})\right]u_x$$

$$+ 2\left\{\left[\left(2\xi_{1,x} - \frac{\partial\eta_1}{\partial u}\right)w + \eta_2\right]u^2 + \left[2\eta_1 - \frac{\partial\eta_1}{\partial w}w\right]uw - i\left(\frac{\partial\eta_1}{\partial w}\eta_2 + \xi_{1,x}\eta_1\right)\right\}$$

$$- \left(\frac{\partial\eta_1}{\partial t} + \frac{\partial^2\eta_1}{\partial x^2}\right) = 0,$$

(5.94)

$$\xi_{1,ww}w_x^3 + \left(2\xi_{1,xw} - 2i\xi_{1,w}\xi_1 - \frac{\partial^2\eta_2}{\partial w^2} + 2\xi_{1,uw}u_x\right)w_x^2 + \left(\xi_{1,uu}w_x - \frac{\partial^2\eta_2}{\partial u^2}\right)u_x^2$$

$$+ 2\left(\xi_{1,xu} - \frac{\partial^2\eta_2}{\partial u\partial w}\right)u_x w_x + 2\left[2\xi_{1,u}uw^2 - \frac{\partial^2\eta_2}{\partial x\partial u} - i\left(\frac{\partial\eta_2}{\partial u}\xi_1 - \xi_{1,u}\eta_2\right)\right]u_x$$

$$+ 2\left[(3\xi_{1,w}w + \xi_{1,u}u)uw - \frac{\partial^2\eta_2}{\partial x\partial w} - i(\xi_{1,u}\eta_1 - \xi_{1,w} + \xi_{1,x}\xi_1\eta_2) - \frac{1}{2}(i\xi_{1,t} - \xi_{1,xx})\right]w_x$$

$$+ 2\left\{\left[\left(2\xi_{1,x} - \frac{\partial\eta_2}{\partial w}\right)u + \eta_1\right]w^2 + \left[2\eta_2 - \frac{\partial\eta_2}{\partial u}u\right]uw + i\left(\frac{\partial\eta_2}{\partial u}\eta_1 + \xi_{1,x}\eta_2\right)\right\}$$

$$- \left(\frac{\partial\eta_2}{\partial t} + \frac{\partial^2\eta_2}{\partial x^2}\right) = 0,$$

(5.95)

whose solutions for the infinitesimals have the following form

$$\xi_1 = \frac{b_1 + x + b_4 t}{b_2 + 2t},$$

$$\xi_2 = 1,$$

$$\eta_1 = \frac{i\frac{b_4}{2}x - 1 + ib_3}{b_2 + 2t}u,$$

$$\eta_2 = -\frac{i\frac{b_4}{2}x + 1 + ib_3}{b_2 + 2t}w,$$

(5.96)

where b_i, $i = 1, \ldots, 4$ are arbitrary parameters. Nonclassical symmetries for NLS (5.96) do not provide further information to the symmetry analysis, and, in fact, they fully coincide with the classical ones (5.88) in that case.

2. $\xi_2 = 0$, $\xi_1 \neq 0$

For this case, we take $\xi_1 = 1$, and the invariant surface conditions (5.64) and their differential invariants provide

$$
\begin{aligned}
u_x &= \eta_1, & u_{xx} &= \mathrm{D}_x(\eta_1), & u_{xt} &= \mathrm{D}_t(\eta_1), \\
w_x &= \eta_2, & w_{xx} &= \mathrm{D}_x(\eta_2), & w_{xt} &= \mathrm{D}_t(\eta_2),
\end{aligned}
\tag{5.97}
$$

which yields the following system of PDEs for the infinitesimals (that do not depend on any derivative of the dependent variables)

$$
\begin{aligned}
&i\frac{\partial \eta_1}{\partial t} + \frac{\partial^2 \eta_1}{\partial x^2} + 2uw\left(\frac{\partial \eta_1}{\partial u}u - \frac{\partial \eta_1}{\partial w}w\right) - 4\eta_1 uw - 2\eta_2 u^2 + \frac{\partial(\eta_1^2)}{\partial w}\frac{\partial \eta_2}{\partial u} + \frac{\partial(\eta_2^2)}{\partial w}\frac{\partial \eta_1}{\partial w} \\
&+ 2\frac{\partial \eta_1}{\partial w}\frac{\partial \eta_2}{\partial x} + 2\eta_1\eta_2\frac{\partial^2 \eta_1}{\partial u \partial w} + 2\eta_1\frac{\partial^2 \eta_1}{\partial x \partial u} + 2\eta_2\frac{\partial^2 \eta_1}{\partial x \partial w} + \eta_1^2\frac{\partial^2 \eta_1}{\partial u^2} + \frac{\partial^2 \eta_1}{\partial w^2}\eta_2^2 = 0, \\
&-i\frac{\partial \eta_2}{\partial t} + \frac{\partial^2 \eta_2}{\partial x^2} - 2uw\left(\frac{\partial \eta_2}{\partial u}u - \frac{\partial \eta_2}{\partial w}w\right) - 4\eta_2 uw - 2\eta_1 w^2 + \frac{\partial(\eta_1^2)}{\partial u}\frac{\partial \eta_2}{\partial u} + \frac{\partial(\eta_2^2)}{\partial u}\frac{\partial \eta_1}{\partial w} \\
&+ 2\frac{\partial \eta_2}{\partial u}\frac{\partial \eta_1}{\partial x} + 2\eta_1\eta_2\frac{\partial^2 \eta_2}{\partial u \partial w} + 2\eta_2\frac{\partial^2 \eta_2}{\partial x \partial w} + 2\eta_1\frac{\partial^2 \eta_2}{\partial x \partial u} + \eta_2^2\frac{\partial^2 \eta_2}{\partial w^2} + \frac{\partial^2 \eta_2}{\partial u^2}\eta_1^2 = 0,
\end{aligned}
\tag{5.98}
$$

whose solution is

$$
\begin{aligned}
\xi_1 &= 1, \\
\eta_1 &= \frac{i(x + c_1)}{c_2 + 2t}\, u, \\
\eta_2 &= -\frac{i(x + c_1)}{c_2 + 2t}\, w,
\end{aligned}
\tag{5.99}
$$

where c_1, c_2 are arbitrary constants. This result constitutes, again, a subset of the classical Lie symmetries for NLS.

The fact that the nonclassical method does not produce any result beyond the set of classical Lie symmetries for NLS equation was already noticed in [272]. Unfortunately, this situation also occurs for other famous integrable equations, such as KdV or mKdV [98]. Nevertheless, some other systems as Burgers equation do possess additional symmetries and reductions due to the nonclassical method [346].

5.3. Symmetry reduction: travelling wave solution

Let us implement, as an illustrative example, the reduction method as a tool to derive solutions for the NLS equation in $1 + 1$ dimensions (5.79). We will be interested in travelling wave solutions, *i.e.* solutions of the form $u(x, t) = f(x - vt)$ with velocity

v. In order to achieve that, let us consider the following vector field

$$X = vX_1 + X_2 + \gamma X_3 = v\frac{\partial}{\partial x} + \frac{\partial}{\partial t} + i\gamma\left(u\frac{\partial}{\partial u} - w\frac{\partial}{\partial w}\right), \tag{5.100}$$

where we have taken $\gamma \neq 0$ in order to allow additional phase shifts in the fields. The associated characteristic system to (5.100) is given by

$$\frac{dx}{v} = \frac{dt}{1} = \frac{du}{i\gamma u} = -\frac{dw}{i\gamma w}. \tag{5.101}$$

The integration by pairs with the variable t of the system above results in the reduced variable and the reduced fields of the form

$$z = x - vt, \qquad U(z) = u(x,t)e^{i\gamma t}, \qquad W(z) = w(x,t)e^{-i\gamma t}. \tag{5.102}$$

Substitution into the original equations (5.79) provides the reduced equations as the coupled system of ODEs

$$
\begin{aligned}
U'' &= ivU' + \gamma U + 2U^2 W, \\
W'' &= -ivW' + \gamma W + 2UW^2,
\end{aligned}
\tag{5.103}
$$

where $U = U(z)$, $W = W(z)$ and the prime denotes derivation with respect to z, $' \equiv \frac{d}{dz}$.

It is immediate to see that throughout this procedure, we have been able to reduce the order by one, starting from the $(1 + 1)$-NLS equation (5.79) and leading to the system of ODEs (5.103) in the reduced variable z. In order to solve (5.103), let us introduce the change of coordinates

$$U(z) = \rho(z)e^{i\varphi(z)}, \qquad W(z) = \rho(z)e^{-i\varphi(z)},$$

where $\rho(z)$ and $\varphi(z)$ are real functions. Then, system (5.103) reads

$$\rho'' = \rho\left(\varphi'^2 - v\rho'\right) + \rho\left(\gamma + 2\rho^2\right), \tag{5.104a}$$

$$\varphi'' = -\left(2\varphi' - v\right)\frac{\rho'}{\rho}. \tag{5.104b}$$

Equation (5.104b) can be easily integrated as

$$\varphi' = \frac{v}{2} + \frac{I_2}{\rho^2}, \qquad I_2 \in \mathbb{R},$$

with I_2 as the arbitrary constant of integration. Thus, equation (5.104a) gives rise

to the following second-order decoupled ODE for ρ

$$\rho'' = 2\rho^3 + \left(\gamma - \frac{v^2}{4}\right)\rho + \frac{I_2^2}{\rho^3}. \tag{5.105}$$

According to the seminal work of P. Painlevé [331,332], one should be able to classify this equation in one of the 50 standard Painlevé type equations. Implementing the ansatz $\rho(z) = R(z)^{\frac{1}{2}}$, equation (5.105) may be easily expressed into the standard form [228]

$$R'' = \frac{R'^2}{2R} + 4R^2 + 2aR + \frac{2I_2^2}{R}, \tag{5.106}$$

where we have defined $a = \gamma - \frac{v^2}{4}$. Equation (5.106) corresponds to equation XXXIII[5] [228], and its first integral is

$$R'^2 = 4\left(R^3 + aR^2 + I_1 R - I_2^2\right), \tag{5.107}$$

where I_1 is the arbitrary constant of integration. This first integral is solvable in terms of Jacobi and Weierstrass elliptic functions, giving rise to a plethora of diverse solutions, depending on the different values of the parameters $\{a, I_2\}$, with different behaviours for $R(z)$. Solutions of this kind have been extensively studied in [179].

Let us examine, for example, the travelling wave solutions arising from the case $a \neq 0$, $I_2 = 0$. The form of the solutions depends on the value of I_1, as well as the nature of the roots of the polynomial in the right-hand side of (5.107). A particular solution[6] for this case reads

$$R(z) = \delta_1^2 \operatorname{sn}^2\left(\delta_2 z + I_3, k\right), \tag{5.108}$$

where $\operatorname{sn}(\cdot)$ denotes the elliptic sine function, I_3 is the constant of integration and the parameters $\{\delta_1, \delta_2\}$ and the elliptic modulus k are given by

$$\delta_1^2 = -\frac{a + \sqrt{a^2 - 4I_1}}{2}, \qquad \delta_2^2 = \frac{-a + \sqrt{-a^2 - 4I_1}}{2},$$
$$k^2 = -\frac{a + \sqrt{a^2 - 4I_1}}{-a + \sqrt{a^2 - 4I_1}}. \tag{5.109}$$

---

[5]This classification arises in the most general case when $a \neq 0$, $I_2 \neq 0$. If $a = 0 = I_2$, equation (5.106) corresponds to equation XVIII in [228], and if $a \neq 0$, $I_2 = 0$, to equation XIX, respectively. The corresponding first integrals for these subcases can be easily recover by selecting the particular values of $\{a, I_2\}$ in (5.107).

[6]This solution is valid in the regime $\frac{a^2}{4} > I_1 > 0$, where the elliptic integral (5.107) has three real and different roots. Taking into account that $R(z) > 0$, the solution (5.108) has been computed following expression 233.00 in [66].

Thus, this result provides

$$\rho(z) = \delta_1 \operatorname{sn}(\delta_2 z + I_3, k), \qquad \varphi(z) = \frac{v}{2} z + I_4, \tag{5.110}$$

being I_4 a constant of integration, which implies that the reduced fields are

$$U(z) = \delta_1 \operatorname{sn}(\delta_2 z + I_3, k) \, e^{i\left(\frac{v}{2} z + I_4\right)}, \quad W(z) = \delta_1 \operatorname{sn}(\delta_2 z + I_3, k) \, e^{-i\left(\frac{v}{2} z + I_4\right)}, \tag{5.111}$$

which identically satisfy relations (5.103). Finally, the solutions for the original fields u and w have the form

$$u(x, t) = \delta_1 \operatorname{sn}[\delta_2 (x - vt) + I_3, k] \, e^{i\left[\frac{v}{2}(x - vt) + I_4 + \gamma t\right]},$$
$$w(x, t) = \delta_1 \operatorname{sn}[\delta_2 (x - vt) + I_3, k] \, e^{-i\left[\frac{v}{2}(x - vt) + I_4 + \gamma t\right]}, \tag{5.112}$$

which constitutes bounded and periodic travelling waves for (5.79) where the parameters are given in (5.109). According to [179], there exist no other regular travelling wave solutions for the $(1 + 1)$-defocusing NLS equation (5.79) with the choice of $I_2 = 0$ arising from symmetry reductions of the form (5.100).

It is worth studying the limiting case $k = 1$ arising from the elliptic functions, which is equivalent to consider $I_1 = \frac{a^2}{4}$. Hence, renaming $a = -2\delta^2$, solutions (5.112) behave as

$$u(x, t) = \delta \tanh^2 [\delta (x - vt) + I_3] \, e^{i\left[\frac{v}{2}(x - vt) + \left(\frac{v^2}{2} - 2\delta^2\right)t + I_4\right]},$$
$$w(x, t) = \delta \tanh^2 [\delta (x - vt) + I_3] \, e^{-i\left[\frac{v}{2}(x - vt) + \left(\frac{v^2}{2} - 2\delta^2\right)t + I_4\right]}, \tag{5.113}$$

which represens a bounded and non-periodic solitary wave known as *kink*.

Chapter 6

Lie symmetries for spectral problems

This Chapter is devoted to the applications of Lie's formalism of classical (and non-classical) symmetries to several integrable models arising from the fields of Mathematical Physics and Materials Sciences. Lie's procedure will yield the direct computation of the classical or nonclassical symmetries, which allows us to characterize the underlying Lie algebra structure of such systems and investigate the associated similarity reductions.

As already mentioned in Chapter 5, Lie symmetries were first introduced by S. Lie [273–275] as a valuable technique to solve differential equations or reduce a system of equations to a simpler form. The overwhelming advantages of this procedure are evidenced by its countless successful applications when dealing with the study of differential equations [52, 323, 378].

Throughout this thesis we will consider a slightly different approach. In this research we will be more concerned about the application of Lie's method to the spectral problem associated with these integrable systems. Lie symmetry analysis for any kind of differential equations has been an extensively studied topic in the last decades. Nevertheless, much less frequent is the identification of Lie symmetries for the associated Lax pair in the case of integrable nonlinear problems. The first contributions to this matter were developed in [268, 292] and a fruitful work in this topic has been conducted by Estévez and collaborators [61, 143, 144, 154–156].

The inspection of Lax pairs is interesting since they are considered a proof of integrability for the corresponding nonlinear differential equation [3, 13]. The associated linear problem can be characterized in terms of a system of PDEs, which linearly depends on a new set of dependent variables, the eigenfunctions, and an additional element: the spectral parameter (either isospectral or nonisospectral). As the Lax pair constitutes an equivalent system to the former nonlinear equation, it is expected that the symmetries of the spectral problem include the symmetries of the starting

PDE [268]. Furthermore, Lie symmetries for Lax pairs yield much more information than just the symmetries of a set of PDEs. This procedure allows us not only to obtain the symmetry transformations of the fields, but also provides how the eigenfunctions and the spectral parameter are transformed under the action of the symmetry group [24, 25, 27, 28, 144, 155].

This fact will play a crucial role in the study of symmetry reductions arising from those systems. As already illustrated in the previous Chapter, each Lie point symmetry leads to a reduced version of the equation with the number of independent variables diminished by one. Similarity reductions, when applied to a given Lax pair, require to consider the eigenfunctions as usual fields and include the spectral parameter as an independent variable for the eigenfunctions. This process allows to simultaneously derive the reduction of the fields together with the reduction of the eigenfunctions and the spectral parameter itself. Hence, this formalism straightforwardly yields new families of integrable differential equations in lower dimensions together with their associated spectral problems. This whole procedure is also applicable for hierarchies of PDEs [154, 155].

Another core component of the symmetry analysis for Lax pairs rests on the spectral parameter. Particular importance should be given to nonisospectral Lax pairs, *i.e.* associated linear problems where the spectral parameter is not constant, but a function of the independent variables of the system. In those cases, the spectral problem generally satisfies an additional differential equation[1] called *nonisospectral condition* or *nonisospectral equation*. Nonisopectral Lax pairs generally appear in multidimensional integrable systems in $2+1$ or higher dimensions, which typically describe more realistic models and display a richer dynamics [247]. From the symmetry point of view, the spectral parameter should be treated as a scalar field and the reduction procedure shall be extended to the nonisospectral condition. Thus, Lie symmetries for nonisospectral Lax pairs allow us to analyze how the nonisospectral condition propagates under the reduction, and eventually obtain nontrivial nonisospectral reduced problems in lower dimensions [24, 155, 156].

This Chapter is organized as follows. Section 1 is aimed at the application of Lie's symmetry method to nonlinear systems in $1 + 1$ dimensions and their associated spectral problems. We analyze three nonlinear differential equations of interest, arising from the integrability analysis for the generalized DNLS equation described in Section 2 from Chapter 3. As it has been shown, each of these equations possesses two equivalent Lax pairs that are isospectral. The associated similarity reductions will provide two equivalent reduced Lax pairs for each reduced ODE, linked by the

---

[1]Notice that the nonisospectral condition and the spectral problem do not necessarily present a linear dependence with the spectral parameter.

reduction of the compatibility condition in each case. The research conducted in this Section is partially covered in publications [23, 28]. Section 2 is dedicated to the symmetry analysis performed over nonlinear models in $2+1$ dimensions. The dynamics in multidimensional systems is richer than in the $(1+1)$-case due to the versatile nature of the spectral parameter. In this Section we will analyze two examples, following the constributions [24, 25], respectively. The first example accounts for an integrable generalization of the NLS equation in $2+1$ dimensions with an associated nonisospectral Lax pair. The second example displays a multi-component generalization of NLS equation with a Lax pair that does not possess a spectral parameter. In this case, the proper process of symmetry reduction for the spectral problem provides a natural way to introduce the spectral parameter in the reduced problem.

1. Lie symmetries for nonlinear systems in $1+1$ dimensions

This Section is devoted to the Lie symmetry analysis applied to nonlinear systems of physical relevance in $1+1$ dimensions and their associated linear problems. In this Chapter, we will not be concerned about the explicit obtention of the Lax pairs for those equations, they will be taken as known. In particular, the spectral problems treated hereafter have been derived by means of the SMM by the author of this manuscript in [28].

1.1. Derivative NLS equation in $1+1$ dimensions

We begin this Section with the symmetry analysis for the generalized DNLS equation in $1+1$ dimension proposed in Section 2 from Chapter 3. Equation (3.90) can be rewritten as the system

$$
\begin{aligned}
im_t - m_{xx} + i(\gamma-2)\, m\,\overline{m}\, m_x + i(\gamma-1)m^2\,\overline{m}_x - \frac{1}{4}\gamma(\gamma-1)m^3\,\overline{m}^2 = 0, \\
-i\overline{m}_t - \overline{m}_{xx} - i(\gamma-2)\, m\,\overline{m}\,\overline{m}_x - i(\gamma-1)\overline{m}^2 m_x - \frac{1}{4}\gamma(\gamma-1)\overline{m}^3\, m^2 = 0,
\end{aligned}
\tag{6.1}
$$

where m and $\overline{m}$ are, as usual, complex conjugates, $|m|^2 = m \cdot \overline{m}$, and γ is a real parameter.

It has been proved that equation (6.1) has an associated spectral problem given by

$$\chi_{xx} = \left[i\lambda - \frac{i(\gamma - 2)}{2}\,|m|^2 + \frac{m_x}{m} \right] \chi_x + \frac{1}{4} \left[2im\overline{m}_x - (\gamma - 1)\,|m|^4 \right] \chi,$$

$$\chi_t = -\left[\lambda + \frac{\gamma - 2}{2}\,|m|^2 + \frac{im_x}{m} \right] \chi_x - \left[i\lambda^2 + \frac{i(\gamma - 1)}{4}\,|m|^4 + \frac{1}{2}m\overline{m}_x \right] \chi,$$

(6.2)

and its complex conjugate

$$\overline{\chi}_{xx} = \left[-i\lambda + \frac{i(\gamma - 2)}{2}\,|m|^2 + \frac{\overline{m}_x}{\overline{m}} \right] \overline{\chi}_x + \frac{1}{4} \left[-2i\overline{m}m_x - (\gamma - 1)\,|m|^4 \right] \overline{\chi},$$

$$\overline{\chi}_t = -\left[\lambda + \frac{\gamma - 2}{2}\,|m|^2 - \frac{i\overline{m}_x}{\overline{m}} \right] \overline{\chi}_x - \left[-i\lambda^2 - \frac{i(\gamma - 1)}{4}\,|m|^4 + \frac{1}{2}\overline{m}m_x \right] \overline{\chi},$$

(6.3)

where λ acts as the spectral parameter and it can be checked that the conjunction of the compatibility conditions $\chi_{xxt} = \chi_{txx}$ and $\overline{\chi}_{xxt} = \overline{\chi}_{txx}$ retrieve (6.1). Equation (6.1) is demonstrated to possess another Lax pair, which reads

$$\psi_{xx} = \left[-i\lambda + \frac{i(\gamma - 2)}{2}\,|m|^2 + \frac{\left((\gamma - 1)\,|m|^2\,\overline{m} - 2i\overline{m}_x \right)_x}{(\gamma - 1)\,|m|^2\,\overline{m} - 2i\overline{m}_x} \right] \psi_x$$

$$+ \frac{1}{4} \left[2im\overline{m}_x - (\gamma - 1)\,|m|^4 \right] \psi,$$

$$\psi_t = -\left[\lambda + \frac{\gamma - 2}{2}\,|m|^2 - \frac{i\left((\gamma - 1)\,|m|^2\,\overline{m} - 2i\overline{m}_x \right)_x}{(\gamma - 1)\,|m|^2\,\overline{m} - 2i\overline{m}_x} \right] \psi_x$$

$$+ \left[i\lambda^2 + \frac{i(\gamma - 1)}{4}\,|m|^4 + \frac{1}{2}m\overline{m}_x \right] \psi,$$

(6.4)

and its complex conjugate

$$\overline{\psi}_{xx} = \left[i\lambda - \frac{i(\gamma - 2)}{2}\,|m|^2 + \frac{\left((\gamma - 1)\,|m|^2\,m + 2im_x \right)_x}{(\gamma - 1)\,|m|^2\,m + 2im_x} \right] \overline{\psi}_x$$

$$- \frac{1}{4} \left[2i\overline{m}m_x + (\gamma - 1)\,|m|^4 \right] \overline{\psi},$$

(6.5)

$$\overline{\psi}_t = -\left[\lambda + \frac{\gamma-2}{2}\,|m|^2 + \frac{i\left((\gamma-1)\,|m|^2\,m + 2im_x\right)_x}{(\gamma-1)\,|m|^2\,m + 2im_x}\right]\overline{\psi}_x$$

$$-\left[i\lambda^2 + \frac{i(\gamma-1)}{4}\,|m|^4 - \frac{1}{2}\overline{m}m_x\right]\overline{\psi}.$$

Both spectral problems (6.2)-(6.3) and (6.4)-(6.5) are linked via the conditions

$$\frac{\psi_x\chi_x}{\psi\chi} - \frac{i}{2}\,m\,\overline{m}_x + \frac{\gamma-1}{4}\,|m|^4 = 0, \qquad \frac{\overline{\psi}_x\overline{\chi}_x}{\overline{\psi}\,\overline{\chi}} + \frac{i}{2}\,\overline{m}\,m_x + \frac{\gamma-1}{4}\,|m|^4 = 0. \qquad (6.6)$$

Lie point symmetries

Thereupon, we proceed to analyze the Lie point symmetries of the spectral problem for equation (6.1), given by the system of PDEs (6.2)-(6.6). Symmetry analysis for Lax pairs requires to consider the spectral problems as a system of PDEs where the eigenfunctions $\{\chi, \overline{\chi}, \psi, \overline{\psi}\}$ are treated as fields depending on the variables (x, t, λ), while $\{m, \overline{m}\}$ exclusively depend on (x, t). The inclusion of the dependence on λ for the eigenfunctions constitutes the novelty and the cornerstone to the whole procedure concerning the extension of Lie symmetries for Lax pairs. This fact will allow us to naturally define the notion of infinitesimal transformation for this parameter as well as for the eigenfunctions, leading to a straightforward framework to apply Lie's formalism. This approach successfully provides the correct criterion of invariance for the spectral problem and the associated similarity reductions. Then, the space of independent variables $\mathscr{X}$ has coordinates (x, t, λ) while the space of dependent variables $\mathscr{U}$ is endowed with coordinates $(m, \overline{m}, \chi, \overline{\chi}, \psi, \overline{\psi})$. Since we are dealing with a second-order Lax pair, the set of PDEs (6.2)-(6.6) defines a submanifold in the second-order jet space $\mathcal{M}_{(2)} = \mathscr{X} \times \mathscr{U}_{(2)}$.

In order to apply Lie's symmetry method to the linear problem (6.2)-(6.6)[2], let us

---

[2]Notice that Lax pairs (6.2)-(6.3) and (6.4)-(6.5) are completely decoupled in terms of the eigenfunctions $\{\chi, \overline{\chi}\}$ and $\{\psi, \overline{\psi}\}$, and that the compatibility condition of each Lax pair successfully provides equation (6.1). Then, we may conclude that each Lax pair should be considered independent from the other, and that conditions (6.6) exclusively link two versions of the same problem. Thus, from the symmetry point of view, it is not necessarily to consider the complete problem (6.2)-(6.6), it suffices to analyze either (6.2)-(6.3) or (6.4)-(6.5) and the resulting symmetries should be consistent by construction with conditions (6.6).

consider the following one-parameter group of infinitesimal transformations

$$\tilde{x} = x + \epsilon\, \xi_x(x, t, \lambda, m, \overline{m}, \chi, \overline{\chi}, \psi, \overline{\psi}) + \mathcal{O}(\epsilon^2),$$
$$\tilde{t} = t + \epsilon\, \xi_t(x, t, \lambda, m, \overline{m}, \chi, \overline{\chi}, \psi, \overline{\psi}) + \mathcal{O}(\epsilon^2),$$
$$\tilde{\lambda} = \lambda + \epsilon\, \xi_\lambda(x, t, \lambda, m, \overline{m}, \chi, \overline{\chi}, \psi, \overline{\psi}) + \mathcal{O}(\epsilon^2),$$
$$\tilde{m} = m + \epsilon\, \eta_m(x, t, \lambda, m, \overline{m}, \chi, \overline{\chi}, \psi, \overline{\psi}) + \mathcal{O}(\epsilon^2),$$
$$\tilde{\overline{m}} = \overline{m} + \epsilon\, \eta_{\overline{m}}(x, t, \lambda, m, \overline{m}, \chi, \overline{\chi}, \psi, \overline{\psi}) + \mathcal{O}(\epsilon^2), \tag{6.7}$$
$$\tilde{\chi} = \chi + \epsilon\, \eta_\chi(x, t, \lambda, m, \overline{m}, \chi, \overline{\chi}, \psi, \overline{\psi}) + \mathcal{O}(\epsilon^2),$$
$$\tilde{\overline{\chi}} = \overline{\chi} + \epsilon\, \eta_{\overline{\chi}}(x, t, \lambda, m, \overline{m}, \chi, \overline{\chi}, \psi, \overline{\psi}) + \mathcal{O}(\epsilon^2),$$
$$\tilde{\psi} = \psi + \epsilon\, \eta_\psi(x, t, \lambda, m, \overline{m}, \chi, \overline{\chi}, \psi, \overline{\psi}) + \mathcal{O}(\epsilon^2),$$
$$\tilde{\overline{\psi}} = \overline{\psi} + \epsilon\, \eta_{\overline{\psi}}(x, t, \lambda, m, \overline{m}, \chi, \overline{\chi}, \psi, \overline{\psi}) + \mathcal{O}(\epsilon^2),$$

where ϵ is the parameter of the group. The associated vector field that generates the transformation above reads as

$$X = \xi_x \frac{\partial}{\partial x} + \xi_t \frac{\partial}{\partial t} + \xi_\lambda \frac{\partial}{\partial \lambda} + \eta_m \frac{\partial}{\partial m} + \eta_{\overline{m}} \frac{\partial}{\partial \overline{m}} + \eta_\chi \frac{\partial}{\partial \chi} + \eta_{\overline{\chi}} \frac{\partial}{\partial \overline{\chi}} + \eta_\psi \frac{\partial}{\partial \psi} + \eta_{\overline{\psi}} \frac{\partial}{\partial \overline{\psi}}. \tag{6.8}$$

This infinitesimal transformation induces a well known one in the derivatives of the fields $\{m, \overline{m}\}$ and the eigenfunctions $\{\chi, \overline{\chi}, \psi, \overline{\psi}\}$ given by (5.55), such that it must preserve the invariance of the starting system of PDEs,

$$\begin{cases} \tilde{m}_{\tilde{x}} = m_x + \epsilon(\eta_m)_x + \mathcal{O}(\epsilon^2), & \tilde{\overline{m}}_{\tilde{x}} = \overline{m}_x + \epsilon(\eta_{\overline{m}})_x + \mathcal{O}(\epsilon^2), \\ \tilde{m}_{\tilde{x}\tilde{x}} = m_{xx} + \epsilon(\eta_m)_{xx} + \mathcal{O}(\epsilon^2), & \tilde{\overline{m}}_{\tilde{x}\tilde{x}} = \overline{m}_{xx} + \epsilon(\eta_{\overline{m}})_{xx} + \mathcal{O}(\epsilon^2), \\ \tilde{\chi}_{\tilde{x}} = \chi_x + \epsilon(\eta_\chi)_x + \mathcal{O}(\epsilon^2), & \tilde{\overline{\chi}}_{\tilde{x}} = \overline{\chi}_x + \epsilon(\eta_{\overline{\chi}})_x + \mathcal{O}(\epsilon^2), \\ \tilde{\chi}_{\tilde{t}} = \chi_t + \epsilon(\eta_\chi)_t + \mathcal{O}(\epsilon^2), & \tilde{\overline{\chi}}_{\tilde{t}} = \overline{\chi}_t + \epsilon(\eta_{\overline{\chi}})_t + \mathcal{O}(\epsilon^2), \\ \tilde{\chi}_{\tilde{x}\tilde{x}} = \chi_{xx} + \epsilon(\eta_\chi)_{xx} + \mathcal{O}(\epsilon^2), & \tilde{\overline{\chi}}_{\tilde{x}\tilde{x}} = \overline{\chi}_{xx} + \epsilon(\eta_{\overline{\chi}})_{xx} + \mathcal{O}(\epsilon^2), \\ \tilde{\psi}_{\tilde{x}} = \psi_x + \epsilon(\eta_\psi)_x + \mathcal{O}(\epsilon^2), & \tilde{\overline{\psi}}_{\tilde{x}} = \overline{\psi}_x + \epsilon(\eta_{\overline{\psi}})_x + \mathcal{O}(\epsilon^2), \\ \tilde{\psi}_{\tilde{t}} = \psi_t + \epsilon(\eta_\psi)_t + \mathcal{O}(\epsilon^2), & \tilde{\overline{\psi}}_{\tilde{t}} = \overline{\psi}_t + \epsilon(\eta_{\overline{\psi}})_t + \mathcal{O}(\epsilon^2), \\ \tilde{\psi}_{\tilde{x}\tilde{x}} = \psi_{xx} + \epsilon(\eta_\psi)_{xx} + \mathcal{O}(\epsilon^2), & \tilde{\overline{\psi}}_{\tilde{x}\tilde{x}} = \overline{\psi}_{xx} + \epsilon(\eta_{\overline{\psi}})_{xx} + \mathcal{O}(\epsilon^2), \end{cases} \tag{6.9}$$

where the different coefficients $(\eta_\Omega)_x, (\eta_\Omega)_t, (\eta_\Omega)_{xx}$ for $\Omega = \{m, \overline{m}, \chi, \overline{\chi}, \psi, \overline{\psi}\}$ can be easily computed by the general prolongation formula (5.54) from Chapter 5. We have analyzed both the classical and nonclassical approaches to Lie symmetries for the spectral problem (6.2)-(6.6), following the theoretical prescriptions given in (5.60)

and (5.70), respectively. Unfortunately, nonclassical Lie symmetries do not provide any further information, since they fully coincide with the classical symmetries, of the form

$$\xi_x(x, t, \lambda, m, \overline{m}, \chi, \overline{\chi}, \psi, \overline{\psi}) = A_1 x + A_2,$$

$$\xi_t(x, t, \lambda, m, \overline{m}, \chi, \overline{\chi}, \psi, \overline{\psi}) = 2A_1 t + A_3,$$

$$\xi_\lambda(x, t, \lambda, m, \overline{m}, \chi, \overline{\chi}, \psi, \overline{\psi}) = -A_1 \lambda,$$

$$\eta_m(x, t, \lambda, m, \overline{m}, \chi, \overline{\chi}, \psi, \overline{\psi}) = \left(-\frac{A_1}{2} + iZ_1(t)\right) m,$$

$$\eta_{\overline{m}}(x, t, \lambda, m, \overline{m}, \chi, \overline{\chi}, \psi, \overline{\psi}) = \left(-\frac{A_1}{2} - iZ_1(t)\right) \overline{m}, \tag{6.10}$$

$$\eta_\chi(x, t, \lambda, m, \overline{m}, \chi, \overline{\chi}, \psi, \overline{\psi}) = K_1(\lambda)\, \chi,$$

$$\eta_{\overline{\chi}}(x, t, \lambda, m, \overline{m}, \chi, \overline{\chi}, \psi, \overline{\psi}) = \overline{K}_1(\lambda)\, \overline{\chi},$$

$$\eta_\psi(x, t, \lambda, m, \overline{m}, \chi, \overline{\chi}, \psi, \overline{\psi}) = K_2(\lambda)\, \psi,$$

$$\eta_{\overline{\psi}}(x, t, \lambda, m, \overline{m}, \chi, \overline{\chi}, \psi, \overline{\psi}) = \overline{K}_2(\lambda)\, \overline{\psi},$$

where A_i, $i = 1, ..., 3$ are arbitrary constants, $Z_1(t)$ is an arbitrary real function of t and $K_i(\lambda)$, $\overline{K}_i(\lambda)$, $i = 1, 2$ are arbitrary complex (and conjugate) functions of λ. It is easy to check that the infinitesimals $\{\xi_x, \xi_t, \eta_m, \eta_{\overline{m}}\}$ in (6.10) provide the full set of Lie point symmetries for the nonlinear equation (6.1). The symmetry associated with A_1 amounts to a scale transformation, A_2 and A_3 define translations in space and time, respectively, whilst $Z_1(t)$ creates a phase translation in the fields m and $\overline{m}$. The Lie symmetries for the associated Lax pair are augmented with the corresponding transformations for λ and the eigenfunctions. The infinitesimal ξ_λ allows to extend the scaling transformation to the spectral parameter. Symmetries associated to $K_i(\lambda)$, $\overline{K}_i(\lambda)$, $i = 1, 2$ represent a phase shift in the eigenfunctions due to the linearity of the spectral problem.

Classification of the associated Lie algebra

Hereafter, we will characterize and classify the Lie algebra associated to the symmetry group for the spectral problem (6.2)-(6.6). Since the symmetry group (6.10) depends on up to eight arbitrary elements, we may construct eight infinitesimal generators associated to these symmetries, listed as

$$X_1 = x\frac{\partial}{\partial x} + 2t\frac{\partial}{\partial t} - \lambda\frac{\partial}{\partial \lambda} - \frac{m}{2}\frac{\partial}{\partial m} - \frac{\overline{m}}{2}\frac{\partial}{\partial \overline{m}},$$

$$X_2 = \frac{\partial}{\partial x},$$

$$X_3 = \frac{\partial}{\partial t},$$

$$Y_{\{Z_1(t)\}} = iZ_1(t) \left(m \frac{\partial}{\partial m} - \overline{m} \frac{\partial}{\partial \overline{m}} \right),$$ (6.11)

$$\Gamma^\chi_{\{K_1(\lambda)\}} = K_1(\lambda) \chi \frac{\partial}{\partial \chi},$$

$$\Gamma^{\overline{\chi}}_{\{\overline{K}_1(\lambda)\}} = \overline{K}_1(\lambda) \overline{\chi} \frac{\partial}{\partial \overline{\chi}},$$

$$\Gamma^\psi_{\{K_2(\lambda)\}} = K_2(\lambda) \psi \frac{\partial}{\partial \psi},$$

$$\Gamma^{\overline{\psi}}_{\{\overline{K}_2(\lambda)\}} = \overline{K}_2(\lambda) \overline{\psi} \frac{\partial}{\partial \overline{\psi}},$$

where generators $X_1 - X_3$ arise exclusively from the arbitrary constants in the symmetries, $Y_{\{Z_1(t)\}}$ depends on an arbitrary function of time and $\Gamma^\rho_{\{\kappa(\lambda)\}} = \kappa(\lambda) \rho \partial_\rho$ denotes the generic generator associated to the arbitrary functions $\kappa(\lambda) = \{K_j(\lambda), \overline{K}_j(\lambda)\}$, $j = 1, 2$ and $\rho = \{\chi, \overline{\chi}, \psi, \overline{\psi}\}$.

It is worthwhile to remark the presence of arbitrary functions in the Lie symmetries. The influence of these arbitrary functions should not be taken lightly and it deserves a meticulous analysis. According to [378], symmetry generators depending on arbitrary constants will give rise to a Lie algebra, while generator depending on arbitrary functions will not, since we are dealing with an infinite-dimensional basis of generators. Notwithstanding this, the commutator of two symmetry generators is still a generator of a symmetry, in a sense that will be highlighted later. Arbitrary functions will play a decisive role in the symmetry analysis of differential equations, specially in the case of higher spatial dimensions, as it is illustrated in Appendix B.

Commutation relations among the symmetry generators may be performed. The results are summarized in the following table, where entry in row i and column j symbolizes the operation $[V_i, V_j]$, with V_i, V_j two generators of the symmetry group,

	X_1	X_2	X_3	$Y_{\{Z_1(t)\}}$	$\Gamma^\rho_{\{\kappa(\lambda)\}}$
X_1	0	$-X_2$	$-2X_3$	$Y_{\left\{2t \frac{dZ_1}{dt}\right\}}$	$\Gamma^\rho_{\left\{-\lambda \frac{d\kappa}{d\lambda}\right\}}$
X_2	X_2	0	0	0	0
X_3	$2X_3$	0	0	$Y_{\left\{\frac{dZ_1}{dt}\right\}}$	0
$Y_{\{\tilde{Z}_1(t)\}}$	$-Y_{\left\{2t \frac{d\tilde{Z}_1}{dt}\right\}}$	0	$-Y_{\left\{\frac{d\tilde{Z}_1}{dt}\right\}}$	0	0
$\Gamma^\rho_{\{\tilde{\kappa}(\lambda)\}}$	$-\Gamma^\rho_{\left\{-\lambda \frac{d\tilde{\kappa}}{d\lambda}\right\}}$	0	0	0	0

Notice that the generic generator $\Gamma^{\rho}_{\{\kappa(\lambda)\}}$ defined above satisfies that $\left[\Gamma^{\rho}_{\{\kappa(\lambda)\}}, \Gamma^{\hat{\rho}}_{\{\hat{\kappa}(\lambda)\}}\right]$ $= 0$ for any combination of the arbitrary functions $\kappa(\lambda)$, $\hat{\kappa}(\lambda)$ and the eigenfunctions ρ, $\hat{\rho}$. As mentioned, it may be observed that every commutator of two infinitesimal generators provides a nontrivial result, due to the presence of the arbitrary functions [323]. However, every commutator can be expressed in terms of other generators of the group, with the appropriate choice of its corresponding arbitrary function. This property provides an analogous notion of closure with the finite-dimensional case. In general terms, these infinitesimal generators do not form a Lie algebra, but it is possible to obtain a finite-dimensional Lie algebra by adopting special values for the arbitrary functions [77, 119].

In order to illustrate this, we will deeply review and classify the Lie algebra associated to the Lie point symmetries of equation (6.1). This can be achieved by considering the ansatz $Z_1(t) = A_4$ constant and $K_1(\lambda) = K_2(\lambda) = 0$. Hence, the resulting Lie algebra has dimension four, and it is generated by the vector fields $X_1 - X_3$ given in (6.11) and $X_4 = im\partial_m - i\overline{m}\,\partial_{\overline{m}}$. Thus, the nontrivial commutation relations among theses generators are

$$[X_1, X_2] = -X_2, \qquad [X_1, X_3] = -2X_3, \tag{6.12}$$

which define a four-dimensional real Lie algebra which is solvable and decomposable $\langle X_1, X_2, X_3 \rangle \oplus X_4$, and it can be classified as $A^a_{3,5} \oplus A_1$ for $a = \frac{1}{2}$ [333, 334]. This Lie algebra in turn constitutes a subalgebra of the Schrödinger algebra $\mathfrak{sch}(1)$ [178, 180].

Similarity reductions

Let us proceed with the explicit computation of the similarity reductions of the spectral problem (6.2)-(6.6). This procedure will allow us to simultaneously obtain the reduction of the eigenfunctions and the spectral parameter. Since we have considered that the eigenfunctions possess a dependence on λ, the arising reduced spectral problem will consist on a system of two PDEs for the reduced eigenfunctions on terms of the reduced variables z, Λ. Nevertheless, the original nonlinear equation (6.1) is described in $1 + 1$ dimensions, and therefore the resulting reduced nonlinear equation, obtained as the compatibility condition of the reduced spectral problem, is expected to be an ODE in the reduced variable z.

In the following, we shall use the next notation regarding the reduced parameters in the upcoming calculations

	Original variables	New reduced variables
Independent variables	x, t, λ	z, Λ
Fields	$m(x,t), \overline{m}(x,t)$	$\mathcal{M}(z), \overline{\mathcal{M}}(z)$
Eigenfunctions	$\chi(x,t,\lambda), \overline{\chi}(x,t,\lambda)$	$\Phi(z,\Lambda), \overline{\Phi}(z,\Lambda)$
	$\psi(x,t,\lambda), \overline{\psi}(x,t,\lambda)$	$\Psi(z,\Lambda), \overline{\Psi}(z,\Lambda)$

Similarity reductions may be computed by solving the characteristic system given by

$$\frac{dx}{\xi_x} = \frac{dt}{\xi_t} = \frac{d\lambda}{\xi_\lambda} = \frac{d\,m}{\eta_m} = \frac{d\,\overline{m}}{\eta_{\overline{m}}} = \frac{d\chi}{\eta_\chi} = \frac{d\overline{\chi}}{\eta_{\overline{\chi}}} = \frac{d\psi}{\eta_\psi} = \frac{d\overline{\psi}}{\eta_{\overline{\psi}}}. \tag{6.13}$$

The symmetries that will yield nontrivial reductions are those present in the transformations of the independent variables, *i.e.* the ones related to the arbitrary constants A_1, A_2 and A_3. The remaining symmetries will provide trivial reductions. Several reductions may emerge for different values of these constants, yielding three different cases with an independent reduction each. The relevant reductions correspond to the cases

(i) $A_1 \neq 0$

(ii) $A_1 = 0$, $A_2 \neq 0$, $A_3 \neq 0$

(iii) $A_1 = 0$, $A_2 = 0$, $A_3 \neq 0$

that should be studied separately.

Without loss of generality and for the sake of simplicity, we may consider all the arbitrary functions $K_j(\lambda)$, $\overline{K}_j(\lambda)$, $j = 1, 2$ to be zero, since the associated symmetries are phase shifts over the eigenfunctions that are trivially satisfied due to the linearity of the Lax pairs. Conversely, the arbitrary function depending on t, $Z_1(t)$, does need to be taken into consideration.

- **Case I. $A_1 \neq 0$**

    By solving the characteristic system (6.13) in the general case, the following results have been obtained

    – Reduced variable and reduced spectral parameter

    $$z = \frac{A_1 x + A_2}{\sqrt{A_1}\sqrt{A_3 + 2A_1 t}}, \qquad \Lambda = \frac{\lambda}{\sqrt{A_1}}\sqrt{A_3 + 2A_1 t}. \tag{6.14}$$

- Reduced fields

$$m(x,t) = \mathcal{M}(z)\,\frac{A_1^{\frac{1}{4}}\,e^{i\int \frac{Z_1(t)}{A_3+2A_1 t}dt}}{(A_3+2A_1 t)^{\frac{1}{4}}},$$

$$\overline{m}(x,t) = \overline{\mathcal{M}}(z)\,\frac{A_1^{\frac{1}{4}}\,e^{-i\int \frac{Z_1(t)}{A_3+2A_1 t}dt}}{(A_3+2A_1 t)^{\frac{1}{4}}}. \tag{6.15}$$

- Reduced eigenfunctions

$$\begin{aligned}
\chi(x,t,\lambda) &= \Phi(z,\Lambda), & \overline{\chi}(x,t,\lambda) &= \overline{\Phi}(z,\Lambda), \\
\psi(x,t,\lambda) &= \Psi(z,\Lambda), & \overline{\psi}(x,t,\lambda) &= \overline{\Psi}(z,\Lambda).
\end{aligned} \tag{6.16}$$

- Reduced Φ-spectral problem (reduction of (6.2)-(6.3))

$$\begin{aligned}
&\Phi_{zz} - \left(i\Lambda - \frac{i}{2}(\gamma-2)\mathcal{M}\overline{\mathcal{M}} + \frac{\mathcal{M}_z}{\mathcal{M}}\right)\Phi_z \\
&\quad + \frac{1}{4}\left((\gamma-1)\mathcal{M}^2\overline{\mathcal{M}}^2 - 2i\mathcal{M}\overline{\mathcal{M}}_z\right)\Phi = 0, \\[2mm]
&\Lambda\Phi_\Lambda - \left(z - \Lambda - \frac{i}{2}(\gamma-2)\mathcal{M}\overline{\mathcal{M}} - i\frac{\mathcal{M}_z}{\mathcal{M}}\right)\Phi_z \\
&\quad + \left(\frac{i}{4}(\gamma-1)\mathcal{M}^2\overline{\mathcal{M}}^2 + \frac{1}{2}\mathcal{M}\overline{\mathcal{M}}_z + i\Lambda^2\right)\Phi = 0,
\end{aligned} \tag{6.17}$$

and its complex conjugate.

The link between the two equivalent reduced Lax pairs is given by the reduction of conditions (6.6), which provides

$$\frac{\Phi_z\Psi_z}{\Phi\Psi} + \frac{1}{4}(\gamma-1)\,|\mathcal{M}|^4 - \frac{i}{2}\mathcal{M}\overline{\mathcal{M}}_z = 0, \qquad \frac{\overline{\Phi}_z\overline{\Psi}_z}{\overline{\Phi}\,\overline{\Psi}} + \frac{1}{4}(\gamma-1)\,|\mathcal{M}|^4 + \frac{i}{2}\overline{\mathcal{M}}\mathcal{M}_z = 0. \tag{6.18}$$

- Reduced Ψ-spectral problem (reduction of (6.4)-(6.5))

$$\begin{aligned}
&\Psi_{zz} + \frac{1}{4}\mathcal{M}\left((\gamma-1)\mathcal{M}\overline{\mathcal{M}}^2 - 2i\overline{\mathcal{M}}_z\right)\Psi \\
&\quad + i\left(\Lambda - \frac{1}{2}(\gamma-2)\mathcal{M}\overline{\mathcal{M}} + \frac{\left(i(\gamma-1)\mathcal{M}\overline{\mathcal{M}}^2 + 2\overline{\mathcal{M}}_z\right)_z}{(\gamma-1)\mathcal{M}\overline{\mathcal{M}}^2 - 2i\overline{\mathcal{M}}_z}\right)\Psi_z = 0,
\end{aligned} \tag{6.19}$$

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$$\Lambda\Psi_\Lambda + \left(\Lambda - z + \frac{1}{2}(\gamma-2)\mathcal{M}\overline{\mathcal{M}} - \frac{\left(i(\gamma-1)\mathcal{M}\overline{\mathcal{M}}^2 + 2\overline{\mathcal{M}}_z\right)_z}{(\gamma-1)\mathcal{M}\overline{\mathcal{M}}^2 - 2i\overline{\mathcal{M}}_z}\right)\Psi_z$$

$$-\frac{i}{4}\left((\gamma-1)\mathcal{M}^2\overline{\mathcal{M}}^2 - 2i\mathcal{M}\overline{\mathcal{M}}_z + 4\Lambda^2\right)\Psi = 0,$$

and its complex conjugate.

– Reduced equation

The compatibility condition between both Lax pairs (6.17) and (6.19) and their complex conjugates provide the reduced equation (and its complex conjugate), which may be integrated as

$$\left[\frac{i\mathcal{M}_{zz}}{\mathcal{M}} - (z + \mathcal{M}\overline{\mathcal{M}})\frac{\mathcal{M}_z}{\mathcal{M}} + (\gamma-1)\left(\mathcal{M}\overline{\mathcal{M}}\right)_z + \frac{i}{4}\gamma(\gamma-1)\mathcal{M}^2\overline{\mathcal{M}}^2\right]_z = 0. \tag{6.20}$$

- **Case II. $A_1 = 0$, $A_2 \neq 0$, $A_3 \neq 0$**

By applying the same procedure, integrating (6.13), we get

– Reduced variable and reduced spectral parameter

$$z = \frac{A_2}{A_3}\left(x - \frac{A_2}{A_3}t\right), \qquad \Lambda = \frac{A_3}{A_2}\lambda. \tag{6.21}$$

– Reduced fields

$$m(x,t) = \sqrt{\frac{A_2}{A_3}}\, e^{\frac{i}{A_3}\int Z_1(t)\,dt}\,\mathcal{M}(z),$$

$$\overline{m}(x,t) = \sqrt{\frac{A_2}{A_3}}\, e^{-\frac{i}{A_3}\int Z_1(t)\,dt}\,\overline{\mathcal{M}}(z). \tag{6.22}$$

– Reduced eigenfunctions

$$\chi(x,t,\lambda) = \Phi(z,\Lambda), \qquad \overline{\chi}(x,t,\lambda) = \overline{\Phi}(z,\Lambda),$$

$$\psi(x,t,\lambda) = \Psi(z,\Lambda), \qquad \overline{\psi}(x,t,\lambda) = \overline{\Psi}(z,\Lambda). \tag{6.23}$$

– Reduced Φ-spectral problem

$$
\Phi_{zz} - \left(i\Lambda - \frac{i}{2}(\gamma - 2)\mathcal{M}\overline{\mathcal{M}} + \frac{\mathcal{M}_z}{\mathcal{M}} \right) \Phi_z
$$

$$
+ \frac{1}{4} \left((\gamma - 1)\mathcal{M}^2\overline{\mathcal{M}}^2 - 2i\mathcal{M}\overline{\mathcal{M}}_z \right) \Phi = 0,
$$

$$
\left(1 - \Lambda - \frac{i}{2}(\gamma - 2)\mathcal{M}\overline{\mathcal{M}} - i\frac{\mathcal{M}_z}{\mathcal{M}} \right) \Phi_z
$$

$$
- \left(\frac{i}{4}(\gamma - 1)\mathcal{M}^2\overline{\mathcal{M}}^2 + \frac{1}{2}\mathcal{M}\overline{\mathcal{M}}_z + i\Lambda^2 \right) \Phi = 0,
$$

(6.24)

and its complex conjugate.

Reduction of conditions (6.6) gives the same equation as (6.18).

– Reduced Ψ-spectral problem

$$
\Psi_{zz} + \frac{1}{4}\mathcal{M} \left((\gamma - 1)\mathcal{M}\overline{\mathcal{M}}^2 - 2i\overline{\mathcal{M}}_z \right) \Psi
$$

$$
+ i \left(\Lambda - \frac{1}{2}(\gamma - 2)\mathcal{M}\overline{\mathcal{M}} + \frac{\left(i(\gamma - 1)\mathcal{M}\overline{\mathcal{M}}^2 + 2\overline{\mathcal{M}}_z \right)_z}{(\gamma - 1)\mathcal{M}\overline{\mathcal{M}}^2 - 2i\overline{\mathcal{M}}_z} \right) \Psi_z = 0,
$$

$$
\left(\Lambda - 1 + \frac{1}{2}(\gamma - 2)\mathcal{M}\overline{\mathcal{M}} - \frac{\left(i(\gamma - 1)\mathcal{M}\overline{\mathcal{M}}^2 + 2\overline{\mathcal{M}}_z \right)_z}{(\gamma - 1)\mathcal{M}\overline{\mathcal{M}}^2 - 2i\overline{\mathcal{M}}_z} \right) \Psi_z
$$

$$
- \frac{i}{4} \left((\gamma - 1)\mathcal{M}^2\overline{\mathcal{M}}^2 - 2i\mathcal{M}\overline{\mathcal{M}}_z + 4\Lambda^2 \right) \Psi = 0,
$$

(6.25)

and its complex conjugate.

– Reduced equation

The compatibility condition between both Lax pairs (6.24) and (6.25) and their complex conjugates provide the reduced equation

$$
\left[\frac{i\mathcal{M}_{zz}}{\mathcal{M}} - \left(1 + \mathcal{M}\overline{\mathcal{M}} \right) \frac{\mathcal{M}_z}{\mathcal{M}} + (\gamma - 1)\left(\mathcal{M}\overline{\mathcal{M}} \right)_z + \frac{i}{4}\gamma(\gamma - 1)\mathcal{M}^2\overline{\mathcal{M}}^2 \right]_z = 0.
$$

(6.26)

- *Case III. $A_1 = 0$, $A_2 = 0$, $A_3 \neq 0$*

By integrating the characteristic system (6.13), the following results arise

– Reduced variable and reduced spectral parameter

$$z = x, \qquad \Lambda = \lambda. \tag{6.27}$$

– Reduced fields

$$
\begin{aligned}
m(x,t) &= e^{\frac{i}{A_3} \int Z_1(t)\, dt} \mathcal{M}(z), \\
\overline{m}(x,t) &= e^{-\frac{i}{A_3} \int Z_1(t)\, dt} \, \overline{\mathcal{M}}(z).
\end{aligned}
\tag{6.28}
$$

– Reduced eigenfunctions

$$
\begin{aligned}
\chi(x,t,\lambda) &= \Phi(z,\Lambda), & \overline{\chi}(x,t,\lambda) &= \overline{\Phi}(z,\Lambda), \\
\psi(x,t,\lambda) &= \Psi(z,\Lambda), & \overline{\psi}(x,t,\lambda) &= \overline{\Psi}(z,\Lambda).
\end{aligned}
\tag{6.29}
$$

– Reduced Φ-spectral problem

$$
\begin{aligned}
&\Phi_{zz} - \left(i\Lambda - \frac{i}{2}(\gamma-2)\mathcal{M}\overline{\mathcal{M}} + \frac{\mathcal{M}_z}{\mathcal{M}} \right) \Phi_z \\
&\quad + \frac{1}{4}\left((\gamma-1)\mathcal{M}^2\overline{\mathcal{M}}^2 - 2i\mathcal{M}\overline{\mathcal{M}}_z \right) \Phi = 0, \\
&\left(\Lambda + \frac{i}{2}(\gamma-2)\mathcal{M}\overline{\mathcal{M}} + i\frac{\mathcal{M}_z}{\mathcal{M}} \right) \Phi_z \\
&\quad - \left(\frac{i}{4}(\gamma-1)\mathcal{M}^2\overline{\mathcal{M}}^2 + \frac{1}{2}\mathcal{M}\overline{\mathcal{M}}_z + i\Lambda^2 \right) \Phi = 0,
\end{aligned}
\tag{6.30}
$$

and its complex conjugate.

Reduction of conditions (6.6) retrieves again equation (6.18).

– Reduced Ψ-spectral problem

$$
\begin{aligned}
&\Psi_{zz} + \frac{1}{4}\mathcal{M}\left((\gamma-1)\mathcal{M}\overline{\mathcal{M}}^2 - 2i\overline{\mathcal{M}}_z \right) \Psi \\
&\quad + i\left(\Lambda - \frac{1}{2}(\gamma-2)\mathcal{M}\overline{\mathcal{M}} + \frac{\left(i(\gamma-1)\mathcal{M}\overline{\mathcal{M}}^2 + 2\overline{\mathcal{M}}_z \right)_z}{(\gamma-1)\mathcal{M}\overline{\mathcal{M}}^2 - 2i\overline{\mathcal{M}}_z} \right) \Psi_z = 0, \\
&\left(\Lambda + \frac{1}{2}(\gamma-2)\mathcal{M}\overline{\mathcal{M}} - \frac{\left(i(\gamma-1)\mathcal{M}\overline{\mathcal{M}}^2 + 2\overline{\mathcal{M}}_z \right)_z}{(\gamma-1)\mathcal{M}\overline{\mathcal{M}}^2 - 2i\overline{\mathcal{M}}_z} \right) \Psi_z \\
&\quad - \frac{i}{4}\left((\gamma-1)\mathcal{M}^2\overline{\mathcal{M}}^2 - 2i\mathcal{M}\overline{\mathcal{M}}_z + 4\Lambda^2 \right) \Psi = 0,
\end{aligned}
\tag{6.31}
$$

and its complex conjugate.

– Reduced equation

Finally, the reduce equation reads

$$\left[\frac{i\mathcal{M}_{zz}}{\mathcal{M}} - \mathcal{M}\mathcal{M}_z + (\gamma - 1)\left(\mathcal{M}\overline{\mathcal{M}}\right)_z + \frac{i}{4}\gamma(\gamma - 1)\mathcal{M}^2\overline{\mathcal{M}}^2 \right]_z = 0. \qquad (6.32)$$

1.2. Conservative equation in $1 + 1$ dimensions

Throughout the Painlevé analysis for DNLS equation (3.90) conducted in Chapter 3, we derived the following nonlinear equation expressed in conservative form

$$\left[\alpha_x^2 - \alpha_t\right]_t = \left[\alpha_{xxx} + \alpha_x^3 - \frac{\alpha_t^2 + \alpha_{xx}^2}{\alpha_x}\right]_x, \qquad (6.33)$$

to be satisfied by the real field $\alpha(x,t)$ that is related with the probability density of DNLS equations as $2\alpha_x = |m|^2$. Equation (6.33) is proved to have two equivalent Lax pairs given by

$$\chi_{xx} = \chi_x \left[i\lambda + \frac{i\alpha_x^2 + \alpha_{xx} + i\alpha_t}{2\alpha_x}\right] + \chi \left[\frac{-\alpha_x^2 + i\alpha_{xx} + \alpha_t}{2}\right],$$

$$\chi_t = \chi_x \left[-\lambda + \frac{\alpha_t - i\alpha_{xx} + \alpha_x^2}{2\alpha_x}\right] + \chi \left[-i\lambda^2 + \frac{i\alpha_t - \alpha_{xx} - i\alpha_x^2}{2}\right], \qquad (6.34)$$

and

$$\psi_{xx} = \psi_x \left[-i\lambda + \frac{i\alpha_x^4 + 2i\alpha_{xxx}\alpha_x - 2\alpha_x^2\alpha_{xx} - i\alpha_t^2 - i\alpha_{xx}^2 + 2\alpha_{xt}\alpha_x}{2\alpha_x \left(-\alpha_x^2 + i\alpha_{xx} + \alpha_t\right)}\right]$$

$$+ \psi \left[\frac{-\alpha_x^2 + i\alpha_{xx} + \alpha_t}{2}\right],$$

$$\psi_t = \psi_x \left[-\lambda - \frac{\alpha_x^4 + 2\alpha_{xxx}\alpha_x + 2i\alpha_x^2\alpha_{xx} - \alpha_t^2 - \alpha_{xx}^2 - 2i\alpha_{xt}\alpha_x}{2\alpha_x \left(-\alpha_x^2 + i\alpha_{xx} + \alpha_t\right)}\right] \qquad (6.35)$$

$$+ \psi \left[i\lambda^2 + \frac{-i\alpha_t + \alpha_{xx} + i\alpha_x^2}{2}\right],$$

where λ represents the spectral parameter and the eigenfunctions satisfy

$$2 \frac{\psi_x \chi_x}{\psi \chi} + \alpha_x^2 - \alpha_t - i\,\alpha_{xx} = 0. \qquad (6.36)$$

We can easily check that the compatibility condition of (6.34) and (6.35), $\chi_{xxt} = \chi_{txx}$, $\psi_{xxt} = \psi_{txx}$ respectively, retrieves equation (6.33) in each case. Since α stands

for a real field, we do not have to additionally consider the complex conjugates of the aforementioned linear problems.

Lie point symmetries

In order to compute the Lie symmetries for the spectral problem (6.34)-(6.36), we analogously proceed by proposing the following infinitesimal transformation

$$
\begin{aligned}
\tilde{x} &= x + \epsilon\,\xi_x(x,t,\lambda,\alpha,\chi,\psi) + \mathcal{O}(\epsilon^2), \\
\tilde{t} &= t + \epsilon\,\xi_t(x,t,\lambda,\alpha,\chi,\psi) + \mathcal{O}(\epsilon^2), \\
\tilde{\lambda} &= \lambda + \epsilon\,\xi_\lambda(x,t,\lambda,\alpha,\chi,\psi) + \mathcal{O}(\epsilon^2), \\
\tilde{\alpha} &= \alpha + \epsilon\,\eta_\alpha(x,t,\lambda,\alpha,\chi,\psi) + \mathcal{O}(\epsilon^2), \\
\tilde{\chi} &= \chi + \epsilon\,\eta_\chi(x,t,\lambda,\alpha,\chi,\psi) + \mathcal{O}(\epsilon^2), \\
\tilde{\psi} &= \psi + \epsilon\,\eta_\psi(x,t,\lambda,\alpha,\chi,\psi) + \mathcal{O}(\epsilon^2),
\end{aligned}
\tag{6.37}
$$

with ϵ as the parameter of the symmetry group. The associated vector field is given by

$$
X = \xi_x \frac{\partial}{\partial x} + \xi_t \frac{\partial}{\partial t} + \xi_\lambda \frac{\partial}{\partial \lambda} + \eta_\alpha \frac{\partial}{\partial \alpha} + \eta_\chi \frac{\partial}{\partial \chi} + \eta_\psi \frac{\partial}{\partial \psi}.
\tag{6.38}
$$

This transformation can be easily extended to the different derivatives of the dependent variables by the usual procedure [323]. Notice that although the Lax pairs are PDEs of second order, there appear third-order derivatives of α in (6.35), so the associated prolongations have to be performed up to third order. Classical and nonclassical Lie symmetries have been computed, finding that although the overdetermined systems for the infinitesimals are different, the corresponding results give rise to the same set of symmetries

$$
\begin{aligned}
\xi_x(x,t,\lambda,\alpha,\chi,\psi) &= B_1 x + B_2, \\
\xi_t(x,t,\lambda,\alpha,\chi,\psi) &= 2B_1 t + B_3, \\
\xi_\lambda(x,t,\lambda,\alpha,\chi,\psi) &= -B_1\,\lambda, \\
\eta_\alpha(x,t,\lambda,\alpha,\chi,\psi) &= B_4, \\
\eta_\chi(x,t,\lambda,\alpha,\chi,\psi) &= K_1(\lambda)\,\chi, \\
\eta_\psi(x,t,\lambda,\alpha,\chi,\psi) &= K_2(\lambda)\,\psi,
\end{aligned}
\tag{6.39}
$$

where B_i, $i = 1,...,4$ are arbitrary real constants and $K_j(\lambda)$, $j = 1,2$ are arbitrary complex functions of λ. The symmetry associated to B_1 generates the usual scale transformations, the ones associated to $B_2 - B_4$ induce translations in the variables $\{x,t,\alpha\}$, respectively, and $K_j(\lambda)$, for all j, are responsible for phase shifts in the

eigenfunctions due to linearity.

Similarity reductions

Similarity reductions can be achieved by integrating the associated characteristic system

$$\frac{dx}{\xi_x} = \frac{dt}{\xi_t} = \frac{d\lambda}{\xi_\lambda} = \frac{d\alpha}{\eta_\alpha} = \frac{d\chi}{\eta_\chi} = \frac{d\psi}{\eta_\psi}, \tag{6.40}$$

giving rise to two nontrivial reductions

(i) $B_1 \neq 0$
(ii) $B_1 = 0$, $B_2 \neq 0$, $B_3 \neq 0$

that should be treated independently.

The reduced variables will be denoted as usual as $\{z, \Lambda\}$, the reduced field will be displayed as $\mathcal{A}(z)$ and the reduced eigenfunctions are $\Phi(z, \Lambda)$, $\Psi(z, \Lambda)$. Without loss of generality, we will consider $K_1(\lambda) = 0$, $K_2(\lambda) = 0$.

- **Case I. $B_1 \neq 0$ ($B_4 = B_1$)**

  - Reduced variable and reduced spectral parameter

    $$z = \frac{B_1 x + B_2}{\sqrt{B_1}\sqrt{B_3 + 2B_1 t}}, \qquad \Lambda = \frac{\lambda}{\sqrt{B_1}}\sqrt{B_3 + 2B_1 t}. \tag{6.41}$$

  - Reduced field
    $$\alpha(x, t) = \frac{B_4}{2B_1} \log(B_3 + 2B_1 t) + \mathcal{A}(z), \tag{6.42}$$

    where $\log(\cdot)$ denotes the natural logarithm.

  - Reduced eigenfunctions

    $$\chi(x, t, \lambda) = \Phi(z, \Lambda), \qquad \psi(x, t, \lambda) = \Psi(z, \Lambda). \tag{6.43}$$

  - Reduced Φ-spectral problem

    $$2\Phi_{zz} - \left(2i\Lambda + i\mathcal{A}_z + \frac{\mathcal{A}_{zz} + i}{\mathcal{A}_z}\right)\Phi_z - \left(i\mathcal{A}_{zz} - \mathcal{A}_z^2 + 1\right)\Phi = 0,$$

    $$2\Lambda\Phi_\Lambda - \left(2z - 2\Lambda + \mathcal{A}_z + \frac{i\mathcal{A}_{zz} + 1}{\mathcal{A}_z}\right)\Phi_z + \left(2i\Lambda^2 + i\mathcal{A}_z^2 - \mathcal{A}_{zz} + i\right)\Phi = 0.$$

    $$\tag{6.44}$$

– Reduced Ψ-spectral problem

$$2\Psi_{zz} - \left(-2i\Lambda + \frac{2\mathcal{A}_z^2\mathcal{A}_{zz} + i\mathcal{A}_{zz}^2 + i}{\mathcal{A}_z\left(\mathcal{A}_z^2 - i\mathcal{A}_{zz} - 1\right)}\right)\Psi_z + \frac{\left(\mathcal{A}_z^2 - 1\right)^2 + \mathcal{A}_{zz}^2}{\mathcal{A}_z^2 - i\mathcal{A}_{zz} - 1}\Psi = 0,$$

$$2i\Lambda\Psi_\Lambda + \left(2i\Lambda - 2iz + \frac{2\mathcal{A}_z^2\mathcal{A}_{zz} + i\mathcal{A}_{zz}^2 + i}{\mathcal{A}_z\left(\mathcal{A}_z^2 - i\mathcal{A}_{zz} - 1\right)}\right)\Psi_z$$
$$+ \left(2\Lambda^2 - \mathcal{A}_z^2 - i\mathcal{A}_{zz} + i\right)\Psi = 0.$$

$$(6.45)$$

The link between these two equivalent reduced Lax pairs is given by

$$2\frac{\Phi_z\Psi_z}{\Phi\Psi} - i\mathcal{A}_{zz} + \mathcal{A}_z^2 + z\mathcal{A}_z - 1 = 0. \tag{6.46}$$

– Reduced equation

$$\left[\mathcal{A}_{zzz} - \frac{\mathcal{A}_{zz}^2 + 1}{\mathcal{A}_z} + \mathcal{A}_z^3 + z(\mathcal{A}_z^2 + \mathcal{A}) - \int (\mathcal{A} - \mathcal{A}_z^2)dz\right]_z = 0. \tag{6.47}$$

- **Case II. $B_1 = 0$, $B_2 \neq 0$, $B_3 \neq 0$ ($B_4 = B_2^2/B_3$)**

  – Reduced variable and reduced spectral parameter

  $$z = \frac{B_2}{B_3}\left(x - \frac{B_2}{B_3}t\right), \qquad \Lambda = \frac{B_3}{B_2}\lambda. \tag{6.48}$$

  – Reduced field
  $$\alpha(x, t) = \frac{B_4}{B_3}t + \mathcal{A}(z). \tag{6.49}$$

  – Reduced eigenfunctions
  $$\chi(x, t, \lambda) = \Phi(z, \Lambda), \qquad \psi(x, t, \lambda) = \Psi(z, \Lambda). \tag{6.50}$$

  – Reduced Φ-spectral problem

  $$2\Phi_{zz} - \left(2i\Lambda + i\mathcal{A}_z + \frac{\mathcal{A}_{zz} + i}{\mathcal{A}_z}\right)\Phi_z - \left(i\mathcal{A}_{zz} - \mathcal{A}_z^2 + 1\right)\Phi = 0,$$
  $$\left(2\Lambda - 2\mathcal{A}_z + i\mathcal{A}_{zz} - \mathcal{A}_z^2 - 1\right)\Phi_z + \mathcal{A}_z\left(2i\Lambda^2 + i\mathcal{A}_z^2 + \mathcal{A}_{zz} - 1\right)\Phi = 0.$$

  $$(6.51)$$

– Reduced Ψ-spectral problem

$$
2\Psi_{zz} - \left(2i\Lambda - \frac{2\mathcal{A}_z^2 \mathcal{A}_{zz} + i\mathcal{A}_{zz}^2 + i}{\mathcal{A}_z \left(\mathcal{A}_z^2 - i\mathcal{A}_{zz} - 1\right)}\right)\Psi_z + \frac{\left(\mathcal{A}_z^2 - 1\right)^2 + \mathcal{A}_{zz}^2}{\mathcal{A}_z^2 - i\mathcal{A}_{zz} - 1}\Psi = 0,
$$

$$
\left(-2\Lambda + \frac{2i\mathcal{A}_z^2 \mathcal{A}_{zz} + \mathcal{A}_{zz}^2 - 2i\mathcal{A}_z^2 \mathcal{A}_{zz} - 2\mathcal{A}_z^3 - \mathcal{A}_z^2 + \left(\mathcal{A}_z + 1\right)^2}{\mathcal{A}_z \left(\mathcal{A}_z^2 - i\mathcal{A}_{zz} - 1\right)}\right)\Psi_z
$$

$$
- \left(2i\Lambda^2 - i\mathcal{A}_z^2 + \mathcal{A}_{zz} + i\right)\Psi = 0,
$$

(6.52)

where the two equivalent Lax pairs are connected via the transformation (6.46).

– Reduced equation

$$
\left[\mathcal{A}_{zzz} - \frac{\mathcal{A}_{zz}^2 + 1}{\mathcal{A}_z} + \mathcal{A}_z^3 + \mathcal{A}_z^2\right]_z = 0.
$$

(6.53)

1.3. Nonlocal Boussinesq equation in $1+1$ dimensions

Painlevé analysis for the generalized DNLS equation in Section 2 from Chapter 3 yielded a third PDE of interest

$$
\left[u_{tt} + u_{xxxx} + 2u_{xx}^2 - \frac{u_{xt}^2 + u_{xxx}^2}{u_{xx}}\right]_x = 0,
$$

(6.54)

which is known as the nonlocal Boussinesq equation [261, 419]. This equations has proved to be integrable and its connection to the Kaup system has been studied in [141]. The SMM provides the following spectral problem for (6.54)

$$
\chi_{xx} = \chi_x \left[i\lambda + \frac{u_{xxx} + iu_{xt}}{2u_{xx}}\right] - u_{xx}\chi,
$$

$$
\chi_t = \chi_x \left[-\lambda + \frac{-iu_{xxx} + u_{xt}}{2u_{xx}}\right] - i\left[\lambda^2 + u_{xx}\right]\chi,
$$

(6.55)

$$
\psi_{xx} = \psi_x \left[-i\lambda + \frac{u_{xxx} - iu_{xt}}{2u_{xx}}\right] - u_{xx}\psi,
$$

$$
\psi_t = \psi_x \left[-\lambda + \frac{iu_{xxx} + u_{xt}}{2u_{xx}}\right] + i\left[\lambda^2 + u_{xx}\right]\psi,
$$

(6.56)

where $\{\chi, \psi\}$ are two complex conjugate eigenfunctions satisfying $\frac{\psi_x \chi_x}{\psi \chi} + u_{xx} = 0$ and λ is the spectral parameter.

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As stated in Chapter 3, the complex conjugate of u, named as $\overline{u}$, satisfies the same equation (6.54). Then, the system (6.55)-(6.56) also constitutes a Lax pair for $\overline{u}$. Notwithstanding, u and $\overline{u}$ must obey the following coupling condition

$$iu_t + u_{xx} - i\overline{u}_t + \overline{u}_{xx} + (u_x - \overline{u}_x)^2 = 0 \tag{6.57}$$

as a consequence of the ansatz (3.98) and relation (3.99).

Lie point symmetries

Lie symmetries for the Lax pair (6.55)-(6.56) and their conjugates have been computed, as the associated spectral problems to the nonlinear equation (6.54) for u and $\overline{u}$, respectively. In this case, we also need to take into account the coupling condition (6.57) in the symmetry analysis. If we introduce a uniparametric infinitesimal transformation of the form

$$
\begin{aligned}
\tilde{x} &= x + \epsilon\,\xi_x(x,t,\lambda,u,\overline{u},\chi,\overline{\chi},\psi,\overline{\psi}) + \mathcal{O}(\epsilon^2),\\
\tilde{t} &= t + \epsilon\,\xi_t(x,t,\lambda,u,\overline{u},\chi,\overline{\chi},\psi,\overline{\psi}) + \mathcal{O}(\epsilon^2),\\
\tilde{\lambda} &= \lambda + \epsilon\,\xi_\lambda(x,t,\lambda,u,\overline{u},\chi,\overline{\chi},\psi,\overline{\psi}) + \mathcal{O}(\epsilon^2),\\
\tilde{u} &= u + \epsilon\,\eta_u(x,t,\lambda,u,\overline{u},\chi,\overline{\chi},\psi,\overline{\psi}) + \mathcal{O}(\epsilon^2),\\
\tilde{\overline{u}} &= \overline{u} + \epsilon\,\eta_{\overline{u}}(x,t,\lambda,u,\overline{u},\chi,\overline{\chi},\psi,\overline{\psi}) + \mathcal{O}(\epsilon^2),\\
\tilde{\chi} &= \chi + \epsilon\,\eta_\chi(x,t,\lambda,u,\overline{u},\chi,\overline{\chi},\psi,\overline{\psi}) + \mathcal{O}(\epsilon^2),\\
\tilde{\overline{\chi}} &= \overline{\chi} + \epsilon\,\eta_{\overline{\chi}}(x,t,\lambda,u,\overline{u},\chi,\overline{\chi},\psi,\overline{\psi}) + \mathcal{O}(\epsilon^2),\\
\tilde{\psi} &= \psi + \epsilon\,\eta_\psi(x,t,\lambda,u,\overline{u},\chi,\overline{\chi},\psi,\overline{\psi}) + \mathcal{O}(\epsilon^2),\\
\tilde{\overline{\psi}} &= \overline{\psi} + \epsilon\,\eta_{\overline{\psi}}(x,t,\lambda,u,\overline{u},\chi,\overline{\chi},\psi,\overline{\psi}) + \mathcal{O}(\epsilon^2),
\end{aligned}
\tag{6.58}
$$

and apply Lie's method to compute the classical Lie symmetries, we obtain

$$
\begin{aligned}
\xi_x(x,t,\lambda,u,\overline{u},\chi,\overline{\chi},\psi,\overline{\psi}) &= C_1 x + 2C_4 t + C_2,\\
\xi_t(x,t,\lambda,u,\overline{u},\chi,\overline{\chi},\psi,\overline{\psi}) &= 2C_1 t + C_3,\\
\xi_\lambda(x,t,\lambda,u,\overline{u},\chi,\overline{\chi},\psi,\overline{\psi}) &= -C_1\,\lambda + C_4,\\
\eta_u(x,t,\lambda,u,\overline{u},\chi,\overline{\chi},\psi,\overline{\psi}) &= \left(C_5 + \frac{iC_4}{2}\right)x + Z_1(t) + iC_6,\\
\eta_{\overline{u}}(x,t,\lambda,u,\overline{u},\chi,\overline{\chi},\psi,\overline{\psi}) &= \left(C_5 - \frac{iC_4}{2}\right)x + Z_1(t) - iC_6,\\
\eta_\chi(x,t,\lambda,u,\overline{u},\chi,\overline{\chi},\psi,\overline{\psi}) &= \left(-2iC_4 t\lambda + K_1(\lambda)\right)\chi,
\end{aligned}
\tag{6.59}
$$

$$\eta_{\overline{\chi}}(x, t, \lambda, u, \overline{u}, \chi, \overline{\chi}, \psi, \overline{\psi}) = (2iC_4 t\lambda + \overline{K}_1(\lambda))\, \overline{\chi},$$
$$\eta_{\psi}(x, t, \lambda, u, \overline{u}, \chi, \overline{\chi}, \psi, \overline{\psi}) = (2iC_4 t\lambda + K_2(\lambda))\, \psi,$$
$$\eta_{\overline{\psi}}(x, t, \lambda, u, \overline{u}, \chi, \overline{\chi}, \psi, \overline{\psi}) = (-2iC_4 t\lambda + \overline{K}_2(\lambda))\, \overline{\psi},$$

where C_i, $i = 1, \ldots, 6$ are arbitrary real constants, $Z_1(t)$ is an arbitrary real function of t and $K_j(\lambda), j = 1, 2$ are arbitrary complex functions of λ. It is worth stressing the presence of a complete new symmetry associated to C_4, induced by the Miura transformation applied over α (3.98). The nonclassical method provides exactly the same Lie symmetries (6.59).

Similarity reductions

Similarity reductions for the spectral problem (6.55)-(6.56) are obtained by the usual procedure described above, which requires the integration of the characteristic system

$$\frac{dx}{\xi_x} = \frac{dt}{\xi_t} = \frac{d\lambda}{\xi_\lambda} = \frac{d\,u}{\eta_u} = \frac{d\,\overline{u}}{\eta_{\overline{u}}} = \frac{d\chi}{\eta_\chi} = \frac{d\overline{\chi}}{\eta_{\overline{\chi}}} = \frac{d\psi}{\eta_\psi} = \frac{d\overline{\psi}}{\eta_{\overline{\psi}}}. \tag{6.60}$$

$\mathcal{U}(z)$ and $\overline{\mathcal{U}}(z)$ will denote the reduced fields for u, $\overline{u}$ respectively, while the remaining reduced variables follow the notation introduced in the above Subsections. In this case we are particularly interested in the role of the constants C_1 and C_4 in the similarity reductions. Hence, three nontrivial cases emerge, listed as

(i) $C_1 \neq 0$
(ii) $C_1 = 0$, $C_3 \neq 0$, $C_4 \neq 0$
(iii) $C_1 = 0$, $C_4 = 0$, $C_3 \neq 0$, $C_5 \neq 0$

We will set $C_6 = 0$, $K_1(\lambda) = 0$, $K_2(\lambda) = 0$ in order to perform the reductions. We have selected particular values for some arbitrary constants in order to simplify the resulting reduced problems.

- *Case I. $C_1 \neq 0$*

    – Reduced variable and reduced spectral parameter

$$z = \frac{C_1^2 x - (2C_4(C_1 t + C_3) - C_1 C_2)}{C_1^{3/2}\sqrt{C_3 + 2C_1 t}}, \qquad \Lambda = \frac{C_1 \lambda - C_4}{C_1^{3/2}}\sqrt{C_3 + 2C_1 t}. \tag{6.61}$$

– Reduced field

$$
\begin{aligned}
u(x,t) = \mathcal{U}(z) &+ \int \frac{Z_1(t)}{C_3 + 2C_1 t} dt \\
&+ \left(C_5 + \frac{iC_4}{2} \right) \left[\frac{z\sqrt{C_3 + 2C_1 t}}{C_1^{3/2}} + \frac{C_4 t}{C_1^2} - \frac{C_1 C_2 - C_3 C_4}{2C_1^3} \log\left(2C_1 t + C_3\right) \right].
\end{aligned}
\tag{6.62}
$$

– Reduced eigenfunctions

$$
\chi(x,t,\lambda) = \left(\frac{C_1 \lambda - C_4 t}{C_1^{3/2}} \right)^{-i\frac{C_3 C_4^2}{C_1^3}} e^{-i\frac{C_4}{C_1^3}(2C_1(C_3+C_1 t) - C_4(C_3+C_1 t))\lambda)} \Phi(z,\Lambda),
$$

$$
\psi(x,t,\lambda) = \left(\frac{C_1 \lambda - C_4 t}{C_1^{3/2}} \right)^{i\frac{C_3 C_4^2}{C_1^3}} e^{i\frac{C_4}{C_1^3}(2C_1(C_3+C_1 t) - C_4(C_3+C_1 t))\lambda)} \Psi(z,\Lambda).
$$

$$
\tag{6.63}
$$

– Reduced spectral problem

$$
\Phi_{zz} + \left(\frac{iz}{2} - i\Lambda + \frac{i\mathcal{U}_z - \mathcal{U}_{zzz}}{2\mathcal{U}_{zz}} \right) \Phi_z - \mathcal{U}_{zz}\Phi = 0,
$$

$$
\Lambda\Phi_\Lambda + \left(-\frac{z}{2} + \Lambda + \frac{\mathcal{U}_z + i\mathcal{U}_{zzz}}{2\mathcal{U}_{zz}} \right) \Phi_z + i\left(\Lambda^2 - \mathcal{U}_{zz} \right) \Phi = 0,
$$

$$
\tag{6.64}
$$

$$
\Psi_{zz} + \left(-\frac{iz}{2} + i\Lambda - \frac{i\mathcal{U}_z + \mathcal{U}_{zzz}}{2\mathcal{U}_{zz}} \right) \Psi_z - \mathcal{U}_{zz}\Psi = 0,
$$

$$
\Lambda\Psi_\Lambda + \left(-\frac{z}{2} + \Lambda + \frac{\mathcal{U}_z - i\mathcal{U}_{zzz}}{2\mathcal{U}_{zz}} \right) \Psi_z - i\left(\Lambda^2 - \mathcal{U}_{zz} \right) \Psi = 0,
$$

$$
\tag{6.65}
$$

that are linked via the relation $\frac{\Phi_z \Psi_z}{\Phi\Psi} - \mathcal{U}_{zz} = 0$.

– Reduced equation

$$
\left[\mathcal{U}_{zzzz} - \frac{\mathcal{U}_{zzz}^2 + \mathcal{U}_z^2}{\mathcal{U}_{zz}} - 2\mathcal{U}_{zz}^2 + z\mathcal{U}_z \right]_z = 0.
\tag{6.66}
$$

Similarity reductions for the field $\overline{\mathcal{U}}$ and its associated spectral problem given in terms of the eigenfunctions $\{\overline{\Phi}, \overline{\Psi}\}$ can be computed by taking the complex conjugate of (6.64)-(6.65) and (6.66), respectively. Besides, the coupling condition (6.57) that relates $\mathcal{U}$ and $\overline{\mathcal{U}}$ reduces as

$$
iz\mathcal{U}_z - \mathcal{U}_{zz} + \mathcal{U}_z^2 - iz\overline{\mathcal{U}}_z - \overline{\mathcal{U}}_{zz} + \overline{\mathcal{U}}_z^2 - 2\mathcal{U}_z\overline{\mathcal{U}}_z = 0.
\tag{6.67}
$$

- **Case II. $C_1 = 0$, $C_3 \neq 0$, $C_4 \neq 0$ $(C_2 = C_3^{2/3} C_4^{1/3}, C_5 = 0)$**

    - Reduced variable and reduced spectral parameter

    $$z = \frac{C_4^{1/3}}{C_3^{1/3}} \left(x - \frac{C_2}{C_3} t - \frac{C_4}{C_3} t^2 \right), \qquad \Lambda = \frac{C_3^{1/3}}{C_4^{1/3}} \left(\lambda - \frac{C_4}{C_3} t \right). \qquad (6.68)$$

    - Reduced field

    $$u(x, t) = \frac{iC_4}{2C_3} \left(zt - \frac{C_2}{2C_3} t^2 - \frac{C_4}{3C_3} t^3 \right) + \int \frac{Z_1(t)}{C_3} dt + \mathcal{U}(z). \qquad (6.69)$$

    - Reduced eigenfunctions

    $$\chi = e^{-i \frac{C_4 t^2}{C_3} \left(\lambda - \frac{C_4}{3C_3} t \right)} \Phi(z, \Lambda), \quad \psi = e^{i \frac{C_4 t^2}{C_3} \left(\lambda - \frac{C_4}{3C_3} t \right)} \Psi(z, \Lambda). \qquad (6.70)$$

    - Reduced spectral problem

    $$\Phi_{zz} + \frac{1}{2} \left(i - 2i\Lambda - \frac{1}{2\mathcal{U}_{zz}} - \frac{\mathcal{U}_{zzz}}{\mathcal{U}_{zz}} \right) \Phi_z - \mathcal{U}_{zz} \Phi = 0,$$
    $$\Phi_\Lambda + \frac{1}{2} \left(1 - 2\Lambda - \frac{i}{2\mathcal{U}_{zz}} - \frac{i\mathcal{U}_{zzz}}{\mathcal{U}_{zz}} \right) \Phi_z - i \left(\Lambda^2 - \mathcal{U}_{zz} \right) \Phi = 0, \qquad (6.71)$$

    $$\Psi_{zz} + \frac{1}{2} \left(-i + 2i\Lambda - \frac{1}{2\mathcal{U}_{zz}} - \frac{\mathcal{U}_{zzz}}{\mathcal{U}_{zz}} \right) \Psi_z - \mathcal{U}_{zz} \Psi = 0,$$
    $$\Psi_\Lambda + \frac{1}{2} \left(1 - 2\Lambda + \frac{i}{2\mathcal{U}_{zz}} + \frac{i\mathcal{U}_{zzz}}{\mathcal{U}_{zz}} \right) \Psi_z + i \left(\Lambda^2 - \mathcal{U}_{zz} \right) \Psi = 0, \qquad (6.72)$$

    such that $\frac{\Phi_z \Psi_z}{\Phi \Psi} - \mathcal{U}_{zz} = 0$.

    - Reduced equation

    $$\left[\mathcal{U}_{zzzz} + \frac{1 - 4\mathcal{U}_{zzz}^2}{4\mathcal{U}_{zz}} - 2\mathcal{U}_{zz}^2 - 2\mathcal{U}_z \right]_z = 0. \qquad (6.73)$$

    The associated reduced spectral problem for $\overline{\mathcal{U}}$ can be deductible by the complex conjugate of (6.71)-(6.72) and (6.73), where the coupling condition (6.57) reduces as

    $$i\mathcal{U}_z - \mathcal{U}_{zz} + \mathcal{U}_z^2 - i\overline{\mathcal{U}}_z - \overline{\mathcal{U}}_{zz} + \overline{\mathcal{U}}_z^2 - z - 2\mathcal{U}_z \overline{\mathcal{U}}_z = 0. \qquad (6.74)$$

- **Case III. $C_1 = 0$, $C_4 = 0$, $C_3 \neq 0$, $C_5 \neq 0$ $(C_2 = C_3^{2/3} C_5^{1/3})$**

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– Reduced variable and reduced spectral parameter

$$z = \frac{C_5^{1/3}}{C_3^{1/3}}\left(x - \frac{C_2}{C_3}t\right), \qquad \Lambda = \frac{C_5^{1/3}}{C_3^{1/3}}\left(\lambda - \frac{C_2}{2C_3}\right). \tag{6.75}$$

– Reduced field

$$u(x,t) = \frac{C_5}{C_3}\left(\frac{C_3^{1/3}}{C_5^{1/3}}zt - \frac{C_2}{2C_3}t^2\right) + \int \frac{Z_1(t)}{C_3}dt + \mathcal{U}(z). \tag{6.76}$$

– Reduced eigenfunctions

$$\chi(x,t,\lambda) = \Phi(z,\Lambda), \qquad \psi(x,t,\lambda) = \Psi(z,\Lambda). \tag{6.77}$$

– Reduced spectral problem

$$\Phi_{zz} - \frac{1}{2}\left(2i\Lambda + \frac{\mathcal{U}_{zzz} - i}{\mathcal{U}_{zz}}\right)\Phi_z - \mathcal{U}_{zz}\Phi = 0,$$
$$(2\Lambda\mathcal{U}_{zz} + i\mathcal{U}_{zzz} + 1)\Phi_z - \left(2i\mathcal{U}_{zz}^2 - \frac{i}{2}(2\Lambda + 1)^2\mathcal{U}_{zz}\right)\Phi = 0, \tag{6.78}$$

$$\Psi_{zz} - \frac{1}{2}\left(-2i\Lambda + \frac{\mathcal{U}_{zzz} + i}{\mathcal{U}_{zz}}\right)\Psi_z - \mathcal{U}_{zz}\Psi = 0,$$
$$(2\Lambda\mathcal{U}_{zz} - i\mathcal{U}_{zzz} + 1)\Psi_z - \left(-2i\mathcal{U}_{zz}^2 + \frac{i}{2}(2\Lambda + 1)^2\mathcal{U}_{zz}\right)\Psi = 0, \tag{6.79}$$

such that $\frac{\Phi_z\Psi_z}{\Phi\Psi} - \mathcal{U}_{zz} = 0$.

– Reduced equation

$$\left[\mathcal{U}_{zzzz} - \frac{\mathcal{U}_{zzz}^2}{\mathcal{U}_{zz}} - \frac{1}{\mathcal{U}_{zz}} - 2\mathcal{U}_{zz}^2\right]_z = 0. \tag{6.80}$$

As usual, the associated reduced spectral problem for $\overline{\mathcal{U}}$ can be obtained as the complex conjugate of (6.78)-(6.79) and (6.80), where $\mathcal{U}$ and $\overline{\mathcal{U}}$ satisfy

$$i\mathcal{U}_z - \mathcal{U}_{zz} + \mathcal{U}_z^2 - i\overline{\mathcal{U}}_z - \overline{\mathcal{U}}_{zz} + \overline{\mathcal{U}}_z^2 - 2\mathcal{U}_z\overline{\mathcal{U}}_z = 0. \tag{6.81}$$

2. Lie symmetries for nonlinear systems in $2+1$ dimensions

This Section is aimed at the application of the Lie symmetry analysis to two systems of PDEs that represent integrable generalizations of NLS equations in $2+1$ dimensions, with their respective linear problems. The treatment of spectral problems in higher spatial dimensions is extremely appealing due to the natural emergence of nonisospectral Lax pairs, and their consequences.

2.1. Generalized NLS equation in $2+1$ dimensions with higher-order dispersive terms

In the present Subsection we consider an integrable system formulated in term of the NLS equation in $2+1$ dimensions with higher-order dispersive contributions. Higher-order dispersion terms emerge in the theoretical description of diverse phenomena in Physics or related disciplines. In particular, NLS equation constitutes a more than suitable candidate for this role, given its myriad applications in the context of nonlinear phenomenology. Several celebrated integrable generalizations of the (1+1)-NLS equation have been proposed in this fashion, such as the Hirota equation [209], the Sasa-Satsuma equation [368] or the Lakshmanan-Porsezian-Daniel equation [258]. The extension to higher spatial dimensions naturally arises in order to include more degrees of freedom in the spatial coordinates and describe more complex scenarios. Various extensions of NLS equation to $2+1$ dimensions can be consulted in [5, 69, 167, 428].

Estévez *et al.* proposed in [142] the following generalized NLS equation with higher-order terms as a promising starting candidate to describe the dynamics of α-helical proteins within the continuum approximation in multiple spatial dimensions,

$$
\begin{aligned}
&iu_t + u_{xy} + 2um_y + i\gamma_2 \left(u_{xxx} - 6uwu_x\right) \\
&\quad + \gamma_1(u_{xxxx} - 8uwu_{xx} - 2u^2w_{xx} - 4uu_xw_x - 6wu_x^2 + 6u^3w^2) = 0, \\
&-iw_t + w_{xy} + 2wm_y - i\gamma_2 \left(w_{xxx} - 6uww_x\right) \\
&\quad + \gamma_1(w_{xxxx} - 8uww_{xx} - 2w^2u_{xx} - 4wu_xw_x - 6uw_x^2 + 6u^2w^3) = 0, \\
&(m_x + uw)_y = 0,
\end{aligned}
\tag{6.82}
$$

where $u = u(x, y, t)$ and $w = w(x, y, t)$ are complex conjugates and $m = m(x, y, t)$ is a real field. This equation can be considered as a higher-order nonlinear Schrödinger equation that includes third ($\gamma_2 \neq 0$) and fourth ($\gamma_1 \neq 0$) order derivatives with

respect to x. The system (6.82) generalizes to $2+1$ dimensions the system proposed by Ankiewitz *et al.* in [34, 36], which contains as particular cases many of the integrable generalizations cited above. Furthermore, for $\gamma_1 = \gamma_2 = 0$, (6.82) reduces to the $(2+1)$-NLS equation described in [69].

The inspection of the integrability properties of (6.82) has been extensively studied in [142] and it has been proved that this model has the Painlevé Property for arbitrary values of γ_1, γ_2. It possesses a two-component Lax pair, deducible through the SMM, and the Darboux transformation approach retrieves a wide family of lump solutions. The associated spectral problem to (6.82) is a nonisospectral Lax pair of the following form

$$
\begin{aligned}
\psi_x &= -i\lambda\psi - u\chi, \\
\chi_x &= i\lambda\chi - w\psi, \\
\psi_t &= 2\lambda\psi_y + \lambda_y\psi + i\,(m_y\psi - u_y\chi) + (\gamma_2 - 2\lambda\gamma_1)A_1(u,w,\lambda,\psi,\chi) \\
&\quad + i\gamma_1 A_2(u,w,\lambda,\psi,\chi), \\
\chi_t &= 2\lambda\chi_y + \lambda_y\chi - i\,(m_y\chi - w_y\psi) + (\gamma_2 - 2\lambda\gamma_1)A_1(w,u,\lambda,\chi,\psi) \\
&\quad - i\gamma_1 A_2(w,u,\lambda,\chi,\psi),
\end{aligned}
\tag{6.83}
$$

where $\psi(x,y,t)$, $\chi(x,y,t)$ are two complex conjugate eigenfunctions, $\lambda = \lambda(y,t)$ is the spectral parameter and

$$
\begin{aligned}
A_1(u,w,\lambda,\psi,\chi) &= 3\left(uw + \lambda^2\right)\psi_x - \psi_{xxx} - 3u_x\chi_x, \\
A_2(u,w,\lambda,\psi,\chi) &= \left(3u^2w^2 + u_xw_x - uw_{xx} - wu_{xx}\right)\psi + \left(6uwu_x - u_{xxx}\right)\chi.
\end{aligned}
\tag{6.84}
$$

The compatibility condition of the linear problem (6.83) yields the starting system (6.82) as well as the nonisospectral condition

$$
\lambda_t - 2\lambda\lambda_y = 0.
\tag{6.85}
$$

Lie symmetries and similarity reductions for $(2+1)$-nonisospectral Lax pairs

One of the crucial aspects of Lax pairs in higher spatial dimensions lies in their non-isospectral nature. In those cases, the spectral parameter is expected to satisfy an additional constraint, given by a (nonlinear) PDE, as in (6.85). Then, Lie's formalism should be slightly modified in order to incorporate this supplementary condition. Lie symmetries should now leave invariant not only the system of equations defining the Lax pair (6.83) but also the nonisospectral condition for λ (6.85). In such scenario, the spectral parameter should be therefore treated as a scalar field of its

corresponding variables. This fact establishes a clear difference between the isospectral and the nonisospectral case from the symmetry point of view. Actually, this distinction enables the possibility of finding nonisospectral linear problems when a nonisospectral Lax pair in higher dimensions is reduced.

Then, the space of independent variables $\mathscr{X}$ has coordinates (x, y, t) while the space of dependent variables $\mathscr{U}$ is endowed with coordinates $(u, w, \lambda, \psi, \chi)$. After some manipulations, the spectral problem (6.83) can be expressed as a zero-curvature Lax pair of first-order in the eigenfunctions. Nevertheless, third-order derivatives for the fields u, w appear in the aforementioned linear problem, which need to be taken into account. Then, the spectral problem (6.83) defines a submanifold in the third-order jet space $\mathcal{M}_{(3)} = \mathscr{X} \times \mathscr{U}_{(3)}$.

Lie's symmetry method for the linear problem (6.83) requires the consideration of the following one-parameter group of infinitesimal transformations

$$
\begin{aligned}
\tilde{x} &= x + \epsilon \xi_1(x, y, t, u, w, m, \lambda, \psi, \chi) + \mathcal{O}(\epsilon^2), \\
\tilde{y} &= y + \epsilon \xi_2(x, y, t, u, w, m, \lambda, \psi, \chi) + \mathcal{O}(\epsilon^2), \\
\tilde{t} &= t + \epsilon \xi_3(x, y, t, u, w, m, \lambda, \psi, \chi) + \mathcal{O}(\epsilon^2), \\
\tilde{u} &= u + \epsilon \eta_1(x, y, t, u, w, m, \lambda, \psi, \chi) + \mathcal{O}(\epsilon^2), \\
\tilde{w} &= w + \epsilon \eta_2(x, y, t, u, w, m, \lambda, \psi, \chi) + \mathcal{O}(\epsilon^2), \\
\tilde{m} &= m + \epsilon \eta_3(x, y, t, u, w, m, \lambda, \psi, \chi) + \mathcal{O}(\epsilon^2), \\
\tilde{\lambda} &= \lambda + \epsilon \eta_4(x, y, t, u, w, m, \lambda, \psi, \chi) + \mathcal{O}(\epsilon^2), \\
\tilde{\psi} &= \psi + \epsilon \phi_1(x, y, t, u, w, m, \lambda, \psi, \chi) + \mathcal{O}(\epsilon^2), \\
\tilde{\chi} &= \chi + \epsilon \phi_2(x, y, t, u, w, m, \lambda, \psi, \chi) + \mathcal{O}(\epsilon^2),
\end{aligned}
\tag{6.86}
$$

where ϵ is the parameter of the group and ξ_1, ξ_2, ξ_3, η_1, η_2, η_3, η_4, ϕ_1 and ϕ_2 are the associated infinitesimals, which define the vector field

$$
X = \xi_1 \frac{\partial}{\partial x} + \xi_2 \frac{\partial}{\partial y} + \xi_3 \frac{\partial}{\partial t} + \eta_1 \frac{\partial}{\partial u} + \eta_2 \frac{\partial}{\partial w} + \eta_3 \frac{\partial}{\partial m} + \eta_4 \frac{\partial}{\partial \lambda} + \phi_1 \frac{\partial}{\partial \psi} + \phi_2 \frac{\partial}{\partial \chi}. \tag{6.87}
$$

The infinitesimal transformation (6.86) can be extended to the different derivatives of the dependent variables as introduced in Chapter 5. This procedure, when applied to (6.83), yields an overdetermined system of PDEs, whose solution provides the Lie symmetries.

We shall remark that Lie's approach, either classical or nonclassical, requires the substitution of the highest-order derivatives in the system of PDEs under study. The order of the highest derivatives in (6.83) is different depending on whether γ_1 and γ_2 are (one or both) null or not. This means that it will necessary to split the

problem in four different cases depending on the different combinations of γ_1 and γ_2, and each of them should be analyzed separately. In this context, the following integrable equations arise:

1. Generalized NLS equation in $2+1$ dimensions [142] ($\gamma_1 \neq 0$, $\gamma_2 \neq 0$)

2. Lakshmanan-Porsezian-Daniel equation [258] in $2+1$ dimensions ($\gamma_1 \neq 0$, $\gamma_2 = 0$)

3. Hirota equation [209] in $2+1$ dimensions ($\gamma_1 = 0$, $\gamma_2 \neq 0$)

4. Standard NLS equation in $2+1$ dimensions [69] ($\gamma_1 = 0$, $\gamma_2 = 0$)

where each equation can be obtained by selecting the proper values of the parameters γ_1, γ_2 in (6.82). The associated spectral problems for each case may be easily found by taking the same ansatz for γ_1, γ_2 in (6.83).

It is worthwhile to mention that we have calculated both the classical and the non-classical symmetries for each case described above. The computational complexity presented in the nonclassical method substantially rises when considering more degrees of freedom, as well as when higher orders of derivation are involved. Therefore, the nonclassical symmetry analysis emerges as an intractable problem, due to the highly nonlinear nature of the overdetermined system for the infinitesimals. When solvable (for the simplest cases with $\xi_i \neq 0$, $i = 1, \ldots, 3$), the overdetermined system for the nonclassical symmetries provides the same result than the classical one. Therefore, the results displayed in the ensuing Sections correspond to the classical Lie symmetries for each pertinent case under consideration.

Hence, in the following we will be dealing with the classical Lie symmetries and reductions for (6.83). These symmetries obviously provide the corresponding symmetries and reductions for the nonlinear equation (6.82) as well as for the nonisospectral condition (6.85). Lie symmetries for any of the aforementioned cases have been analyzed in deep and their associated commutation relations are reflected in Appendix B.1. Once the classical symmetries (6.86) have been computed, we may conduct the symmetry reduction procedure to obtain the associated similarity reductions by the standard method described in Chapter 5. We can achieve that by solving the characteristic system

$$\frac{dx}{\xi_1} = \frac{dy}{\xi_2} = \frac{dt}{\xi_3} = \frac{du}{\eta_1} = \frac{dw}{\eta_2} = \frac{dm}{\eta_3} = \frac{d\lambda}{\eta_4} = \frac{d\psi}{\phi_1} = \frac{d\chi}{\phi_2}. \tag{6.88}$$

Similarity reductions for (6.83) will provide reduced problems defined as systems of PDEs depending on two independent variables p, q. $\Lambda(p,q)$ represents the reduced

spectral problem, whose isospectral or nonisospectral nature is yet to be determined, and it will depend on the reduction considered in each particular case. $F(p,q)$, $H(p,q)$, $N(p,q)$, $\Phi(p,q,\Lambda)$ and $\Omega(p,q,\Lambda)$ denote the invariants that arise from the integration of the charasteristic system (6.88). They correspond to the integration in u, w, m, ψ and χ, respectively. These notation can be summarized in the table:

	Original variables	New reduced variables
Independent variables	x, y, t	p, q
Spectral parameter	$\lambda(y, t)$	$\Lambda(p, q)$
Fields	$u(x, y, t), w(x, y, t)$	$F(p, q)$, $H(p, q)$
	$m(x, y, t)$	$N(p, q)$
Eigenfunctions	$\psi(x, y, t, \lambda), \chi(x, y, t, \lambda)$	$\Omega(p, q, \Lambda)$, $\Phi(p, q, \Lambda)$

The following lines present the Lie symmetry analysis and different similarity reductions for the four cases under consideration mentioned above. We will proceed in each case as described in the previous paragraphs.

2.1.1. Generalized NLS equation in $2 + 1$ dimensions ($\gamma_1 \neq 0$, $\gamma_2 \neq 0$)

This case corresponds to the most general scenario for the integrable system (6.82), where both parameters γ_1, γ_2 are considered to be nonzero. Then, by applying the infinitesimal transformation (6.86) to (6.83) and following the standard procedure already described, the classical Lie symmetries for the spectral problem (6.83) turn out to be

$$
\begin{aligned}
\xi_1 &= K_1\left(t\right), \\
\xi_2 &= \alpha_1, \\
\xi_3 &= \alpha_2, \\
\eta_1 &= iu\left(\dot{K}_1(t)y + 2\,K_2\left(t\right)\right), \\
\eta_2 &= -iw\left(\dot{K}_1(t)y + 2\,K_2\left(t\right)\right), \\
\eta_3 &= \frac{1}{4}\ddot{K}_1(t)y^2 + \dot{K}_2(t)y + \delta(x,t), \\
\eta_4 &= 0, \\
\phi_1 &= \psi\left(\frac{i}{2}\dot{K}_1(t)y + iK_2\left(t\right) + \zeta\left(y,t,\lambda\right)\right), \\
\phi_2 &= \chi\left(-\frac{i}{2}\dot{K}_1(t)y - iK_2\left(t\right) + \zeta\left(y,t,\lambda\right)\right),
\end{aligned}
\tag{6.89}
$$

where we have omitted the dependence of the infinitesimals in the variables $\{x, y, t, u,$ $w, m, \lambda, \psi, \chi\}$ for the sake of simplicity. The dot notation implies derivation with respect to t, $\cdot \equiv \frac{d}{dt}$.

The set of Lie symmetries is described in terms of two arbitrary constants α_1, α_2, two arbitrary real functions of t, $K_1(t)$, $K_2(t)$, one arbitrary real function of $\{x, t\}$, $\delta(x, t)$, and the arbitrary function $\zeta(y, t, \lambda)$ that satisfies the nonlinear differential equation

$$\frac{\partial \zeta}{\partial t} - 2\lambda \frac{\partial \zeta}{\partial y} = 0. \tag{6.90}$$

Classical Lie symmetries for the most general case (6.89) therefore depend on up to six arbitrary parameters. It can be easily checked that both (6.82) and the nonisospectral condition (6.85) remain invariant under the action of this symmetry group. This fact is straightforwardly depicted in the associated reductions. Commutation relations among the associated infinitesimal generators to (6.89) are displayed in Appendix B.1.1.

The symmetries that yield nontrivial reductions are those related to $K_1(t)$, α_1, α_2 and henceforth, we exclusively analyze the reductions associated to these three cases. In what follows, we consider diverse subcases depending on whether these parameters are different from zero or not. In each subcase only a single arbitrary constant or function will be taken as nonzero, whilst the remaining ones are set as null.

- *Case I. $K_1(t) = 1$*

  - Reduced variables
  $$p = y, \qquad q = t. \tag{6.91}$$

  - Reduced spectral parameter
  $$\lambda(y, t) = \Lambda(p, q). \tag{6.92}$$

  - Reduced fields
  $$u(x, y, t) = F(p, q), \qquad w(x, y, t) = H(p, q),$$
  $$m(x, y, t) = N(p, q). \tag{6.93}$$

  - Reduced eigenfunctions
  $$\psi(x, y, t, \lambda) = \Phi(p, q, \Lambda), \qquad \chi(x, y, t, \lambda) = \Omega(p, q, \Lambda). \tag{6.94}$$

  Nevertheless, this is not an interesting reduction because the reduced equa-

tions can be easily integrated providing a trivial solution

$$
\begin{aligned}
F(p,q) &= b_0 e^{iZ(p,q)}, \\
H(p,q) &= b_0 e^{-iZ(p,q)}, \\
N(p,q) &= -3\gamma_1 b_0^4 p + \frac{1}{2}\int \frac{\partial Z(p,q)}{\partial q}\, dp,
\end{aligned}
\tag{6.95}
$$

where b_0 is an arbitrary constant and $Z(p,q)$ is an arbitrary function.

- **Case II. $\alpha_1 \neq 0$**

  – Reduced variables
  $$
  p = x, \qquad q = t.
  \tag{6.96}
  $$

  – Reduced spectral parameter
  $$
  \lambda(y,t) = \Lambda(q).
  \tag{6.97}
  $$

  – Reduced fields
  $$
  \begin{aligned}
  u(x,y,t) &= F(p,q), \qquad w(x,y,t) = H(p,q), \\
  m(x,y,t) &= N(p,q).
  \end{aligned}
  \tag{6.98}
  $$

  – Reduced eigenfunctions
  $$
  \psi(x,y,t,\lambda) = \Phi(p,q,\Lambda), \qquad \chi(x,y,t,\lambda) = \Omega(p,q,\Lambda).
  \tag{6.99}
  $$

  – Reduced spectral problem
  $$
  \begin{aligned}
  &\Phi_p + i\Lambda\,\Phi + F\Omega = 0, \\
  &\Omega_p - i\Lambda\,\Omega + H\Phi = 0, \\
  &\Phi_q + \Big[i\gamma_1 B_1(F,H,\Lambda) + (2\Lambda\gamma_1 - \gamma_2)B_2(F,H) + 2i\Lambda\gamma_2(2\Lambda^2 + FH)\Big]\Phi \\
  &\quad + \Big[i\gamma_1 B_3(F,H,\Lambda) + (2\Lambda\gamma_1 - \gamma_2)B_4(F,H,\Lambda) + 2i\Lambda\gamma_2 F_p\Big]\Omega = 0, \\
  &\Omega_q - \Big[i\gamma_1 B_1(F,H,\Lambda) + (2\Lambda\gamma_1 + \gamma_2)B_2(F,H) + 2i\Lambda\gamma_2(2\Lambda^2 + FH)\Big]\Omega \\
  &\quad + \Big[-i\gamma_1 B_3(H,F,\Lambda) + (2\Lambda\gamma_1 - \gamma_2)B_4(H,F,\Lambda) - 2i\Lambda\gamma_2 H_p\Big]\Phi = 0,
  \end{aligned}
  \tag{6.100}
  $$

where

$$
\begin{aligned}
B_1(F, H, \Lambda) &= -8\Lambda^4 - 4\Lambda^2 FH - 3F^2 H^2 + HF_{pp} + FH_{pp} - F_p H_p, \\
B_2(F, H) &= F_p H - H_p F, \\
B_3(F, H, \Lambda) &= F_{ppp} - 6FHF_p - 4\Lambda^2 F_p, \\
B_4(F, H, \Lambda) &= F_{pp} - 2F^2 H - 4\Lambda^2 F.
\end{aligned}
\tag{6.101}
$$

The compatibility condition of (6.100) requires that the resulting reduced problem is isospectral

$$
\Lambda = \text{constant.}
\tag{6.102}
$$

– Reduced equations

$$
\begin{aligned}
&\gamma_1 \left(8\, FHF_{pp} + 6\, HF_p{}^2 + 2\, F^2 H_{pp} + 4\, FF_p H_p - F_{pppp} - 6\, F^3 H^2 \right) \\
&+ i\gamma_2 \left(6\, FHF_p - F_{ppp} \right) - iF_q = 0, \\
&\gamma_1 \left(8\, FHH_{pp} + 6\, FH_p{}^2 + 2\, H^2 F_{pp} + 4\, HF_p H_p - H_{pppp} - 6\, F^2 H^3 \right) \\
&- i\gamma_2 \left(6\, FHH_p - H_{ppp} \right) + iH_q = 0.
\end{aligned}
\tag{6.103}
$$

This reduction successfully yields the equations and the isospectral Lax pair of reference [36].

• **Case III. $\alpha_2 \neq 0$**

– Reduced variables

$$
p = x, \qquad q = y.
\tag{6.104}
$$

– Reduced spectral parameter

$$
\lambda(y, t) = \Lambda(q).
\tag{6.105}
$$

– Reduced fields

$$
\begin{aligned}
u(x, y, t) &= F(p, q), \qquad w(x, y, t) = H(p, q), \\
m(x, y, t) &= N(p, q).
\end{aligned}
\tag{6.106}
$$

– Reduced eigenfunctions

$$
\psi(x, y, t, \lambda) = \Phi(p, q, \Lambda), \qquad \chi(x, y, t, \lambda) = \Omega(p, q, \Lambda).
\tag{6.107}
$$

– Reduced spectral problem

$$\Phi_p + i\Lambda\,\Phi + F\Omega = 0,$$
$$\Omega_p - i\Lambda\,\Omega + H\Phi = 0,$$
$$2\Lambda\Phi_q = \Big[i\gamma_1 B_1(F,H,\Lambda) + (2\Lambda\gamma_1 - \gamma_2)B_2(F,H) + 2i\Lambda\gamma_2(2\Lambda^2 + FH) - iN_q\Big]\Phi$$
$$+ \Big[i\gamma_1 B_3(F,H,\Lambda) + (2\Lambda\gamma_1 - \gamma_2)B_4(F,H,\Lambda) + 2i\Lambda\gamma_2 F_p + i\,F_q\Big]\Omega,$$
$$2\Lambda\Omega_q = -\Big[i\gamma_1 B_1(F,H,\Lambda) + (2\Lambda\gamma_1 + \gamma_2)B_2(F,H) + 2i\Lambda\gamma_2(2\Lambda^2 + FH) - iN_q\Big]\Omega$$
$$+ \Big[-i\gamma_1 B_3(H,F,\Lambda) + (2\Lambda\gamma_1 - \gamma_2)B_4(H,F,\Lambda) - 2i\Lambda\gamma_2 H_p - iH_p\Big]\Phi,$$

$$\text{(6.108)}$$

where $B_1 - B_4$ are defined in (6.101). The compatibility condition of this Lax pair implies $\Lambda = $ constant, *i.e.* the spectral problem defined above is isospectral.

– Reduced equations

$$\gamma_1\left(8\,FHF_{pp} + 6\,HF_p{}^2 + 2\,F^2 H_{pp} + 4\,FF_p H_p - F_{pppp} - 6\,F^3 H^2\right)$$
$$+ i\gamma_2\left(6\,FHF_p - F_{ppp}\right) - 2\,FN_q - F_{pq} = 0,$$

$$\gamma_1\left(8\,FHH_{pp} + 6\,FH_p{}^2 + 2\,H^2 F_{pp} + 4\,HF_p H_p - H_{pppp} - 6\,F^2 H^3\right) \quad \text{(6.109)}$$
$$- i\gamma_2\left(6\,FHH_p - H_{ppp}\right) - 2\,HN_q - H_{pq} = 0,$$

$$FH_q + HF_q + N_{pq} = 0.$$

2.1.2. Lakshmanan-Porsezian-Daniel equation in $2 + 1$ dimensions ($\gamma_1 \neq 0, \gamma_2 = 0$)

In this Subsection we are considering (6.83) when $\gamma_2 = 0$. The resulting nonlinear PDE and its spectral Lax pair constitute a generalization to $2 + 1$ dimensions of the Lakshmanan-Porsezian-Daniel equation [258]. If we apply the infinitesimal transformation (6.86), then we obtain the following classical Lie symmetries

$$\xi_1 = K_1(t) + \alpha_3,$$
$$\xi_2 = \alpha_1 + 3\alpha_3\,y,$$
$$\xi_3 = \alpha_2 + 4\alpha_3\,t,$$
$$\eta_1 = iu\left(\dot{K}_1(t)y + 2\,K_2(t)\right) - \alpha_3\,u,$$
$$\eta_2 = -iw\left(\dot{K}_1(t)y + 2\,K_2(t)\right) - \alpha_3\,w, \quad \text{(6.110)}$$

$$\eta_3 = \frac{1}{4}\ddot{K}_1(t)y^2 + \dot{K}_2(t)y + \delta(x,t) - \alpha_3\, m,$$

$$\eta_4 = \alpha_3\, \lambda,$$

$$\phi_1 = \psi\left(\frac{i}{2}\dot{K}_1(t)y + iK_2(t) + \zeta(y,t,\lambda)\right),$$

$$\phi_2 = \chi\left(-\frac{i}{2}\dot{K}_1(t)y - iK_2(t) + \zeta(y,t,\lambda)\right),$$

where α_i, $i = 1,\ldots,3$ are arbitrary constants and K_j, $j = 1,2$ and δ are arbitrary real functions of the indicated variables. The function $\zeta(y,t,\lambda)$ satisfies the equation (6.90). These Lie symmetries depend on seven arbitrary parameters, which coincide with the infinitesimals in (6.89), except for those terms related to the additional constant α_3, representing the scaling symmetry. The computation of the commutation relations among the elements of the symmetry group (6.110) are addressed in Appendix B.1.2.

According to (6.110), we have four nontrivial reductions related to $K_1(t)$, α_1, α_2, and α_3, but only the last one gives us a new result. The other three reductions provide the same results as in the previous Subsection, by imposing $\gamma_2 = 0$ in equations (6.95), (6.100) and (6.108). This extra symmetry yields a nonisospectral Lax pair in $1+1$ dimensions after the symmetry reduction process.

- **Case I. $\alpha_3 \neq 0$**

  - Reduced variables

  $$p = t^{-\frac{1}{4}}\, x, \qquad q = \frac{t^{\frac{3}{4}}}{y}. \tag{6.111}$$

  - Reduced spectral parameter

  $$\lambda(y,t) = t^{-\frac{1}{4}}\, \Lambda(q). \tag{6.112}$$

  - Reduced fields

  $$u(x,y,t) = t^{-\frac{1}{4}} F(p,q), \qquad w(x,y,t) = t^{-\frac{1}{4}} H(p,q),$$
  $$m(x,y,t) = t^{-\frac{1}{4}} N(p,q). \tag{6.113}$$

  - Reduced eigenfunctions

  $$\psi(x,y,t,\lambda) = \Phi(p,q,\Lambda), \qquad \chi(x,y,t,\lambda) = \Omega(p,q,\Lambda). \tag{6.114}$$

– Reduced spectral problem

$$\Phi_p + i\Lambda\,\Phi + F\Omega = 0,$$
$$\Omega_p - i\Lambda\,\Omega + H\Phi = 0,$$
$$\left(\frac{8\,\Lambda\,q+3}{4}\right)q\,\Phi_q + i\left(\frac{\Lambda\,p}{4} + q^2\,N_q\right)\Phi + q^2\Lambda_q\Phi + \left(\frac{pF}{4} - iq^2F_q\right)\Omega$$
$$+ i\gamma_1\Big[B_1(F,H,\Lambda)\Phi + B_3(F,H,\Lambda)\Omega\Big] + 2\Lambda\gamma_1\Big[B_2(F,H)\Phi + B_4(F,H,\Lambda)\Omega\Big] = 0,$$
$$\left(\frac{8\,\Lambda\,q+3}{4}\right)q\,\Omega_q - i\left(\frac{\Lambda\,p}{4} + q^2\,N_q\right)\Omega + q^2\Lambda_q\Omega + \left(\frac{Hp}{4} + iq^2H_q\right)\Phi$$
$$- i\gamma_1\Big[B_1(F,H,\Lambda)\Omega + B_3(H,F,\Lambda)\Phi\Big] + 2\Lambda\gamma_1\Big[B_2(H,F)\Omega + B_4(H,F,\Lambda)\Phi\Big] = 0,$$

$$(6.115)$$

where $B_1 - B_4$ are defined in (6.101). The compatibility of the above equations (6.115) implies the following:

– Nonisospectral condition

$$\Lambda_q = \frac{\Lambda}{q\,(8\,\Lambda\,q+3)}.$$

$$(6.116)$$

– Reduced equations

$$\gamma_1\left(-6\,F^3H^2 + 2\,F^2H_{pp} + 8\,FHF_{pp} + 4\,FF_pH_p + 6\,HF_p^2 - F_{pppp}\right)$$
$$+ 2q^2\,FN_q + q^2F_{pq} + \frac{i}{4}\left(pF_p - 3\,qF_q + F\right) = 0,$$

$$\gamma_1\left(-6\,F^2H^3 + 2\,H^2F_{pp} + 8\,FHH_{pp} + 4\,HF_pH_p + 6\,FH_p{}^2 - H_{pppp}\right)$$
$$+ 2q^2\,HN_q + q^2H_{pq} - \frac{i}{4}\left(pH_p - 3\,qH_q + H\right) = 0,$$

$$FH_q + HF_q + N_{pq} = 0.$$

$$(6.117)$$

2.1.3. Hirota equation in $2+1$ dimensions $(\gamma_1 = 0,\ \gamma_2 \neq 0)$

In this Subsection we analyze the spectral problem (6.83) when $\gamma_1 = 0$. The resulting integrable system can be regarded as a generalization of the Hirota equation [209] to higher dimensions. Lie's formalism for this system yields the infinitesimals listed below

$$\xi_1 = K_1\,(t) + \alpha_3\,x,$$
$$\xi_2 = \alpha_1 + 2\alpha_3\,y,$$

$$\xi_3 = \alpha_2 + 3\alpha_3\,t,$$

$$\eta_1 = iu\left(\dot{K}_1(t)y + 2\,K_2\,(t)\right) - \alpha_3\,u,$$

$$\eta_2 = -iw\left(\dot{K}_1(t)y + 2\,K_2\,(t)\right) - \alpha_3\,w,\tag{6.118}$$

$$\eta_3 = \frac{1}{4}\ddot{K}_1(t)y^2 + \dot{K}_2(t)y + K_3(x,t) - \alpha_3\,m,$$

$$\eta_4 = -\alpha_3\,\lambda,$$

$$\phi_1 = \psi\left(\frac{i}{2}\dot{K}_1(t)y + iK_2\,(t) + \zeta\,(y,t,\lambda)\right),$$

$$\phi_2 = \chi\left(-\frac{i}{2}\dot{K}_1(t)y - iK_2\,(t) + \zeta\,(y,t,\lambda)\right),$$

where α_i, $i = 1,\ldots,3$ are arbitrary constants, K_j, $j = 1,2$, and δ are arbitrary real functions, while $\zeta(y,t,\lambda)$ obeys relation (6.90). We can easily see that the scaling symmetry mediated by the constant α_3 appears once more in this case. The associated commutation relations can be found in Appendix B.1.3.

From results (6.118) we can realize that, as in the previous Subsection, we have four nontrivial reductions related to $K_1(t)$, α_1, α_2, and α_3. The first three reductions provide the same results of Subsection 2.1.1 by setting $\gamma_1 = 0$. Conversely, the reduction associated to the new symmetry α_3 represents the counterpart to the reduction of the prior Subsection for a NLS equation in $2+1$ dimensions with fourth-order derivatives. This symmetry also gives out a nonisospectral reduced Lax pair.

- *Case I. $\alpha_3 \neq 0$*

  – Reduced variables

  $$p = t^{-\frac{1}{3}}x, \qquad q = \frac{t^{\frac{2}{3}}}{y}.\tag{6.119}$$

  – Reduced spectral parameter

  $$\lambda(y,t) = t^{-\frac{1}{3}}\Lambda(q).\tag{6.120}$$

  – Reduced fields

  $$u(x,y,t) = t^{-\frac{1}{3}}F(p,q), \qquad w(x,y,t) = t^{-\frac{1}{3}}H(p,q),$$
  $$m(x,y,t) = t^{-\frac{1}{3}}N(p,q).\tag{6.121}$$

– Reduced eigenfunctions

$$\psi(x, y, t, \lambda) = \Phi(p, q, \Lambda), \qquad \chi(x, y, t, \lambda) = \Omega(p, q, \Lambda). \tag{6.122}$$

– Reduced spectral problem

$$\Phi_p + i\Lambda\,\Phi + F\Omega = 0,$$
$$\Omega_p - i\Lambda\,\Omega + H\Phi = 0,$$
$$\left(\frac{2}{3}\,q + 2\,\Lambda\,q^2\right)\Phi_q + \frac{p}{3}\,(F\Omega + i\Lambda\,\Phi) + q^2\,(\Lambda_q\Phi + i\,N_q\Phi - i\,F_q\Omega)$$
$$+ \gamma_2\Big[2i\Lambda^2(\Lambda^2 + FH) - B_2(F, H)\Big]\Phi + \gamma_2\Big[2i\Lambda F_p - B_4(F, H, \Lambda)\Big]\Omega = 0,$$
$$\left(\frac{2}{3}\,q + 2\,\Lambda\,q^2\right)\Omega_q + \frac{p}{3}\,(H\Phi - i\Lambda\,\Omega) + q^2\,(\Lambda_q\Omega - i\,N_q\Omega + i\,H_q\Phi)$$
$$- \gamma_2\Big[2i\Lambda^2(\Lambda^2 + FH) - B_2(F, H)\Big]\Omega - \gamma_2\Big[2i\Lambda H_p + B_4(H, F, \Lambda)\Big]\Phi = 0, \tag{6.123}$$

with $B_1 - B_4$ being defined in (6.101). The compatibility of this Lax pair retrieves the following:

– Nonisospectral condition

$$\Lambda_q = \frac{1}{2}\,\frac{\Lambda}{q\,(3\,\Lambda\,q + 1)}. \tag{6.124}$$

– Reduced equations

$$i\gamma_2\,(6\,FHF_p - F_{ppp}) + \frac{i}{3}\,(F + pF_p - 2qF_q) + q^2\,(F_{pq} + FN_q) = 0,$$

$$-i\gamma_2\,(6\,FHH_p - H_{ppp}) - \frac{i}{3}\,(H + pH_p - 2qH_q) + q^2\,(H_{pq} + HN_q) = 0,$$

$$FH_q + HF_q + N_{pq} = 0. \tag{6.125}$$

2.1.4. Standard NLS equation in $2 + 1$ dimensions ($\gamma_1 = 0$, $\gamma_2 = 0$)

The generalization of NLS equation to $2+1$ dimensions proposed by Calogero in [69] and its associated spectral problem are comprised in (6.83) by taking $\gamma_1 = 0$ and $\gamma_2 = 0$. The symmetries obtained in this case are give by

$$\xi_1 = K_1\left(t\right) + \alpha_3\, x + 2\alpha_5\, xt,$$

$$\xi_2 = \alpha_1 + \alpha_4\, y + 2\alpha_6\, t + 2\alpha_5\, yt,$$

$$\xi_3 = \alpha_2 + \alpha_3 t + \alpha_4 t + 2\alpha_5 t^2,$$

$$\eta_1 = iu\left(\dot{K}_1(t)y + 2\,K_2\left(t\right) + 2x(\alpha_5 y + \alpha_6)\right) - \left(\alpha_3 + 2\alpha_5 t\right)u,$$

$$\eta_2 = -iw\left(\dot{K}_1(t)y + 2\,K_2\left(t\right) + 2x(\alpha_5 y + \alpha_6)\right) - \left(\alpha_3 + 2\alpha_5 t\right)w,$$

$$\eta_3 = \frac{1}{4}\ddot{K}_1(t)y^2 + \dot{K}_2(t)y + \delta(x,t) - \left(\alpha_3 + 2\alpha_5 t\right)m,$$

$$\eta_4 = -\alpha_3\,\lambda - \alpha_5\left(2t\lambda + y\right) - \alpha_6,$$

$$\phi_1 = \psi\left(\frac{i}{2}\dot{K}_1(t)y + iK_2\left(t\right) + \zeta\left(y,t,\lambda\right)\right) + \psi\left(ix\left(\alpha_5 y + \alpha_6\right) - \alpha_5 t\right),$$

$$\phi_2 = \chi\left(-\frac{i}{2}\dot{K}_1(t)y - iK_2\left(t\right) + \zeta\left(y,t,\lambda\right)\right) + \chi\left(-ix\left(\alpha_5 y + \alpha_6\right) - \alpha_5 t\right),$$

$$(6.126)$$

where α_i, $i = 1,\ldots,6$ are arbitrary constants and K_j, $j = 1,2$, and δ are arbitrary real functions of the indicated variables, whilst $\zeta(y,t,\lambda)$ should additionally satisfy equation (6.90). The results obtained depend on up to ten arbitrary parameters, four of which, $\alpha_3 - \alpha_6$, represent new symmetries compared to the ones found in (6.89). The symmetry associated with α_3 defines a scale transformation, α_4 stands for a hyperbolic rotation in the (y,t)-plane, α_5 describes a kind of a conformal point symmetry and α_6 represents a Galilean boost. The commutation relations for (6.126) are given in Appendix B.1.4.

In accordance with the previous Subsections, these additional symmetries will allow us to obtain four novel nontrivial reductions. The similarity reduction associated to α_3 retrieves again a nonisospectral Lax pair in $1 + 1$ dimensions.

- *Case I. $\alpha_3 \neq 0$*

  − Reduced variables
  $$p = \frac{x}{t}, \qquad q = y. \tag{6.127}$$

  − Reduced spectral parameter
  $$\lambda(y,t) = \frac{\Lambda(q)}{t}. \tag{6.128}$$

– Reduced fields

$$u(x, y, t) = \frac{F(p, q)}{t}, \qquad w(x, y, t) = \frac{H(p, q)}{t},$$
$$m(x, y, t) = \frac{N(p, q)}{t}. \tag{6.129}$$

– Reduced eigenfunctions

$$\psi(x, y, t, \lambda) = \Phi(p, q, \Lambda), \qquad \chi(x, y, t, \lambda) = \Omega(p, q, \Lambda). \tag{6.130}$$

– Reduced spectral problem

$$\Phi_p + i\Lambda \Phi + F\Omega = 0,$$
$$\Omega_p - i\Lambda \Omega + H\Phi = 0,$$
$$2\Lambda \Phi_q = \left(\frac{1}{2} + ip\Lambda - iN_q\right) \Phi + (pF + iF_q) \Omega, \tag{6.131}$$
$$2\Lambda \Omega_q = \left(\frac{1}{2} - ip\Lambda + iN_q\right) \Omega + (pH - iH_q) \Phi.$$

The compatibility of the system (6.131) implies the following results:

– Nonisospectral condition

$$\Lambda_q = -\frac{1}{2}. \tag{6.132}$$

– Reduced equations

$$2\,FN_q + F_{pq} - ipF_p - iF = 0,$$
$$2\,HN_q + H_{pq} + ipH_p + iH = 0, \tag{6.133}$$
$$FH_q + HF_q + N_{pq} = 0.$$

• *Case II. $\alpha_4 \neq 0$*

– Reduced variables

$$p = x, \qquad q = \frac{t}{y}. \tag{6.134}$$

– Reduced spectral parameter

$$\lambda(y, t) = \Lambda(q). \tag{6.135}$$

– Reduced fields

$$u(x, y, t) = F(p, q), \qquad w(x, y, t) = H(p, q),$$
$$m(x, y, t) = N(p, q).$$

(6.136)

– Reduced eigenfunctions

$$\psi(x, y, t, \lambda) = \Phi(p, q), \qquad \chi(x, y, t, \lambda) = \Omega(p, q).$$

(6.137)

– Reduced spectral problem

$$\Phi_p + i\Lambda\,\Phi + F\Omega = 0,$$
$$\Omega_p - i\Lambda\,\Omega + H\Phi = 0,$$
$$\Phi_q - iq\,\Omega\,F_q + iq\,\Phi\,N_q + 2\,\Lambda\,q\,\Phi_q = 0,$$
$$\Omega_q + iq\,\Phi\,H_q - iq\,\Omega\,N_q + 2\,\Lambda\,q\,\Omega_q = 0,$$

(6.138)

whose compatibility condition requires its isospectrality.

– Reduced equations

$$-iF_q + 2q\,FN_q + qF_{pq} = 0,$$
$$iH_q + 2q\,HN_q + qH_{pq} = 0,$$
$$H_qF + HF_q + N_{pq} = 0.$$

(6.139)

- **Case III. $\alpha_5 \neq 0$**

   – Reduced variables

$$p = \frac{x}{t}, \qquad q = \frac{y}{t}.$$

(6.140)

   – Reduced spectral parameter

$$\lambda(y, t) = \frac{-qt + \Lambda(p, q)}{2t}.$$

(6.141)

   – Reduced fields

$$u(x, y, t) = \frac{F(p, q)}{t}\,\exp\left(itpq\right), \quad w(x, y, t) = \frac{H(p, q)}{t}\,\exp\left(-itpq\right),$$
$$m(x, y, t) = \frac{N(p, q)}{t}.$$

(6.142)

  – Reduced eigenfunctions

$$\psi(x,y,t,\lambda) = \frac{\Phi(p,q)}{\sqrt{t}} \, \exp\left(\frac{itpq}{2}\right), \quad \chi(x,y,t,\lambda) = \frac{\Omega(p,q)}{\sqrt{t}} \, \exp\left(\frac{-itpq}{2}\right).$$
(6.143)

  – Reduced spectral problem

$$\Phi_p + \frac{1}{2} \, i\, \Lambda\Phi + F\Omega = 0,$$
$$\Omega_p - \frac{1}{2} \, i\, \Lambda\Omega + H\Phi = 0,$$
$$\Lambda\,\Phi_q - i\, F_q\Omega + i\, N_q\Phi = 0,$$
$$\Lambda\,\Omega_q + i\, H_q\Phi - iN_q\Omega = 0,$$
(6.144)

  whose compatibility condition implies that we are dealing with an isospectral linear problem, $\Lambda = $ constant.

  – Reduced equations
$$2\,FN_q + F_{pq} = 0,$$
$$2\,HN_q + H_{pq} = 0,$$
$$FH_q + HF_q + N_{pq} = 0.$$
(6.145)

- *Case IV. $\alpha_6 \neq 0$*

  – Reduced variables
$$p = x, \qquad q = t.$$
(6.146)

  – Reduced spectral parameter

$$\lambda(y,t) = \frac{2\Lambda(p,q) - y}{2q}.$$
(6.147)

  – Reduced fields

$$u(x,y,t) = F(p,q) \, \exp\left(i\,\frac{p}{q}\,y\right), \qquad w(x,y,t) = H(p,q) \, \exp\left(-i\,\frac{p}{q}\,y\right),$$
$$m(x,y,t) = N(p,q).$$
(6.148)

  – Reduced eigenfunctions

$$\psi(x,y,t,\lambda) = \Phi(p,q) \, \exp\left(i\,\frac{p}{2q}\,y\right), \quad \chi(x,y,t,\lambda) = \Omega(p,q) \, \exp\left(-i\,\frac{p}{2q}\,y\right).$$
(6.149)

– Reduced spectral problem

$$
\begin{aligned}
q\,\Phi_p + q\,F\,\Omega + i\Lambda\,\Phi &= 0, \\
q\,\Omega_p + q\,H\,\Phi - i\Lambda\,\Omega &= 0, \\
-2\,q^2\Phi_q + 2\,pq\,F\Omega + 2ip\,\Lambda\,\Phi - q\,\Phi &= 0, \\
-2\,q^2\Omega_q + 2\,pq\,H\Phi - 2ip\,\Lambda\,\Omega - q\,\Omega &= 0,
\end{aligned}
\tag{6.150}
$$

which is an isospectral Lax pair.

– Reduced equations

$$
\begin{aligned}
pF_p + qF_q + F &= 0, \\
pH_p + qH_q + H &= 0.
\end{aligned}
\tag{6.151}
$$

2.2. Multi-component NLS equations in $2+1$ dimensions

This Subsection is devoted to the study of the $(2+1)$-dimensional multi-component NLS equation introduced in Section 1 from Chapter 4, which can be regarded as an extension to higher dimensions of the Manakov system [286], also known as vector NLS [8]. This equation reads

$$
\begin{aligned}
i\boldsymbol{\alpha}_t + \boldsymbol{\alpha}_{xx} + 2\,m_x\,\boldsymbol{\alpha} &= 0, \\
-i\boldsymbol{\alpha}^\dagger_t + \boldsymbol{\alpha}^\dagger_{xx} + 2\,m_x\,\boldsymbol{\alpha}^\dagger &= 0, \\
\left(m_y + \boldsymbol{\alpha}\,\boldsymbol{\alpha}^\dagger\right)_x &= 0,
\end{aligned}
\tag{6.152}
$$

where $\boldsymbol{\alpha}$ has two components, $\boldsymbol{\alpha}(x,y,t) = \begin{pmatrix} \alpha_1(x,y,t) \\ \alpha_2(x,y,t) \end{pmatrix}$, $\boldsymbol{\alpha}^\dagger$ stands for its complex conjugate and $m(x,y,t)$ is a real scalar function.

System (6.152) has proved to be integrable in the Painlevé sense and a three-component Lax pair has been successfully derived via the SMM by the author of this manuscript in [25]. The associated spectral problem has the expression

$$
\begin{aligned}
\psi_y &= -\alpha^\dagger_1\chi - \alpha^\dagger_2\rho, \\
\chi_x &= -\alpha_1\psi, \\
\rho_x &= -\alpha_2\psi,
\end{aligned}
\tag{6.153a}
$$

$$
\begin{aligned}
\psi_t &= -i\psi_{xx} - 2im_x\psi, \\
\chi_t &= -i\left(\alpha_1\right)_x \chi + i\alpha_1\psi_x, \\
\rho_t &= -i\left(\alpha_2\right)_x \rho + i\alpha_2\psi_x,
\end{aligned}
\tag{6.153b}
$$

where $\psi(x,y,t)$, $\chi(x,y,t)$, $\rho(x,y,t)$ are the three complex eigenfunctions, (6.153a) describes the spatial part of the Lax pair and (6.153b) stands for the temporal counterpart. It is important to remark that we should need to consider the complex conjugate of (6.153) in order to fully characterize the spectral problem associated to (6.152), by virtue of the SMM procedure developed in Section 1 of Chapter 4. It is also worth noticing that this spectral problem is not explicitly written in terms of a spectral parameter.

Classical Lie symmetries

Lie symmetries for the spectral problem (6.153) can be straightforwardly calculated following the prescription indicated in the previous Sections. The space of independent variables $\mathscr{X}$ is endowed with the usual coordinates (x, y, t) and the space of dependent variables $\mathscr{U}$ is now characterize with local coordinates $(\alpha_1, \alpha_2, \alpha_1^\dagger, \alpha_2^\dagger, m, \psi, \chi, \rho, \psi^\dagger, \chi^\dagger, \rho^\dagger)$. Since (6.153) is a second-order Lax pair, it can be understood as a submanifold of the second-order jet space $\mathcal{M}_{(2)} = \mathscr{X} \times \mathscr{U}_{(2)}$.

Let us consider a one-parameter Lie group of infinitesimal transformations given by

$$
\begin{aligned}
\tilde{x} &= x + \epsilon\, \xi_1(x,y,t,\alpha_1,\alpha_2,\alpha_1^\dagger,\alpha_2^\dagger,m,\psi,\chi,\rho,\psi^\dagger,\chi^\dagger,\rho^\dagger) + \mathcal{O}(\epsilon^2), \\
\tilde{y} &= y + \epsilon\, \xi_2(x,y,t,\alpha_1,\alpha_2,\alpha_1^\dagger,\alpha_2^\dagger,m,\psi,\chi,\rho,\psi^\dagger,\chi^\dagger,\rho^\dagger) + \mathcal{O}(\epsilon^2), \\
\tilde{t} &= t + \epsilon\, \xi_3(x,y,t,\alpha_1,\alpha_2,\alpha_1^\dagger,\alpha_2^\dagger,m,\psi,\chi,\rho,\psi^\dagger,\chi^\dagger,\rho^\dagger) + \mathcal{O}(\epsilon^2), \\
\tilde{\alpha}_1 &= \alpha_1 + \epsilon\, \eta_1(x,y,t,\alpha_1,\alpha_2,\alpha_1^\dagger,\alpha_2^\dagger,m,\psi,\chi,\rho,\psi^\dagger,\chi^\dagger,\rho^\dagger) + \mathcal{O}(\epsilon^2), \\
\tilde{\alpha}_2 &= \alpha_2 + \epsilon\, \eta_2(x,y,t,\alpha_1,\alpha_2,\alpha_1^\dagger,\alpha_2^\dagger,m,\psi,\chi,\rho,\psi^\dagger,\chi^\dagger,\rho^\dagger) + \mathcal{O}(\epsilon^2), \\
\tilde{\alpha}_1^\dagger &= \alpha_1^\dagger + \epsilon\, \eta_3(x,y,t,\alpha_1,\alpha_2,\alpha_1^\dagger,\alpha_2^\dagger,m,\psi,\chi,\rho,\psi^\dagger,\chi^\dagger,\rho^\dagger) + \mathcal{O}(\epsilon^2), \\
\tilde{\alpha}_2^\dagger &= \alpha_2^\dagger + \epsilon\, \eta_4(x,y,t,\alpha_1,\alpha_2,\alpha_1^\dagger,\alpha_2^\dagger,m,\psi,\chi,\rho,\psi^\dagger,\chi^\dagger,\rho^\dagger) + \mathcal{O}(\epsilon^2), \\
\tilde{m} &= m + \epsilon\, \eta_5(x,y,t,\alpha_1,\alpha_2,\alpha_1^\dagger,\alpha_2^\dagger,m,\psi,\chi,\rho,\psi^\dagger,\chi^\dagger,\rho^\dagger) + \mathcal{O}(\epsilon^2), \\
\tilde{\psi} &= \psi + \epsilon\, \phi_1(x,y,t,\alpha_1,\alpha_2,\alpha_1^\dagger,\alpha_2^\dagger,m,\psi,\chi,\rho,\psi^\dagger,\chi^\dagger,\rho^\dagger) + \mathcal{O}(\epsilon^2), \\
\tilde{\chi} &= \chi + \epsilon\, \phi_2(x,y,t,\alpha_1,\alpha_2,\alpha_1^\dagger,\alpha_2^\dagger,m,\psi,\chi,\rho,\psi^\dagger,\chi^\dagger,\rho^\dagger) + \mathcal{O}(\epsilon^2), \\
\tilde{\rho} &= \rho + \epsilon\, \phi_3(x,y,t,\alpha_1,\alpha_2,\alpha_1^\dagger,\alpha_2^\dagger,m,\psi,\chi,\rho,\psi^\dagger,\chi^\dagger,\rho^\dagger) + \mathcal{O}(\epsilon^2), \\
\tilde{\psi}^\dagger &= \psi^\dagger + \epsilon\, \phi_4(x,y,t,\alpha_1,\alpha_2,\alpha_1^\dagger,\alpha_2^\dagger,m,\psi,\chi,\rho,\psi^\dagger,\chi^\dagger,\rho^\dagger) + \mathcal{O}(\epsilon^2), \\
\tilde{\chi}^\dagger &= \chi^\dagger + \epsilon\, \phi_5(x,y,t,\alpha_1,\alpha_2,\alpha_1^\dagger,\alpha_2^\dagger,m,\psi,\chi,\rho,\psi^\dagger,\chi^\dagger,\rho^\dagger) + \mathcal{O}(\epsilon^2), \\
\tilde{\rho}^\dagger &= \rho^\dagger + \epsilon\, \phi_6(x,y,t,\alpha_1,\alpha_2,\alpha_1^\dagger,\alpha_2^\dagger,m,\psi,\chi,\rho,\psi^\dagger,\chi^\dagger,\rho^\dagger) + \mathcal{O}(\epsilon^2),
\end{aligned}
\tag{6.154}
$$

where ϵ is the group parameter and ξ_i, η_j, ϕ_k, $i = 1, \ldots, 3$, $j = 1, \ldots, 5$, $k = 1, \ldots, 6$ are the components of the related vector field

$$X = \xi_1 \frac{\partial}{\partial x} + \xi_2 \frac{\partial}{\partial y} + \xi_3 \frac{\partial}{\partial t} + \eta_1 \frac{\partial}{\partial \alpha_1} + \eta_2 \frac{\partial}{\partial \alpha_2} + \eta_3 \frac{\partial}{\partial \alpha_1^\dagger} + \eta_4 \frac{\partial}{\partial \alpha_2^\dagger} + \eta_5 \frac{\partial}{\partial m}$$
$$+ \phi_1 \frac{\partial}{\partial \psi} + \phi_2 \frac{\partial}{\partial \chi} + \phi_3 \frac{\partial}{\partial \rho} + \phi_4 \frac{\partial}{\partial \psi^\dagger} + \phi_5 \frac{\partial}{\partial \chi^\dagger} + \phi_6 \frac{\partial}{\partial \rho^\dagger}.$$

(6.155)

This infinitesimal transformation induces a well known one in the derivatives of the fields, as shown in Chapter 5, and it must leave invariant the set of solutions of (6.153). At this point we should decide between either the classical or nonclassical approach. We have proceeded with both methods, and the nonclassical formalism, when the associated overdetermined system of PDEs for the infinitesimal is solvable, provides the same results than the classical setting in a remarkably more complicated way. Then, we will be essentially dealing with the classical Lie symmetries for (6.153) and its complex conjugate problem, which result in

$$\xi_1 = 4\,\dot{K}_1(t)\,x + 2\,K_2(t),$$
$$\xi_2 = 2\,C_1(y),$$
$$\xi_3 = 8\,K_1(t),$$
$$\eta_1 = \left[i\left(\ddot{K}_1(t)\,x^2 + \dot{K}_2(t)\,x + K_3(t) + C_2(y) \right) - 2\,\dot{K}_1(t) - C_1'(y) \right] \alpha_1$$
$$+ \left[C_4(y) + i\,C_5(y) \right] \alpha_2,$$
$$\eta_2 = \left[i\left(\ddot{K}_1(t)\,x^2 + \dot{K}_2(t)\,x + K_3(t) + C_3(y) \right) - 2\,\dot{K}_1(t) - C_1'(y) \right] \alpha_2$$
$$- \left[C_4(y) - i\,C_5(y) \right] \alpha_1,$$
$$\eta_3 = \left[-i\left(\ddot{K}_1(t)\,x^2 + \dot{K}_2(t)\,x + K_3(t) + C_2(y) \right) - 2\,\dot{K}_1(t) - C_1'(y) \right] \alpha_1^\dagger$$
$$+ \left[C_4(y) - i\,C_5(y) \right] \alpha_2^\dagger,$$

(6.156)

$$\eta_4 = \left[-i\left(\ddot{K}_1(t)\,x^2 + \dot{K}_2(t)\,x + K_3(t) + C_3(y) \right) - 2\,\dot{K}_1(t) - C_1'(y) \right] \alpha_2^\dagger$$
$$- \left[C_4(y) + i\,C_5(y) \right] \alpha_1^\dagger,$$
$$\eta_5 = -\,4\,\dot{K}_1(t)\,m + \frac{1}{6}\dddot{K}_1(t)\,x^3 + \frac{1}{4}\ddot{K}_2(t)\,x^2 + \frac{1}{2}\dot{K}_3(t)\,x + \delta(y,t),$$
$$\phi_1 = \left[-i\left(\ddot{K}_1(t)\,x^2 + \dot{K}_2(t)\,x + K_3(t) \right) - 2\,\dot{K}_1(t) + \lambda \right] \psi,$$
$$\phi_2 = \left[i\,C_2(y) - C_1'(y) + \lambda \right] \chi + \left[C_4(y) + i\,C_5(y) \right] \rho,$$
$$\phi_3 = \left[i\,C_3(y) - C_1'(y) + \lambda \right] \rho - \left[C_4(y) - i\,C_5(y) \right] \chi,$$
$$\phi_4 = \left[i\left(\ddot{K}_1(t)\,x^2 + \dot{K}_2(t)\,x + K_3(t) \right) - 2\,\dot{K}_1(t) + \lambda^\dagger \right] \psi^\dagger,$$

$$\phi_5 = \left[-i\, C_2(y) - C_1'(y) + \lambda^\dagger\right]\chi^\dagger + \left[C_4(y) - i\, C_5(y)\right]\rho^\dagger,$$

$$\phi_6 = \left[-i\, C_3(y) - C_1'(y) + \lambda^\dagger\right]\rho^\dagger - \left[C_4(y) + i\, C_5(y)\right]\chi^\dagger,$$

where we have used the convention $\dot{} \equiv \frac{d}{dt}$ and $' \equiv \frac{d}{dy}$.

These Lie symmetries depend on a set of nine arbitrary real functions and two arbitrary complex constants, listed as,

- Three arbitrary real functions $K_j(t)$, $j = 1, ..., 3$, which depend exclusively on the temporal coordinate t.

- Five arbitrary real functions $C_j(y)$, $j = 1, ..., 5$, which depend on the coordinate y.

- An arbitrary real function $\delta(y, t)$.

- Furthermore, these symmetries include two arbitrary constants λ, $\lambda^\dagger$, which can be taken as complex conjugates given that we have considered the spectral problem (6.153) and its complex conjugate. We shall prove later that this complex constant will play the role of the spectral parameter in the $(1+1)$-reductions of the Lax pair under study.

Lie symmetries for the Lax pair (6.153) generalize, extend and include all the Lie symmetries obtained for the multi-component NLS (6.152). Symmetries given by $\xi_1 - \eta_5$ in (6.156) can be analogously derived by implementing a similar procedure over the starting system of PDEs (6.152), whereas symmetries $\phi_1 - \phi_6$ correspond to the transformation of the eigenfunctions of the Lax pair. It is also worthwhile to remark that the only additional symmetry that corresponds strictly to the Lax pair itself is the one associated with the arbitrary constant λ. The symmetry group (6.156) is spanned by eleven infinitesimal generators, whose explicit expressions and commutation relations are given in Appendix B.2.

Similarity reductions

Similarity reductions may be achieved by solving the characteristic system

$$\frac{dx}{\xi_1} = \frac{dy}{\xi_2} = \frac{dt}{\xi_3} = \frac{d\alpha_1}{\eta_1} = \frac{d\alpha_2}{\eta_2} = \frac{d\alpha_1^\dagger}{\eta_3} = \frac{d\alpha_2^\dagger}{\eta_4} = \frac{dm}{\eta_5}$$
$$= \frac{d\psi}{\phi_1} = \frac{d\chi}{\phi_2} = \frac{d\rho}{\phi_3} = \frac{d\psi^\dagger}{\phi_4} = \frac{d\chi^\dagger}{\phi_5} = \frac{d\rho^\dagger}{\phi_6}.$$

(6.157)

The process of symmetry reduction provides the corresponding reduced problem with the number of independent variables diminished by one, leading to similarity reductions in 1+1 dimensions. We shall summarize the notation used for the reduced parameters in the following table.

	Original variables	New reduced variables
Indep. variables	x, y, t	p, q
Fields	$\alpha_1(x, y, t), \alpha_2(x, y, t)$	$F(p, q)$, $H(p, q)$
	$\alpha_1^\dagger(x, y, t), \alpha_2^\dagger(x, y, t)$	$F^\dagger(p, q)$, $H^\dagger(p, q)$
	$m(x, y, t)$	$N(p, q)$
Eigenfunctions	$\psi(x, y, t), \chi(x, y, t), \rho(x, y, t)$	$\Phi(p, q), \Sigma(p, q), \Omega(p, q)$
	$\psi^\dagger(x, y, t), \chi^\dagger(x, y, t), \rho^\dagger(x, y, t)$	$\Phi^\dagger(p, q), \Sigma^\dagger(p, q), \Omega^\dagger(p, q)$

The symmetries that will yield nontrivial reductions are those related to the arbitrary functions $K_1(t)$, $K_2(t)$ and $C_1(y)$, associated to the transformations of the independent variables. The rest of the symmetries provide trivial reductions (phase shifts and rotations for the fields and the eigenfunctions). Several reductions may emerge for different values of K_1, K_2, C_1, giving rise to three independent reductions.

We introduce the shorthand notation to be used in the forthcoming calculations:

$$I_0(t) = \frac{1}{4} \int \frac{K_2(t)}{K_1(t)^{\frac{3}{2}}} \, dt, \quad I_1(t) = \frac{1}{4} \int \frac{K_2(t)^2}{K_1(t)^2} \, dt, \quad I_2(t) = \frac{1}{512} \int \frac{K_2(t)^3}{K_1(t)^{\frac{5}{2}}} \, dt.$$
(6.158)

For the sake of simplicity, in the following we will only display the similarity reductions arising from (6.153), while the reduced version of the complex conjugate problem can be easily derived by conjugation in the results presented hereafter.

- **Case I. $K_1(t) \neq 0$, $K_2(t) \neq 0$, $C_1(y) \neq 0$**

  By solving the characteristic system (6.157), the following results have been obtained

  – Reduced variables

$$p = \frac{x}{K_1(t)^{\frac{1}{2}}} - I_0(t), \qquad q = 4 \int \frac{dy}{C_1(y)} - \int \frac{dt}{K_1(t)}.$$
(6.159)

– Reduced fields

$$\alpha_1(x,y,t) = \frac{2\,F(p,q)}{K_1(t)^{\frac{1}{4}}\,C_1(y)^{\frac{1}{2}}}\,e^{\left\{\frac{i}{8}\left[\frac{\dot{K}_1(t)}{K_1(t)}x^2+\frac{K_2(t)}{K_1(t)}x-I_1(t)\right]\right\}},$$

$$\alpha_2(x,y,t) = \frac{2\,H(p,q)}{K_1(t)^{\frac{1}{4}}\,C_1(y)^{\frac{1}{2}}}\,e^{\left\{\frac{i}{8}\left[\frac{\dot{K}_1(t)}{K_1(t)}x^2+\frac{K_2(t)}{K_1(t)}x-I_1(t)\right]\right\}},$$

$$m(x,y,t) = \frac{x^3}{24\,K_1(t)^{\frac{1}{2}}}\left[K_1(t)^{\frac{1}{2}}\right]_{tt} + \frac{x^2}{32\,K_1(t)^{\frac{1}{2}}}\left[\frac{K_2(t)}{K_1(t)^{\frac{1}{2}}}\right]_t$$

$$-\frac{x}{32}\,\dot{I}_1(t) + \frac{N(p,q)+I_2(t)}{K_1(t)^{\frac{1}{2}}}.$$

$$(6.160)$$

where the subscript $(\cdot)_t$ denotes the derivative with respect to the coordinate t.

– Reduced eigenfunctions

$$\psi(x,y,t) = \frac{\Phi(p,q)}{2\,K_1(t)^{\frac{1}{4}}}\,e^{\left\{-\frac{i}{8}\left[\frac{\dot{K}_1(t)}{K_1(t)}x^2+\frac{K_2(t)}{K_1(t)}x-I_1(t)\right]+\frac{\lambda}{8}\int\frac{dt}{K_1(t)}\right\}},$$

$$\chi(x,y,t) = \frac{\Sigma(p,q)}{C_1(y)^{\frac{1}{2}}}\,e^{\left\{\frac{\lambda}{8}\int\frac{dt}{K_1(t)}\right\}},$$

$$(6.161)$$

$$\rho(x,y,t) = \frac{\Omega(p,q)}{C_1(y)^{\frac{1}{2}}}\,e^{\left\{\frac{\lambda}{8}\int\frac{dt}{K_1(t)}\right\}}.$$

– Reduced spectral problem

$$\Phi_{pp} + \left(2\,N_p - \frac{i}{8}\,\lambda\right)\Phi - i\,F^\dagger\,\Sigma - i\,H^\dagger\,\Omega = 0,$$

$$\Sigma_p + F\,\Phi = 0,$$

$$\Omega_p + H\,\Phi = 0,$$

$$(6.162a)$$

$$\Phi_q + F^\dagger\,\Sigma + H^\dagger\,\Omega = 0,$$

$$\Sigma_q + i\,(F\,\Phi_p - F_p\,\Phi) - \frac{\lambda}{8}\,\Sigma = 0,$$

$$(6.162b)$$

$$\Omega_q + i\,(H\,\Phi_p - H_p\,\Phi) - \frac{\lambda}{8}\,\Omega = 0,$$

and its complex conjugate.

It is worth mentioning that the arbitrary constant λ can be interpreted as the spectral parameter of the reduced linear problem described above. Hence, it

is the symmetry reduction process itself that naturally introduces the spectral parameter in the reduced problem. And the spectral parameter happens to be precisely the arbitrary constant λ arising from the symmetries of the primal Lax pair. This scenario is repeated in the subsequent cases for each similarity reduction when applied to (6.153).

– Reduced equations

The compatibility condition between (6.162a)-(6.162b) will provide the reduced equations (and its complex conjugate)

$$
\begin{aligned}
iF_q - F_{pp} - 2FN_p &= 0, \\
iH_q - H_{pp} - 2HN_p &= 0, \\
\left(N_q + FF^\dagger + HH^\dagger\right)_p &= 0,
\end{aligned}
\tag{6.163}
$$

which are proved to be a nonlocal multi-component NLS system in $1 + 1$ dimensions, expressed for the complex conjugate fields $\{F^\dagger, H^\dagger\}$ with density of probability N_q. This reduction corresponds to the Manakov system [8,286].

We may remark that the same reductions for the Lax pair, and consequently for the equations, arise by performing the similarity reduction with $K_1(t) \neq 0$, $C_1(y) \neq 0$, $K_2(t) = 0$. In this case, the reductions for the independent variables, fields and eigenfunctions are obtained by setting $K_2(t) = 0$ in (6.159), (6.160) and (6.161), respectively.

- *Case II. $K_1(t) \neq 0$, $K_2(t) \neq 0$, $C_1(y) = 0$*

Integration of (6.157) provides the following results

   – Reduced variables

$$
p = \frac{x}{K_1(t)^{\frac{1}{2}}} - I_0(t), \qquad q = y.
\tag{6.164}
$$

   – Reduced fields

$$
\alpha_1(x, y, t) = \frac{F(p, q)}{K_1(t)^{\frac{1}{4}}} e^{\left\{\frac{i}{8}\left[\frac{\dot{K}_1(t)}{K_1(t)} x^2 + \frac{K_2(t)}{K_1(t)} x - I_1(t)\right]\right\}},
$$

$$
\alpha_2(x, y, t) = \frac{H(p, q)}{K_1(t)^{\frac{1}{4}}} e^{\left\{\frac{i}{8}\left[\frac{\dot{K}_1(t)}{K_1(t)} x^2 + \frac{K_2(t)}{K_1(t)} x - I_1(t)\right]\right\}},
\tag{6.165}
$$

$$m(x,y,t) = \frac{x^3}{24\,K_1(t)^{\frac{1}{2}}}\left[K_1(t)^{\frac{1}{2}}\right]_{tt} + \frac{x^2}{32\,K_1(t)^{\frac{1}{2}}}\left[\frac{K_2(t)}{K_1(t)^{\frac{1}{2}}}\right]_t$$
$$- \frac{x}{32}\,\dot{I}_1(t) + \frac{N(p,q) + I_2(t)}{K_1(t)^{\frac{1}{2}}}.$$

– Reduced eigenfunctions

$$\psi(x,y,t) = \frac{\Phi(p,q)}{K_1(t)^{\frac{1}{4}}}\,e^{\left\{-\frac{i}{8}\left[\frac{\dot{K}_1(t)}{K_1(t)}x^2 + \frac{K_2(t)}{K_1(t)}x - I_1(t)\right] + \frac{\lambda}{8}\int\frac{dt}{K_1(t)}\right\}},$$

$$\chi(x,y,t) = \Sigma(p,q)\,e^{\left\{\frac{\lambda}{8}\int\frac{dt}{K_1(t)}\right\}},$$ (6.166)

$$\rho(x,y,t) = \Omega(p,q)\,e^{\left\{\frac{\lambda}{8}\int\frac{dt}{K_1(t)}\right\}}.$$

– Reduced spectral problem

$$\Phi_{pp} + \left(2N_p - \frac{i}{8}\lambda\right)\Phi = 0,$$

$$\Sigma_p + F\,\Phi = 0,$$ (6.167a)

$$\Omega_p + H\,\Phi = 0,$$

$$\Phi_q + F^\dagger\,\Sigma + H^\dagger\,\Omega = 0,$$

$$\lambda\,\Sigma - 8\,i\,(F\,\Phi_p - F_p\,\Phi) = 0,$$ (6.167b)

$$\lambda\,\Omega - 8\,i\,(H\,\Phi_p - H_p\,\Phi) = 0,$$

and its complex conjugate. The previous system of PDEs can be expressed equivalently to the following scalar Lax pair in $1 + 1$ dimensions

$$\Phi_{pp} + \left(2\,N_p - \frac{i}{8}\lambda\right)\Phi = 0,$$
$$\lambda\,\Phi_q - 8\,i\left[\left(F^\dagger\,F_p + H^\dagger\,H_p\right)\Phi + N_q\,\Phi_p\right] = 0,$$ (6.168)

$$\Phi_{pp}^\dagger + \left(2\,N_p + \frac{i}{8}\lambda^\dagger\right)\Phi^\dagger = 0,$$
$$\lambda^\dagger\,\Phi_q^\dagger + 8\,i\left[\left(F\,F_p^\dagger + H\,H_p^\dagger\right)\Phi^\dagger + N_q\,\Phi_p^\dagger\right] = 0.$$ (6.169)

– Reduced equations

The compatibility condition between (6.168) and (6.169) yields the reduced

equations

$$F_{pp} + 2FN_p = 0,$$
$$H_{pp} + 2HN_p = 0,$$
$$\left(N_q + FF^\dagger + HH^\dagger\right)_p = 0,$$

(6.170)

and its complex conjugate.

- **Case III. $K_2(t) \neq 0$, $C_1(y) \neq 0$, $K_1(t) = 0$**

  The following reductions arise from the integration of (6.157),

  – Reduced variables

  $$p = \frac{x}{K_2(t)} - \int \frac{dy}{C_1(y)}, \qquad q = \int \frac{dt}{K_2(t)^2}. \tag{6.171}$$

  – Reduced fields

  $$\alpha_1(x,y,t) = \frac{F(p,q)}{K_2(t)^{\frac{1}{2}} C_1(y)^{\frac{1}{2}}} e^{\left\{\frac{i}{4}\left[\frac{\dot{K}_2(t)}{K_2(t)} x^2 + 2p - q\right]\right\}},$$

  $$\alpha_2(x,y,t) = \frac{H(p,q)}{K_2(t)^{\frac{1}{2}} C_1(y)^{\frac{1}{2}}} e^{\left\{\frac{i}{4}\left[\frac{\dot{K}_2(t)}{K_2(t)} x^2 + 2p - q\right]\right\}}, \tag{6.172}$$

  $$m(x,y,t) = \frac{x^3}{24} \frac{\ddot{K}_2(t)}{K_2(t)} + \frac{N(p,q)}{K_2(t)}.$$

  – Reduced eigenfunctions

  $$\psi(x,y,t) = \frac{\Phi(p,q)}{K_2(t)^{\frac{1}{2}}} e^{\left\{-\frac{i}{4}\left[\frac{\dot{K}_2(t)}{K_2(t)} x^2 - q\right] + \frac{\lambda}{2} \int \frac{dy}{C_1(y)}\right\}},$$

  $$\chi(x,y,t) = \frac{\Sigma(p,q)}{C_1(y)^{\frac{1}{2}}} e^{\left\{\frac{\lambda}{2} \int \frac{dy}{C_1(y)} + i\frac{p}{2}\right\}}, \tag{6.173}$$

  $$\rho(x,y,t) = \frac{\Omega(p,q)}{C_1(y)^{\frac{1}{2}}} e^{\left\{\frac{\lambda}{2} \int \frac{dy}{C_1(y)} + i\frac{p}{2}\right\}}.$$

  – Reduced spectral problem

  $$\Phi_p - \left(F^\dagger \Sigma + H^\dagger \Omega\right) - \frac{\lambda}{2} \Phi = 0,$$

  $$\Sigma_p + F \Phi + \frac{i}{2} \Sigma = 0, \tag{6.174a}$$

  $$\Omega_p + H \Phi + \frac{i}{2} \Omega = 0,$$

$$\Phi_q + \left(i\, H_p^\dagger + \frac{i\,\lambda + 1}{2}\, H^\dagger \right) \Omega + \left(i\, F_p^\dagger + \frac{i\,\lambda + 1}{2}\, F^\dagger \right) \Sigma$$

$$- i \left(FF^\dagger + HH^\dagger - 2\, N_p - \frac{\lambda^2 + 1}{4} \right) \Phi = 0,$$

$$\Sigma_q - \left(\frac{i\,\lambda + 1}{2}\, F - i\, F_p \right) \Phi - i\, FF^\dagger \Sigma - i\, FH^\dagger \Omega = 0, \qquad (6.174\text{b})$$

$$\Omega_q - \left(\frac{i\,\lambda + 1}{2}\, H - i\, H_p \right) \Phi - i\, HF^\dagger \Sigma - i\, HH^\dagger \Omega = 0,$$

and its complex conjugate.

– Reduced equations

The compatibility condition between (6.174a)-(6.174b) provides the following
system of PDEs

$$i\, F_q + (F_p + i\, F)_p + 2\, FN_p = 0,$$

$$i\, H_q + (H_p + i\, H)_p + 2\, HN_p = 0, \qquad (6.175)$$

$$\left(N_p - FF^\dagger - HH^\dagger \right)_p = 0,$$

and its complex conjugate.

Chapter 7

Conclusions

This final Chapter addresses the main general conclusions obtained from the accomplishment of this doctoral thesis. Subsequently, the Chapter closes with a Section of open research topics and future prospects.

1.   Conclusions

The research conducted in this doctoral thesis has focused on the study and characterization of the integrability properties of nonlinear systems, described by nonlinear PDEs of interest arising from various scientific fields, mainly from Mathematical Physics and other related disciplines. This issue has been addressed from four different but complementary approaches, which constitutes the cornerstones of the present dissertation:

1. Integrability analysis by means of techniques based on the Painlevé Property

2. The singular manifold method as a fruitful tool to derive Lax pairs

3. Binary Darboux transformations and τ-formalism as a procedure to obtain iterative solutions, particularly soliton-like solutions

4. Lie symmetries and similarity reductions

As it has already illustrated, NLS equation plays a critical role in the primeval conception of this doctoral thesis, since a significant proportion of the analyzed integrable models constitutes generalizations of diverse kinds for this celebrated nonlinear equation, with copious applications in distinct fields, among which it is worth mentioning Material Sciences and Biology.

The main contributions of this thesis are therefore described in detail hereunder.

Chapter 2

Chapter 2 presents a comprehensive review of some of the remarkable properties of nonlinear integrable PDEs exploited throughout this thesis. Of capital importance was the description of the so-called Painlevé Property, as well as the characterization of the algorithmic tests based on it, which allow us to conjecture the possible integrability of a given differential equation. No less significant is the singular manifold method, which provides a systematic procedure to construct and identify several primary features for those integrable systems, such as auto-Bäcklund transformations, the associated spectral problem or Darboux transformations. The conjunction of all these elements yields a solid and ideal machinery to analyze nonlinear integrable equation, especially when oriented towards the obtention of analytical solutions of solitonic nature.

This procedure has been insightfully illustrated by means of its application to the ubiquitous NLS equation in $1 + 1$ dimensions.

Chapter 3

Chapter 3 is devoted to the applications of the methodology described in the previous Chapter to differential equations in $1 + 1$ dimensions. The Painlevé test, based on the Painlevé Property and following the WTC algorithmic prescription, has proved to be an extremely valuable criterion of integrability. This fact has been more than confirmed by the subsequent application of the SMM, which has successfully retrieved the associated spectral problems for the considered nonlinear systems. Furthermore, binary Darboux transformations are determined to constitute the perfect enhancement for the aforementioned methodology. They allow us to derive in a straightforward approach a plethora of soliton-like solutions of diverse nature.

- Firstly, we have proposed a nonlinear model in $1 + 1$ dimensions describing the electron spin dynamics in deformable helical molecules [26, 124], which generalizes the linear model introduced in [125, 198]. The deformability of the molecule enables the formation of an entirely novel dynamics, by means of the addition of a cubic nonlinearity of NLS-type. This extra contribution allows us to transform the former linear model into a nonlinear system of PDEs that turns out to be a generalization of the Manakov system, a vector extension of NLS equation with two components.

  The integrability of this model has been analyzed by means of the Painlevé test and we have successfully employed the SMM to derive a novel three-component and isospectral Lax pair for this system. Binary Darboux transformations have been

applied to this model, straightforwardly yielding the definition of the τ-function and an iterative algorithmic method to construct recursive analytic solutions. The choice of exponential seed solutions and eigenfunctions for the spectral problem has retrieved a wide class of soliton-like solutions in different regimes. For the defocusing case, we have obtained dark solitons that generalize the corresponding soliton solutions for the Manakov system. For the focusing case, generalizations of the Akhmediev and Kuznetsov-Ma breathers have been explicitly derived. Moreover, this procedure has also resulted in the obtention of two families of rogue waves for the focusing case, linked to a nontrivial polynomial expression for the τ-function. The analysis of cnoidal waves in the hyperbolic limit has yielded stable bright solitons in the focusing case. This latter result constitutes the generalization of the Davydov soliton, a theoretical solution that is prescribed to appear in the continuous limit within the adiabatic approximation describing the dynamics of deformable organic molecules with helical conformation. The helicity of this soliton is proved to be well-defined and preserved alongside its propagation across the helical molecule. The obtention of this result might shed some light on the relevance of the nonlinear interaction regarding the theoretical description of the spin dynamics in helical molecules, as a means to explain the chiral-induced spin selectivity phenomenon reported in experiments.

- On the other hand, we have introduced a modified $(1 + 1)$-NLS equation with derivative-type nonlinearities. The proposed system depends on an arbitrary real parameter γ, which includes as particular cases three celebrated equations of DNLS-type, the Kaup-Newell system ($\gamma = 0$), the Chen-Lee-Liu equation ($\gamma = 1$) and the Gerdjikov-Ivanov equation ($\gamma = 2$). These three nonlinear PDEs, together with the system proposed by the author, are equivalent via a $U(1)$-gauge transformation. We have exploited this gauge invariance property to analyze the integrability, construct the Lax pair and derive rational soliton solutions for these equations. Besides, these results can be straightforwardly extended to any integrable DNLS equation that can be related by a $U(1)$-gauge transformation to the prior ones.

We have successfully reviewed the integrability of this generalized DNLS equation. The Painlevé test was not properly applicable to this equation since the leading-order of the series expansion was rational. Therefore, the introduction of up to two changes of variable has been necessary in order to transform the generalized DNLS equation into a suitable PDE liable to pass the Painlevé test. The first change of variables yields a conservative differential equation for the probability density α of the initial DNLS, with integer leading index but two branches of expansion. The best method to overcome this inconvenience requires the introduction of a Miura transformation that retrieves a nonlocal Boussinesq-like equation for both

the new fields u and its complex conjugate $\overline{u}$. It is worth stressing the crucial role of the Miura transformation, which allows us to connect the aforementioned three differential equations and finally transform the original DNLS into a suitable PDE with a unique branch of expansion that possesses the Painlevé Property. We then ratify the success of this technique to deal with integrable PDEs with several branches of expansion in Painlevé integrability contexts.

Regarding the subtleties of the Painlevé analysis, we should now highlight the fact that the leading index at constant level for the nonlocal Boussinesq equation is zero. Therefore, the Painlevé test has had to be modified in order to incorporate a finite number of logarithmic terms in the series expansion so as to check the integrability of such equation. Then, it is found that the appearance of the logarithmic term does not contradict the assertion of the Painlevé conjecture, since it is precisely the first derivatives of the field u_x, u_t the ones that are expressed in terms of the Laurent series in a neighbourhood of the singular manifold $\phi = 0$.

By means of the SMM we have been able to find the spectral problem for these three systems: the starting DNLS equation, the conservative PDE for the probability density α and the nonlocal Boussinesq equation for u. We have obtained not just one, but two equivalent and isospectral Lax pairs for each of these equations of interest. The associated spectral problems are defined up to a coupling constraint between the eigenfunctions and the fields involved. This condition is inherent to the splitting process of the field α and the introduction of the Miura transformation. Then, this situation will be reflected when determining the associated spectral problems for the initial generalized DNLS equation. Moreover, the obtained linear problems allow to recover the Lax pairs known in the literature for the aforementioned particular DNLS systems.

Binary Darboux transformations have been implemented. Taking an ansatz with exponential seed solution and eigenfunctions allows us to straightforwardly construct a polynomial τ-function. This result inevitably leads to the obtention of interacting solitons of rational type, where the corresponding one and two soliton solution have been deeply investigated.

Chapter 4

Chapter 4 constitutes a continuation of the preceding Chapter, addressing the applications of the theoretical foundations exhibited in Chapter 2 for integrable models in $2 + 1$ dimensions. The analysis of solutions is focused on the characterization of a new kind of localized structures proper of higher spatial dimensions: rationally decaying solitons, known as lumps.

- Primarily, we have analyzed an integrable multi-component NLS equation in $2+1$ dimensions. This PDE can be either regarded as a generalization of the Manakov system to $2+1$ dimensions or as a vector generalization for the Fokas system with two components.

  Its integrability directly follows from the application of the Painlevé test. Furthermore, the SMM has enabled us to derive a nontrivial three-component Lax pair for this system. Lump soliton solutions can be directly constructed after the consequent application of the binary Darboux transformations over the spectral problem. In order to obtain rational solutions for the τ-function, it is desirable to seek seed eigenfunctions expressed as a product of exponential functions and a polynomial expression of arbitrary degree. The substitution of this ansatz into the spectral problem yields recursive relation for those polynomials, allowing the possibility to obtain an infinite number of eigenfunctions given in terms of integers associated to the degree of the polynomials. Therefore, it is expected that the nature of the arising lump solutions strongly depends on the value of such integers. The soliton solutions in this case are found to depend on two integers, $N, M \in \mathbb{N}$.

  Furthermore, the second iteration retrieves a τ-function $\tau_{1,2}$ that can be written in terms of a unique wavenumber, which means that the associated solution shall be interpreted as a one-soliton solution. Nevertheless, the corresponding dynamics to the one-soliton solution turns out to be particularly rich, since it displays the interaction of a nontrivial number of travelling nonlinear waves, of lump-type with equal amplitude, which depends on the choice of the integers N, M. We have explored the one-soliton solutions associated to all possible combinations of $N, M = 0, 1$. These choices retrieve three cases of interest, leading to the interaction of one, two and three lump-like waves with the same amplitude, respectively.

- Last but not least, the second nonlinear PDE analyzed in this Chapter is the so-called Nizhnik-Novikov-Veselov equation, which constitutes a symmetric generalization of the KdV equation to $2 + 1$ dimensions.

  Subsequent application of the Painlevé test retrieves that this nonlinear equation may be considered as integrable. Besides, the SMM allows us to successfully derive the associated spectral problem, where the eigenfunction turns out to be the singular manifold itself. This is due to the fact that the singular manifold equations can be fully integrated without the need of an intermediate linearization ansatz.

  Binary Darboux transformations can be implemented in order to obtain lump soliton solutions. As in the previous case, the consideration of eigenfunctions of the form polynomial expression times exponential functions retrieves an infinite number of possible eigenfunctions that depends on the degree n of such polynomial.

Once again, the nature of the arising soliton solutions is determined by the value of this integer.

The second iteration provides soliton solutions in terms of a sole wavenumber and its complex conjugate, giving rise to the aforementioned one-soliton solution. The dynamics for the one-soliton solution with one ($n = 1$) and two ($n = 2$) components of lump-type of equal amplitude has been profoundly analyzed, as it was done for the previous system.

In this case, we have also studied the two-soliton solution, which requires to consider up to the fourth iteration in the fields, with associated τ-function $\tau_{1,2,3,4}$. The process is constructed such that the first and the third iteration introduce two different wavenumbers k_1 and k_3, whilst the second and the fourth iteration provide the respective complex conjugates, $k_2 = k_1^\dagger$ and $k_4 = k_3^\dagger$. We have studied the simplest case, with $n = 1$, which has allowed us to characterize the dynamics and interaction of two different and independent lumps, of distinct amplitude.

Chapter 5

Chapter 5 is devoted to the theoretical characterization from a geometric point of view of the theory of Lie symmetries for differential equations. An extensive description of both the classical and nonclassical method to compute Lie symmetries is straightforwardly provided. Besides, the process regarding the similarity reduction method is thoroughly investigated. The last Section of this Chapter is aimed at the particular application of this whole procedure, as an example, to the NLS equation in $1 + 1$ dimensions.

Chapter 6

Finally, Chapter 6 addresses the applications of Lie's formalism of classical (and nonclassical) symmetries to diverse integrable models in several dimensions. The novel approach considered in this dissertation primarily targets the application of Lie's method to the spectral problems for these integrable systems. Therefore, it is expected that the isospectral or nonisospectral nature of the spectral parameter associated to those Lax pairs plays a crucial role on the symmetry analysis. This fact has proved to bear special relevance for systems in $2 + 1$ dimensions. Similarity reductions for Lax pairs constitute another critical element of this formulation, since they may yield new families of integrable differential equations in lower dimensions.

- Regarding integrable systems in $1+1$ dimensions, we have analyzed the three non-linear PDEs arising from the Painlevé analysis for the generalized DNLS equation

performed in Chapter 3. We have successfully computed both classical and non-classical Lie symmetries for these systems and their respective spectral problems. The nonclassical method has not retrieved any further results than the classical one, for each case under consideration. The main advantage of performing this procedure directly over the spectral problem is that it allows us to get simultaneously the symmetries related to the independent variables, fields and those associated to the eigenfunctions and the spectral parameter. Hence, Lie symmetries of the associated linear problem provide us more valuable information than the single analysis over the PDE. Finally, similarity reductions have been computed in each case. We have obtained the following:

– Lie symmetries for DNLS equation are given in terms of three arbitrary constants, a single arbitrary real function of t and two arbitrary complex functions on λ, the spectral parameter. The commutation relations among the associated generators have been studied and the Lie algebra has been identified for a particular choice of the arbitrary functions, giving rise to a subalgebra of the Schrödinger algebra $\mathfrak{sch}(1)$. Besides, three nontrivial similarity reductions arise from the reduction process.

– Lie symmetries for the conservative PDE for the probability density α depends on up to four arbitrary real constants and two arbitrary complex functions of λ. We have also studied two nontrivial reductions for this equation.

– The symmetry group of the nonlocal Boussinesq equation for u is presented in terms of six arbitrary real constants, a single arbitrary real function of t and two arbitrary complex functions of λ. It is worth stressing the presence of a complete new symmetry induced by the associated Miura transformation. Finally, three additional reductions of interest emerge for this case.

• Nonlinear PDEs in $2+1$ dimensions exhibit richer results regarding the symmetry analysis than their counterparts in $1+1$ dimensions. This phenomenon may be due to both the addition of an extra spatial dimension and the versatile nature of the spectral parameter for Lax pairs in higher dimensions.

– We have first determined the classical Lie symmetries of an integrable generalization of the NLS equation in $2+1$ dimensions with higher order terms. This integrable system depends on two arbitrary parameters γ_1, γ_2, and contains as particular cases four integrable renowned PDEs: generalized NLS equation in $2+1$ dimensions ($\gamma_1 \neq 0$, $\gamma_2 \neq 0$), Lakshmanan-Porsezian-Daniel equation in $2+1$ dimensions ($\gamma_1 \neq 0$, $\gamma_2 = 0$), Hirota equation in $2+1$ dimensions ($\gamma_1 = 0$, $\gamma_2 \neq 0$) and a standard NLS equation in $2+1$ dimensions ($\gamma_1 = 0$, $\gamma_2 = 0$). This equation has proved to be integrable and possesses a nonisospectral Lax pair.

Four different sets of symmetries can be obtained depending on whether the parameters γ_1, γ_2 are zero or different from zero. These symmetries may depend on up to six arbitrary constants and four arbitrary functions. The reductions associated to each set of symmetries have been successfully identified, yielding several similarity reductions of interest. This procedure provides not only the reductions for the equations, but also the reduced spectral problems. Besides, it is immediately shown how the nonisospectral condition is propagated under the reductions. We have obtained three special cases where the associated reduced spectral problem in $1 + 1$ dimensions is yet nonisospectral. It is worthwhile to remark that the reduced equations are proved to be integrable systems in $1 + 1$, being most of them quite complicated and non autonomous. Then, we may conclude that symmetry techniques also constitute a valuable procedure when identifying the integrability of nonlinear systems.

The commutation relations among the Lie symmetry generators have also been studied, in order to characterize the resulting Lie algebra. It is found that the presence of arbitrary functions in the symmetry generators prevents them from forming a Lie algebra. Nevertheless, it can be proved that the commutator of two symmetry generators is also a generator of a symmetry in an intriguing way, by means of an appropriate choice of the arbitrary functions. This fact provides an analogous notion of closure with the finite-dimensional case. Besides, it is possible to obtain a finite-dimensional Lie algebra by adopting special values for the arbitrary functions.

– We have also studied the multi-component NLS equation introduced in Chapter 4 from the symmetry point of view. As already illustrated, this integrable system possesses a three-component Lax pair, with no explicit spectral parameter.

We have determined the classical Lie symmetries for this system and its spectral problem. The resulting symmetries include nine arbitrary functions of the independent variables and two arbitrary complex constants, which play the role of the spectral parameter when the spectral problem is reduced to $1+1$ dimensions. The commutation relations among the generators associated to each symmetry have been widely analyzed, obtaining a similar result to the one stated for the previous example.

Three nontrivial reductions in $1 + 1$ dimensions have been derived. The reduced equations and the reduced spectral problems have been simultaneously obtained. The reduced spectral problems in $1 + 1$ do possess a spectral parameter, which arises naturally in the process of constructing the reductions due to the symmetry procedure itself.

2.   Future work

The research conducted in this thesis presents a unified methodology based on Painlevé integrability, the SMM and Lie symmetries to approach nonlinear models described by integrable differential equations. Nevertheless, there are several open research topics left that may be suggested as possible future work in this line of investigation:

- The application of this unified scheme has proved to be highly convenient when it comes to describing and obtaining solutions of interest for realistic nonlinear models arising from disciplines as Mathematical Physics, Materials Sciences and Biology. We firmly believe that the application of this type of analytical techniques may help to understand the behaviour of such systems. Therefore, it is worth exploring further research fields related to nonlinear phenomenology in the quest of integrable models that could account for those scenarios.

- As has been amply illustrated, algorithmic tests based on the Painlevé Property provide an accurate integrability criterion for nonlinear differential equations when they are applicable. On the other hand, these methods present several limitations, such as coordinate-dependence on the form of the PDE, the restriction to positive integer dominant indices, the issue associated with multiple expansion branches, etc. These kinds of problems are also reflected in the subsequent application of the SMM. Throughout this work, various procedures have been exhibited to overcome these inconveniences, but they are still far from being general or fully understood. It would be interesting to delve into these ideas in more depth.

  A promising line of research in this regard requires a detailed consideration of the different existing transformations among families of integrable systems, such as Bäcklund transformations, Miura transformations, hodograph or reciprocal transformations. The author of this manuscript has carried out further research on reciprocal transformations and the composition of Miura-reciprocal transformations [27, 30], not included in this dissertation. Reciprocal transformations have proved to be a remarkably fruitful technique to identify the integrability of PDEs, derive Lax pairs and may have an impact on the obtention of solutions of interest for those systems. This subject still remains as an open research topic.

- The SMM, as applied and understood throughout this thesis, may be slightly improved in relation to the derivation of the singular manifold equations. The SMM typically requires that every coefficient arising from the truncation ansatz of the Painlevé expansions should be equated to zero. Nonetheless, it is possible to weaken this condition and find solutions for ϕ that do not necessarily vanish each

coefficient but the set of all of them. This fact could increase the applicability of the SMM to a broader class of nonlinear integrable systems.

Moreover, this procedure leads to a more general solution for the singular manifold, which is subsequently reflected in the corresponding spectral problem and the associated solutions. In particular, this technique might allow the introduction of additional arbitrary functions in the Lax pair, which could eventually play the role of the spectral parameter in case the former linear problem does not possess any.

- In connection with the latter idea, the role of the spectral parameter in Lax pairs, both in $1 + 1$ and higher dimensions, is a matter of keen interest that would be worth studying. Specifically, the question regarding the introduction of a "true spectral parameter", *i.e.* nonremovable after a gauge transformation, constitutes an open problem in this research area. Both the SMM and the symmetry analysis by means of group techniques could shed some light in this regard.

- Another future research plan could involve the extension of our Lie symmetry approach, both classical and nonclassical, to other kinds of generalized scenarios, such as contact symmetries, Lie-Bäckund symmetries, etc. The treatment of spectral problems under this new perspective could retrieve valuable information of the integrable systems under consideration, as well as the analysis of their associated similarity reductions.

Appendices

Appendix A

Complementary calculations for Chapter 5

1. General prolongation formula

The *general prolongation formula*, formulated in equation (5.54), is given by

$$\eta_\sigma^j(x, u, \cdots, u_\sigma) = D^\sigma \left(\eta^j(x, u) - \sum_{i=1}^n \xi_i(x, u) \frac{\partial u^j}{\partial x_i} \right) + \sum_{i=1}^n \xi_i(x, u) \frac{\partial u_\sigma^j}{\partial x_i}. \qquad (A.1)$$

This result is well-known in the literature of Lie symmetries [51, 323, 378] and it can be demonstrated by induction.

Proof. Firstly, let us start with the case of $p = 1$, the prolongation up to first-order derivatives. The first prolongation of the vector field $X^{(1)}$ is given by

$$X^{(1)} = \sum_{i=1}^n \xi_i(x, u) \frac{\partial}{\partial x_i} + \sum_{j=1}^m \eta^j(x, u) \frac{\partial}{\partial u^j} + \sum_{i=1}^n \sum_{j=1}^m (\eta^j)_{x_i}(x, u_{(1)}) \frac{\partial}{\partial u_{x_i}^j}, \qquad (A.2)$$

and the associated one-parameter Lie group of transformations reads

$$\begin{cases} \tilde{x}_i = x_i + \epsilon \xi_i(x, u) + \mathcal{O}(\epsilon^2), \\ \tilde{u}^j = u^j + \epsilon \eta^j(x, u) + \mathcal{O}(\epsilon^2), \\ \tilde{u}_{\tilde{x}_i}^j = u_{x_i}^j + \epsilon (\eta^j)_{x_i}(x, u_{(1)}) + \mathcal{O}(\epsilon^2), \end{cases} \qquad (A.3)$$

where $u_{x_i}^j = \frac{\partial u^j}{\partial x_i}$ and $(\eta^j)_{x_i}$ is the infinitesimal associated to $u_{x_i}^j$, for all $i = 1, \ldots, n$, $j = 1, \ldots, m$. For $p = 1$, the total derivative operator with respect to x_i has the form

$$D_{x_i} = \frac{\partial}{\partial x_i} + \sum_{j=1}^m \frac{\partial u^j}{\partial x_i} \frac{\partial}{\partial u^j} + \sum_{j=1}^m \sum_{l=1}^n \frac{\partial^2 u^j}{\partial x_i \partial x_l} \frac{\partial}{\partial u_{x_l}^j}. \qquad (A.4)$$

Then, it is straightforward to see that

$$
d\tilde{x}_i = \sum_{l=1}^{n}(\mathrm{D}_{x_l}\,\tilde{x}_i)dx_l = \sum_{l=1}^{n}(\mathrm{D}_{x_l}\,x_i + \epsilon\,\mathrm{D}_{x_l}\,\xi_i)dx_l = \sum_{l=1}^{n}(\delta_{il} + \epsilon\,\mathrm{D}_{x_l}\,\xi_i)dx_l,
$$

$$
d\tilde{u}^j = \sum_{k=1}^{n}(\mathrm{D}_{x_k}\,\tilde{u}^j)dx_k = \sum_{k=1}^{n}(\mathrm{D}_{x_k}\,u^j + \epsilon\,\mathrm{D}_{x_k}\,\eta^j)dx_k = \sum_{k=1}^{n}\left(\frac{\partial u^j}{\partial x_k} + \epsilon\,\mathrm{D}_{x_k}\,\eta^j\right)dx_k,
$$

$$\tag{A.5}$$

where δ_{ij} stands for the usual Kronecker delta. Thus, we have

$$
\tilde{u}^j_{\tilde{x}_i} = \frac{\partial \tilde{u}^j}{\partial \tilde{x}_i} = \frac{\sum\limits_{k=1}^{n}\left(u^j_{x_k} + \epsilon\,\mathrm{D}_{x_k}\,\eta^j\right) + \mathcal{O}(\epsilon^2)}{\sum\limits_{l=1}^{n}(\delta_{il} + \epsilon\,\mathrm{D}_{x_l}\,\xi_i) + \mathcal{O}(\epsilon^2)}\,\delta_{kl}. \tag{A.6}
$$

Computing the inverse of the formal power series up to first order in ϵ, we get

$$
\tilde{u}^j_{\tilde{x}_i} = \sum_{k=1}^{n}\left(u^j_{x_k} + \epsilon\,\mathrm{D}_{x_k}\,\eta^j + \mathcal{O}(\epsilon^2)\right)\left(\delta_{ik} - \epsilon\,\mathrm{D}_{x_i}\,\xi_k + \mathcal{O}(\epsilon^2)\right)
$$

$$
= u_{x_i} + \epsilon\left(\mathrm{D}_{x_i}\,\eta^j - \sum_{k=1}^{n}\frac{\partial u^j}{\partial x_k}\,\mathrm{D}_{x_i}\,\xi_k\right) + \mathcal{O}(\epsilon^2). \tag{A.7}
$$

Comparing this result with (A.3), we can conclude that

$$
(\eta^j)_{x_i}(x, u_{(1)}) = \mathrm{D}_{x_i}\,\eta^j - \sum_{k=1}^{n}\frac{\partial u^j}{\partial x_k}\,\mathrm{D}_{x_i}\,\xi_k, \qquad i = 1,\ldots,n,\ j = 1,\ldots,m. \tag{A.8}
$$

Reordering terms and using the definition of (A.4), we find that

$$
(\eta^j)_{x_i} = \mathrm{D}_{x_i}\,\eta^j - \sum_{k=1}^{n}\left(\frac{\partial u^j}{\partial x_k}\,\mathrm{D}_{x_i}\,\xi_k + \xi_k\frac{\partial^2 u^j}{\partial x_i \partial x_k}\right) + \sum_{k=1}^{n}\xi_k\frac{\partial^2 u^j}{\partial x_i \partial x_k}
$$

$$
= \mathrm{D}_{x_i}\left(\eta^j - \sum_{k=1}^{n}\xi_k\frac{\partial u^j}{\partial x_k}\right) + \sum_{k=1}^{n}\xi_k\frac{\partial^2 u^j}{\partial x_i \partial x_k}, \tag{A.9}
$$

which corresponds to the expression provided in (A.1) for $p = 1$.

In order to prove the general case by induction, we assume that the general prolongation formula (A.1) is valid for the extension up to pth-order derivatives, such that the prolonged vector field $X^{(p)}$ is given by (5.50) and the associated Lie group of transformations is (5.55). Given a coordinate $u^j, j = 1,\ldots,m$, there exist $\frac{(n+p-1)!}{p!(n-1)!}$

derivatives or order p, and from them, it is possible to construct n derivatives of order $p+1$ as $(u_\sigma^j)_{x_l} = \frac{\partial u_\sigma^j}{\partial x_l}$, $l = 1, \ldots, n$, $|\sigma| = p$. In this sense, we can define the extension of (5.55) up to $(p+1)$-th order derivatives as[1]

$$(\tilde{u}_\sigma^j)_{\tilde{x}_l} = (u_\sigma^j)_{x_l} + \epsilon (\eta_\sigma^j)_{x_l}(x, u_{(p+1)}), \tag{A.10}$$

for all $l = 1, \ldots, n$, $|\sigma| = p$ and $(\tilde{u}_\sigma^j)_{\tilde{x}_l} = \frac{\partial \tilde{u}_\sigma^j}{\partial \tilde{x}_l} = \frac{\partial^{p+1} u^j}{\partial \tilde{x}_1^{i_1} \cdots \partial \tilde{x}_l^{i_l+1} \cdots \partial \tilde{x}_n^{i_n}}$.

Proceeding by complete analogy with (A.5)-(A.7), we have

$$
\begin{aligned}
(\tilde{u}_\sigma^j)_{\tilde{x}_l} = \frac{\partial \tilde{u}_\sigma^j}{\partial \tilde{x}_l} &= \frac{\sum_{k=1}^{n} \left(\frac{\partial u_\sigma^j}{\partial x_l} + \epsilon \, \mathrm{D}_{x_k} \eta_\sigma^j \right) + \mathcal{O}(\epsilon^2)}{\sum_{r=1}^{n} \left(\delta_{lr} + \epsilon \, \mathrm{D}_{x_r} \xi_l \right) + \mathcal{O}(\epsilon^2)} \delta_{kr} \\
&= \frac{\partial u_\sigma^j}{\partial x_l} + \epsilon \left(\mathrm{D}_{x_l} \eta_\sigma^j - \sum_{k=1}^{n} \frac{\partial u_\sigma^j}{\partial x_k} \, \mathrm{D}_{x_l} \xi_k \right) + \mathcal{O}(\epsilon^2).
\end{aligned}
\tag{A.11}
$$

Hence, it is immediate that

$$(\eta_\sigma^j)_{x_l} = \mathrm{D}_{x_l} \eta_\sigma^j - \sum_{k=1}^{n} \frac{\partial u_\sigma^j}{\partial x_k} \, \mathrm{D}_{x_l} \xi_k, \qquad l = 1, \ldots, n, \; j = 1, \ldots, m, \tag{A.12}$$

which provides the well-known recursion relation for the extended infinitesimal.

Finally, inserting the expression for η_σ^j given by the prolongation formula (A.1) in the relation above we get

$$
\begin{aligned}
(\eta_\sigma^j)_{x_l} &= \mathrm{D}_{x_l} \left[\mathrm{D}^\sigma \left(\eta^j - \sum_{k=1}^{n} \xi_k \frac{\partial u^j}{\partial x_k} \right) + \sum_{k=1}^{n} \xi_k \frac{\partial u_\sigma^j}{\partial x_k} \right] - \sum_{k=1}^{n} \frac{\partial u_\sigma^j}{\partial x_k} \, \mathrm{D}_{x_l} \xi_k \\
&= (\mathrm{D}_{x_l} \cdot \mathrm{D}^\sigma) \left(\eta^j - \sum_{k=1}^{n} \xi_k \frac{\partial u^j}{\partial x_k} \right) + \sum_{k=1}^{n} \left(\frac{\partial u_\sigma^j}{\partial x_k} \mathrm{D}_{x_l} \xi_k + \xi_k \frac{\partial^2 u_\sigma^j}{\partial x_l \partial x_k} \right) - \sum_{k=1}^{n} \frac{\partial u_\sigma^j}{\partial x_k} \, \mathrm{D}_{x_l} \xi_k \\
&= (\mathrm{D}_{x_l} \cdot \mathrm{D}^\sigma) \left(\eta^j - \sum_{k=1}^{n} \xi_k \frac{\partial u^j}{\partial x_k} \right) + \sum_{k=1}^{n} \xi_k \frac{\partial^2 u_\sigma^j}{\partial x_l \partial x_k},
\end{aligned}
\tag{A.13}
$$

which identically coincides with the expression given in (A.1) for the multi-index $\hat{\sigma} = (i_1, \ldots, i_l + 1, \ldots, i_n)$ associated to the specific derivative $(u_\sigma^j)_{x_l} \equiv u_{\hat{\sigma}}^j$, when

---

[1] From the geometric point of view, this fact means that the $(p+1)$-th jet space $M_{(p+1)}$ may be regarded as a subspace of the first jet space $(M_{(p)})_{(1)}$.

$\sigma = (i_1, \ldots, i_l, \ldots, i_n)$ such that $|\sigma| = p$, $|\hat{\sigma}| = p+1$ and for all $l = 1, \ldots, n$. $\qquad\qquad$ $\square$

2. Proof of equation (5.67)

Let us start from the definition of the invariant surface conditions given in (5.64)

$$\Delta^j = \eta^j - \sum_{i=1}^{n} \xi_i \frac{\partial u^j}{\partial x_i}, \qquad j = 1, \ldots, m. \tag{A.14}$$

According to (A.2), the first prolongation for a vector field of the form $X = \sum_{k=1}^{n} \xi_k \frac{\partial}{\partial x_k} + \sum_{l=1}^{m} \eta^l \frac{\partial}{\partial u^l}$ reads

$$X^{(1)} = \sum_{i=k}^{n} \xi_k \frac{\partial}{\partial x_i} + \sum_{l=1}^{m} \eta^l \frac{\partial}{\partial u^l} + \sum_{k=1}^{n}\sum_{l=1}^{m} (\eta^l)_{x_k} \frac{\partial}{\partial u^l_{x_k}}, \tag{A.15}$$

where the general prolongation formula (A.1) establishes that

$$(\eta^l)_{x_k} = D_{x_k}\left(\eta^l - \sum_{p=1}^{n} \xi_p \frac{\partial u^l}{\partial x_p} \right) + \sum_{p=1}^{n} \xi_p \frac{\partial^2 u^l}{\partial x_k \partial x_p}, \tag{A.16}$$

being $D_{x_k} = \frac{\partial}{\partial x_k} + \sum_{q=1}^{m} \frac{\partial u^q}{\partial x_k} \frac{\partial}{\partial u^q} + \sum_{q=1}^{m}\sum_{p=1}^{n} \frac{\partial^2 u^q}{\partial x_k \partial x_p} \frac{\partial}{\partial u^q_{x_p}}$ the total derivative operator.

Then, for every $j = 1, \ldots, m$, we can compute

$$X^{(1)}\left[\Delta^j\right] = \sum_{k=1}^{n} \xi_k \frac{\partial \Delta^j}{\partial x_k} + \sum_{l=1}^{m} \eta^l \frac{\partial \Delta^j}{\partial u^l} + \sum_{k=1}^{n}\sum_{l=1}^{m} \left(\eta^j\right)_{x_k} \frac{\partial \Delta^j}{\partial u^l_{x_k}}. \tag{A.17}$$

Equation (A.16) may be written as

$$\begin{aligned}
(\eta^l)_{x_k} &= D_{x_k}\left(\Delta^l\right) + \sum_{p=1}^{n} \xi_p \frac{\partial^2 u^l}{\partial x_k \partial x_p} \\
&= \frac{\partial \Delta^l}{\partial x_k} + \sum_{q=1}^{m} \frac{\partial \Delta^l}{\partial u^q} \frac{\partial u^q}{\partial x_k} + \sum_{p=1}^{n}\sum_{q=1}^{m} \frac{\partial^2 u^q}{\partial x_k \partial x_p} \frac{\partial \Delta^l}{\partial u^q_{x_p}} + \sum_{p=1}^{n} \xi_p \frac{\partial^2 u^l}{\partial x_k \partial x_p}.
\end{aligned} \tag{A.18}$$

From its definition (A.14), we get that $\frac{\partial \Delta^l}{\partial u^q_{x_p}} = -\xi_i \delta_{lq}\delta_{ip}$, with δ_{ij} being the usual

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Kronecker delta, so that (A.18) takes the form

$$(\eta^l)_{x_k} = \frac{\partial \Delta^l}{\partial x_k} + \sum_{q=1}^{m} \frac{\partial \Delta^l}{\partial u^q} \frac{\partial u^q}{\partial x_k}, \tag{A.19}$$

and finally (A.17) provides

$$\begin{aligned}
X^{(1)}\left[\Delta^j\right] &= \sum_{k=1}^{n} \xi_k \frac{\partial \Delta^j}{\partial x_k} + \sum_{l=1}^{m} \eta^l \frac{\partial \Delta^j}{\partial u^l} - \sum_{k=1}^{n} \xi_k \left(\eta^j\right)_{x_k} \\
&= \sum_{k=1}^{n} \xi_k \frac{\partial \Delta^j}{\partial x_k} + \sum_{l=1}^{m} \eta^l \frac{\partial \Delta^j}{\partial u^l} - \sum_{k=1}^{n} \xi_k \left(\frac{\partial \Delta^j}{\partial x_k} + \sum_{l=1}^{m} \frac{\partial \Delta^j}{\partial u^l} \frac{\partial u^l}{\partial x_k}\right) \\
&= \sum_{l=1}^{m} \left(\eta^l - \sum_{k=1}^{n} \frac{\partial u^l}{\partial x_k}\right) \frac{\partial \Delta^j}{\partial u^l} = \sum_{l=1}^{m} \Delta^l \frac{\partial \Delta^j}{\partial u^l},
\end{aligned} \tag{A.20}$$

which is precisely (5.67).

Appendix B

Commutation relations from the Lie symmetry analysis of PDEs in $2+1$ dimensions

This Appendix addresses the commutation relations of the infinitesimal generators associated to the classical Lie symmetries for the differential equations in $2 + 1$ dimensions studied in the present doctoral thesis (c.f. Section 2 of Chapter 6). The results are displayed in tables. Each $\{i, j\}$-element of the table (*i.e.* the entry in row i and column j) represents the operation $[V_i, V_j]$, with V_i, V_j two generators of the symmetry group.

Since the systems of PDEs are defined in $2+1$ dimensions, the associated Lie symmetries are expected to be expressed in terms of arbitrary functions of the independent variables. By experience, this fact implies that the infinitesimal generators will depend on arbitrary functions, leading to infinite-dimensional subalgebras. According to [378], the infinitesimal generators that depend on arbitrary functions do not form a Lie algebra. Nonetheless, it can be proved that the commutator of two symmetry generators is also a generator of a symmetry in an intriguing way. In general, the operation of commutation between two infinitesimal generators of this kind yields a result that also depends on a combination of the arbitrary functions involved. Then, it is possible to express this commutator as a combination of the generators of the symmetry group by means of an appropriate choice of the arbitrary functions, as it will be illustrated in this Appendix. The presence of arbitrary functions in the symmetry generators therefore provides nontrivial commutation relations among these generators, with peculiar properties. For example, the commutator of a symmetry generator with itself does not necessarily vanish. For this reason, these commutation relations should be carefully studied.

In the present Appendix we will use the following notation. A vector field with a latin subindex as $\mathcal{X}_i$ represents a Lie symmetry associated to an arbitrary constant α_i, $i \in \mathbb{N}$. Conversely, an operator of the form $\mathcal{X}^{[j]}_{\{\kappa_j\}}$, $j \in \mathbb{N}$, is associated to the

symmetry of an arbitrary function κ_j. The dependence on the independent variables of the functions κ_j (which can be x, y, t or λ) will be omitted in the tables for a greater usability for the reader, but it will be explicitly specified in advance for each case of study.

The matter regarding the classification of the resulting Lie algebras falls beyond the scope of this thesis and it is left open for a future work.

1. Generalized NLS equation in $2 + 1$ dimensions with higher-order terms

This Section can be understood as a continuation of the symmetry analysis conducted in Subsection 2.1 of Chapter 6 for the nonlinear PDE (6.82) and its associated spectral problem (6.83). Equation (6.82) depends on two parameters γ_1, γ_2 that provide the following integrable renowned PDEs as particular cases:

1. Generalized NLS equation in $2 + 1$ dimensions ($\gamma_1 \neq 0$, $\gamma_2 \neq 0$)

2. Lakshmanan-Porsezian-Daniel equation in $2 + 1$ dimensions ($\gamma_1 \neq 0$, $\gamma_2 = 0$)

3. Hirota equation in $2 + 1$ dimensions ($\gamma_1 = 0$, $\gamma_2 \neq 0$)

4. Standard NLS equation in $2 + 1$ dimensions ($\gamma_1 = 0$, $\gamma_2 = 0$)

The symmetry analysis of these four systems gives rise to four different sets of Lie symmetries, which have been studied independently.

Generally, we may resume the following common notation for every case:

- $K_i(t)$, $\tilde{K}_j(t)$, $i, j = 1, 2$ denote arbitrary real functions of the variable t, and $Y^{[i]}_{\{K_i\}}$, $Y^{[j]}_{\{\tilde{K}_j\}}$ are their infinitesimal generators, respectively.

- $\delta(x, t)$, $\tilde{\delta}(x, t)$ are arbitrary real functions of $\{x, t\}$ and their associated infinitesimal generators are represented as $Z_{\{\delta\}}$, $Z_{\{\tilde{\delta}\}}$.

- $\zeta(y, t, \lambda)$, $\tilde{\zeta}(y, t, \lambda)$ are arbitrary complex functions of $\{y, t, \lambda\}$ satisfying the differential equations

$$\frac{\partial \zeta}{\partial t} - 2\lambda \frac{\partial \zeta}{\partial y} = 0, \qquad \frac{\partial \tilde{\zeta}}{\partial t} - 2\lambda \frac{\partial \tilde{\zeta}}{\partial y} = 0,$$

whilst $\Gamma_{\{\zeta\}}$, $\Gamma_{\{\tilde{\zeta}\}}$ stand for their associated symmetry generators.

1.1. Generalized NLS equation in $2+1$ dimensions ($\gamma_1 \neq 0$, $\gamma_2 \neq 0$)

Lie symmetries for this case are listed in (6.89) and the associated six infinitesimal generators are

$$X_1 = \partial_y, \quad X_2 = \partial_t,$$

$$Y_{\{K_1\}}^{[1]} = K_1\partial_x + \frac{iy}{2}\dot{K}_1\left(2u\partial_u - 2w\partial_w + \psi\partial_\psi - \chi\partial_\chi\right) + \frac{y^2}{4}\ddot{K}_1\partial_m,$$

$$Y_{\{K_2\}}^{[2]} = iK_2\left(2u\partial_u - 2w\partial_w + \psi\partial_\psi - \chi\partial_\chi\right) + \dot{K}_2 y\partial_m,$$

$$Z_{\{\delta\}} = \delta\partial_m, \quad \Gamma_{\{\zeta\}} = \zeta\left(\psi\partial_\psi + \chi\partial_\chi\right),$$

whose commutation relations read

	X_1	X_2	$Y_{\{K_1\}}^{[1]}$	$Y_{\{K_2\}}^{[2]}$	$Z_{\{\delta\}}$	$\Gamma_{\{\zeta\}}$
X_1	0	0	$\frac{1}{2}Y_{\{\partial_t K_1\}}^{[2]}$	$Z_{\{\partial_t K_2\}}$	0	$\Gamma_{\{\partial_y\zeta\}}$
X_2	0	0	$Y_{\{\partial_t K_1\}}^{[1]}$	$Y_{\{\partial_t K_2\}}^{[2]}$	$Z_{\{\partial_t\delta\}}$	$\Gamma_{\{\partial_t\zeta\}}$
$Y_{\{\tilde{K}_1\}}^{[1]}$	$-\frac{1}{2}Y_{\{\partial_t\tilde{K}_1\}}^{[2]}$	$-Y_{\{\partial_t\tilde{K}_1\}}^{[1]}$	0	0	$Z_{\{\tilde{K}_1\partial_x\delta\}}$	0
$Y_{\{\tilde{K}_2\}}^{[2]}$	$-Z_{\{\partial_t\tilde{K}_2\}}$	$-Y_{\{\partial_t\tilde{K}_2\}}^{[2]}$	0	0	0	0
$Z_{\{\tilde{\delta}\}}$	0	$-Z_{\{\partial_t\tilde{\delta}\}}$	$-Z_{\{K_1\partial_x\tilde{\delta}\}}$	0	0	0
$\Gamma_{\{\tilde{\zeta}\}}$	$-\Gamma_{\{\partial_y\tilde{\zeta}\}}$	$-\Gamma_{\{\partial_t\tilde{\zeta}\}}$	0	0	0	0

Table B.1: *Case $\gamma_1 \neq 0$, $\gamma_2 \neq 0$*

1.2. Lakshmanan-Porsezian-Daniel equation in $2+1$ dimensions ($\gamma_1 \neq 0$, $\tilde{\gamma}_2 = 0$)

Lie symmetries for this case are displayed in (6.110) and the associated seven infinitesimal generators are

$$X_1 = \partial_y,\ X_2 = \partial_t,\ X_3 = 3y\partial_y + 4t\partial_t - u\partial_u - w\partial_w - m\partial_m - \lambda\partial_\lambda,$$

$$Y^{[1]}_{\{K_1\}} = K_1\partial_x + \frac{iy}{2}\dot{K}_1\left(2u\partial_u - 2w\partial_w + \psi\partial_\psi - \chi\partial_\chi\right) + \frac{y^2}{4}\ddot{K}_1\partial_m,$$

$$Y^{[2]}_{\{K_2\}} = iK_2\left(2u\partial_u - 2w\partial_w + \psi\partial_\psi - \chi\partial_\chi\right) + \dot{K}_2 y\partial_m,$$

$$Z_{\{\delta\}} = \delta\partial_m,\ \Gamma_{\{\zeta\}} = \zeta\left(\psi\partial_\psi + \chi\partial_\chi\right).$$

If we introduce the differential operators $\hat{D}_1 = 4t\partial_t - \mathrm{Id},\ \hat{D}_2 = \mathrm{Id} + x\partial_x + 4t\partial_t,\ \hat{D}_3 = 3y\partial_y + 4t\partial_t - \lambda\partial_\lambda$, where Id stands for the identity, the commutation relations result in

	X_1	X_2	X_3	$Y^{[1]}_{\{K_1\}}$	$Y^{[2]}_{\{K_2\}}$	$Z_{\{\delta\}}$	$\Gamma_{\{\zeta\}}$
X_1	0	0	$3X_1$	$\frac{1}{2}Y^{[1]}_{\{\partial_t K_1\}}$	$Z_{\{\partial_t K_2\}}$	0	$\Gamma_{\{\partial_y\zeta\}}$
X_2	0	0	$4X_2$	$Y^{[1]}_{\{\partial_t K_1\}}$	$Y^{[2]}_{\{\partial_t K_2\}}$	$Z_{\{\partial_t\delta\}}$	$\Gamma_{\{\partial_t\zeta\}}$
X_3	$-3X_1$	$-4X_2$	0	$Y^{[1]}_{\{\hat{D}_1 K_1\}}$	$Y^{[2]}_{\{4t\partial_t K_2\}}$	$Z_{\{\hat{D}_2\delta\}}$	$\Gamma_{\{\hat{D}_3\zeta\}}$
$Y^{[1]}_{\{\tilde{K}_1\}}$	$-\frac{1}{2}Y^{[2]}_{\{\partial_t\tilde{K}_1\}}$	$-Y^{[1]}_{\{\partial_t\tilde{K}_1\}}$	$-Y^{[1]}_{\{\hat{D}_1\tilde{K}_1\}}$	0	0	$Z_{\{\tilde{K}_1\partial_x\delta\}}$	0
$Y^{[2]}_{\{\tilde{K}_2\}}$	$-Z_{\{\partial_t\tilde{K}_2\}}$	$-Y^{[2]}_{\{\partial_t\tilde{K}_2\}}$	$-Y^{[2]}_{\{4t\partial_t\tilde{K}_2\}}$	0	0	0	0
$Z_{\{\tilde{\delta}\}}$	0	$-Z_{\{\partial_t\tilde{\delta}\}}$	$-Z_{\{\hat{D}_2\tilde{\delta}\}}$	$-Z_{\{K_1\partial_x\delta\}}$	0	0	0
$\Gamma_{\{\tilde{\zeta}\}}$	$-\Gamma_{\{\partial_y\tilde{\zeta}\}}$	$-\Gamma_{\{\partial_t\tilde{\zeta}\}}$	$-\Gamma_{\{\hat{D}_3\tilde{\zeta}\}}$	0	0	0	0

Table B.2: *Case $\gamma_1 \neq 0$, $\tilde{\gamma}_2 = 0$*

1.3. Hirota equation in $2+1$ dimensions ($\gamma_1 = 0$, $\gamma_2 \neq 0$)

Lie symmetries for this case are given in (6.118) and the associated seven infinitesimal generators are

$$X_1 = \partial_y, \quad X_2 = \partial_t, \quad X_3 = 2y\partial_y + 3t\partial_t - u\partial_u - w\partial_w - m\partial_m - \lambda\partial_\lambda,$$

$$Y^{[1]}_{\{K_1\}} = K_1\partial_x + \frac{iy}{2}\dot{K}_1\left(2u\partial_u - 2w\partial_w + \psi\partial_\psi - \chi\partial_\chi\right) + \frac{y^2}{4}\ddot{K}_1\partial_m,$$

$$Y^{[2]}_{\{K_2\}} = iK_2\left(2u\partial_u - 2w\partial_w + \psi\partial_\psi - \chi\partial_\chi\right) + \dot{K}_2 y\partial_m,$$

$$Z_{\{\delta\}} = \delta\partial_m, \quad \Gamma_{\{\zeta\}} = \zeta\left(\psi\partial_\psi + \chi\partial_\chi\right).$$

If we introduce the differential operators $\check{D}_1 = 3t\partial_t - \mathrm{Id}$, $\check{D}_2 = \mathrm{Id} + x\partial_x + 3t\partial_t$, $\check{D}_3 = 2y\partial_y + 3t\partial_t - \lambda\partial_\lambda$, with Id being the identity, the commutation relations are given by

	X_1	X_2	X_3	$Y^{[1]}_{\{K_1\}}$	$Y^{[2]}_{\{K_2\}}$	$Z_{\{\delta\}}$	$\Gamma_{\{\zeta\}}$
X_1	0	0	$2X_1$	$\frac{1}{2}Y^{[1]}_{\{\partial_t K_1\}}$	$Z_{\{\partial_t K_2\}}$	0	$\Gamma_{\{\partial_y \zeta\}}$
X_2	0	0	$3X_2$	$Y^{[2]}_{\{\partial_t K_1\}}$	$Y^{[2]}_{\{\partial_t K_2\}}$	$Z_{\{\partial_t \delta\}}$	$\Gamma_{\{\partial_t \zeta\}}$
X_3	$-2X_1$	$-3X_2$	0	$Y^{[1]}_{\{\check{D}_1 K_1\}}$	$Y^{[2]}_{\{3t\partial_t K_2\}}$	$Z_{\{\check{D}_2 \delta\}}$	$\Gamma_{\{\check{D}_3 \zeta\}}$
$Y^{[1]}_{\{\tilde{K}_1\}}$	$-\frac{1}{2}Y^{[2]}_{\{\partial_t \tilde{K}_1\}}$	$-Y^{[1]}_{\{\partial_t \tilde{K}_1\}}$	$-Y^{[1]}_{\{\check{D}_1 \tilde{K}_1\}}$	0	0	$Z_{\{\tilde{K}_1 \partial_x \delta\}}$	0
$Y^{[2]}_{\{\tilde{K}_2\}}$	$-Z_{\{\partial_t \tilde{K}_2\}}$	$-Y^{[2]}_{\{\partial_t \tilde{K}_2\}}$	$-Y^{[2]}_{\{3t\partial_t \tilde{K}_2\}}$	0	0	0	0
$Z_{\{\tilde{\delta}\}}$	0	$-Z_{\{\partial_t \tilde{\delta}\}}$	$-Z_{\{\check{D}_2 \tilde{\delta}\}}$	$-Z_{\{K_1 \partial_x \tilde{\delta}\}}$	0	0	0
$\Gamma_{\{\tilde{\zeta}\}}$	$-\Gamma_{\{\partial_y \tilde{\zeta}\}}$	$-\Gamma_{\{\partial_t \tilde{\zeta}\}}$	$-\Gamma_{\{\check{D}_3 \tilde{\zeta}\}}$	0	0	0	0

Table B.3: *Case $\gamma_1 = 0$, $\gamma_2 \neq 0$*

1.4. Standard NLS equation in $2+1$ dimensions ($\gamma_1 = 0$, $\gamma_2 = 0$)

Lie symmetries for this case can be found in (6.126) and the associated ten infinitesimal generators are

$$X_1 = \partial_y,\ X_2 = \partial_t,\ X_3 = x\partial_x + t\partial_t - u\partial_u - w\partial_w - m\partial_m - \lambda\partial_\lambda,\ X_4 = y\partial_y + t\partial_t,\ Z_{\{\delta\}} = \delta\partial_m,\ \Gamma_{\{\zeta\}} = \zeta\left(\psi\partial_\psi + \chi\partial_\chi\right),$$

$$X_5 = 2xt\partial_x + 2yt\partial_y + 2t^2\partial_t - 2(t - ixy)u\partial_u - 2(t + ixy)w\partial_w - 2tm\partial_m - (2\lambda t + y)\partial_\lambda - (t - ixy)\psi\partial_\psi$$
$$- (t + ixy)\chi\partial_\chi,\ X_6 = 2t\partial_y + 2ixu\partial_u - 2ixw\partial_w - \partial_\lambda + ix\psi\partial_\psi - ix\chi\partial_\chi,$$

$$Y^{[1]}_{\{K_1\}} = K_1\partial_x + \frac{iy}{2}\dot{K}_1\left(2u\partial_u - 2w\partial_w + \psi\partial_\psi - \chi\partial_\chi\right) + \frac{y^2}{4}\ddot{K}_1\partial_m,\ Y^{[2]}_{\{K_2\}} = iK_2\left(2u\partial_u - 2w\partial_w + \psi\partial_\psi - \chi\partial_\chi\right) + \dot{K}_2 y\partial_m.$$

If we introduce the differential operators $D_1 = t\partial_t - \mathrm{Id}$, $D_2 = y\partial_y + t\partial_t$, $D_3 = \mathrm{Id} + x\partial_x + t\partial_t$, $D_4 = t\partial_t - \lambda\partial_\lambda$, $D_5 = 2t\partial_t - \partial_\lambda$, $D_6 = 2t\,D_4 + y\,D_5 = 2yt\partial_y + 2t^2\partial_t - (2\lambda t + y)\partial_\lambda$, where Id is the identity, the commutation relations read

	X_1	X_2	X_3	X_4	X_5	X_6	$Y^{[1]}_{\{K_1\}}$	$Y^{[2]}_{\{K_2\}}$	$Z_{\{\delta\}}$	$\Gamma_{\{\zeta\}}$
X_1	0	0	0	X_1	X_6	0	$\frac{1}{2}Y^{[2]}_{\{\partial_t K_1\}}$	$Z_{\{\partial_t K_2\}}$	0	$\Gamma_{\{\partial_y \zeta\}}$
X_2	0	0	X_3	X_2	$2X_3 + 2X_4 - \Gamma_{\{1\}}$	$2X_1$	$Y^{[1]}_{\{\partial_t K_1\}}$	$Y^{[2]}_{\{\partial_t K_2\}}$	$Z_{\{\partial_t \delta\}}$	$\Gamma_{\{\partial_t \zeta\}}$
X_3	0	$-X_3$	0	0	X_5	X_6	$Y^{[1]}_{\{D_1 K_1\}}$	$Y^{[2]}_{\{t\partial_t K_2\}}$	$Z_{\{D_3 \delta\}}$	$\Gamma_{\{D_4 \zeta\}}$
X_4	$-X_1$	$-X_2$	0	0	$-X_5$	0	$Y^{[1]}_{\{t\partial_t K_1\}}$	$Y^{[2]}_{\{t\partial_t K_2\}}$	$Z_{\{t\partial_t \delta\}}$	$\Gamma_{\{D_2 \zeta\}}$
X_5	$-X_6$	$-2X_3 - 2X_4 + \Gamma_{\{1\}}$	$-X_5$	$-X_5$	0	0	$Y^{[1]}_{\{2t D_1 K_1\}}$	$Y^{[2]}_{\{2t^2 \partial_t K_2\}}$	$Z_{\{2t D_3 \delta\}}$	$\Gamma_{\{D_6 \zeta\}}$
X_6	0	$-2X_1$	$-X_6$	0	0	0	$Y^{[1]}_{\{D_1 K_1\}}$	$Z_{\{2t\partial_t K_2\}}$	0	$\Gamma_{\{D_5 \zeta\}}$
$Y^{[1]}_{\{\tilde K_1\}}$	$-\frac{1}{2}Y^{[2]}_{\{\partial_t \tilde K_1\}}$	$-Y^{[1]}_{\{\partial_t \tilde K_1\}}$	$-Y^{[1]}_{\{D_1 \tilde K_1\}}$	$-Y^{[1]}_{\{t\partial_t \tilde K_1\}}$	$-Y^{[1]}_{\{2t D_1 \tilde K_1\}}$	$-Y^{[1]}_{\{D_1 \tilde K_1\}}$	0	0	$Z_{\{\tilde K_1 \partial_x \delta\}}$	0
$Y^{[2]}_{\{\tilde K_2\}}$	$-Z_{\{\partial_t \tilde K_2\}}$	$-Y^{[2]}_{\{\partial_t \tilde K_2\}}$	$-Y^{[2]}_{\{t\partial_t \tilde K_2\}}$	$-Y^{[2]}_{\{t\partial_t \tilde K_2\}}$	$-Y^{[2]}_{\{2t^2 \partial_t \tilde K_2\}}$	$-Z_{\{2t\partial_t \tilde K_2\}}$	0	0	0	0
$Z_{\{\tilde\delta\}}$	0	$-Z_{\{\partial_t \tilde\delta\}}$	$-Z_{\{D_3 \tilde\delta\}}$	$-Z_{\{t\partial_t \tilde\delta\}}$	$-Z_{\{2t D_3 \tilde\delta\}}$	0	$-Z_{\{K_1 \partial_x \tilde\delta\}}$	0	0	0
$\Gamma_{\{\tilde\zeta\}}$	$-\Gamma_{\{\partial_y \tilde\zeta\}}$	$-\Gamma_{\{\partial_t \tilde\zeta\}}$	$-\Gamma_{\{D_4 \tilde\zeta\}}$	$-\Gamma_{\{D_2 \tilde\zeta\}}$	$-\Gamma_{\{D_6 \tilde\zeta\}}$	$-\Gamma_{\{D_5 \tilde\zeta\}}$	0	0	0	0

Table B.4: *Case $\gamma_1 = 0$, $\gamma_2 = 0$*

2. Multi-component NLS equation in $2+1$ dimensions

This Section is devoted to the analysis of the commutation relations concerning the Lie symmetries derived in Subsection 2.2 of Chapter 6. The system under consideration in this case is the spectral problem of the multi-component NLS equation in $2+1$ dimensions (6.152), described in (6.153). Lie symmetries for this system, displayed in (6.156), depend on a set of nine arbitrary functions of the different independent variables and a solely arbitrary constant and its complex conjugate. The following convention in terms of notation will be assumed:

- $K_i(t)$, $H_j(t)$, $i, j = 1, \ldots, 3$ represent arbitrary real functions of the variable t, and we shall denote $X_{\{K_i\}}^{[j]}$, $X_{\{H_j\}}^{[j]}$ as their associated infinitesimal generators.
- $C_k(y)$, $J_l(y)$, $k, l = 1, \ldots, 5$ are consider arbitrary functions of the spatial coordinate y, with generators $Y_{\{C_k\}}^{[l]}$, $Y_{\{J_l\}}^{[l]}$ respectively.
- $\delta(y, t)$, $\gamma(y, t)$ are arbitrary real functions of $\{y, t\}$ and their associated infinitesimal generators are $Z_{\{\delta\}}$, $Z_{\{\gamma\}}$.
- Finally, we have defined $\Lambda_{\{\lambda\}}$, $\overline{\Lambda}_{\{\lambda^\dagger\}}$ as the infinitesimal generators related to the arbitrary complex conjugate constants λ, $\lambda^\dagger$.

With this notation, we have the following eleven symmetry generators:

$$X_{\{K_1\}}^{[1]} = \frac{1}{6}x^3 \dddot{K}_1 \partial_m + ix^2 \ddot{K}_1 \left(\alpha_1 \partial_{\alpha_1} + \alpha_2 \partial_{\alpha_2} - \alpha_1^\dagger \partial_{\alpha_1^\dagger} - \alpha_2^\dagger \partial_{\alpha_2^\dagger} - \psi \partial_\psi + \psi^\dagger \partial_\psi^\dagger \right)$$

$$+ 2\dot{K}_1 \left(2x\partial_x - \alpha_1 \partial_{\alpha_1} - \alpha_2 \partial_{\alpha_2} - \alpha_1^\dagger \partial_{\alpha_1^\dagger} - \alpha_2^\dagger \partial_{\alpha_2^\dagger} - 2m\partial_m - \psi \partial_\psi - \psi^\dagger \partial_\psi^\dagger \right) + 8K_1 \partial_t,$$

$$X_{\{K_2\}}^{[2]} = \frac{1}{4}x^2 \ddot{K}_2 \partial_m + ix\dot{K}_2 \left(\alpha_1 \partial_{\alpha_1} + \alpha_2 \partial_{\alpha_2} - \alpha_1^\dagger \partial_{\alpha_1^\dagger} - \alpha_2^\dagger \partial_{\alpha_2^\dagger} - \psi \partial_\psi + \psi^\dagger \partial_\psi^\dagger \right) + 2K_2 \partial_x,$$

$$X_{\{K_3\}}^{[3]} = \frac{1}{2}x\dot{K}_3 \frac{\partial}{\partial m} + iK_3 \left(\alpha_1 \partial_{\alpha_1} + \alpha_2 \partial_{\alpha_2} - \alpha_1^\dagger \partial_{\alpha_1^\dagger} - \alpha_2^\dagger \partial_{\alpha_2^\dagger} - \psi \partial_\psi + \psi^\dagger \partial_\psi^\dagger \right),$$

$$Y_{\{C_1\}}^{[1]} = -C_1' \left(\alpha_1 \partial_{\alpha_1} + \alpha_2 \partial_{\alpha_2} + \alpha_1^\dagger \partial_{\alpha_1^\dagger} + \alpha_2^\dagger \partial_{\alpha_2^\dagger} + \chi \partial_\chi + \rho \partial_\rho + \chi^\dagger \partial_{\chi^\dagger} + \rho^\dagger \partial_{\rho^\dagger} \right) + 2C_1 \partial_y,$$

$$Y_{\{C_2\}}^{[2]} = iC_2 \left(\alpha_1 \partial_{\alpha_1} - \alpha_1^\dagger \partial_{\alpha_1^\dagger} + \chi \partial_\chi - \chi^\dagger \partial_{\chi^\dagger} \right),$$

$$Y_{\{C_3\}}^{[3]} = iC_3 \left(\alpha_2 \partial_{\alpha_2} - \alpha_2^\dagger \partial_{\alpha_2^\dagger} + \rho \partial_\rho - \rho^\dagger \partial_{\rho^\dagger} \right),$$

$$Y_{\{C_4\}}^{[4]} = C_4 \left(\alpha_2 \partial_{\alpha_1} - \alpha_1 \partial_{\alpha_2} + \alpha_2^\dagger \partial_{\alpha_1^\dagger} - \alpha_1^\dagger \partial_{\alpha_2^\dagger} + \rho \partial_\chi - \chi \partial_\rho + \rho^\dagger \partial_{\chi^\dagger} - \chi^\dagger \partial_{\rho^\dagger} \right),$$

$$Y_{\{C_5\}}^{[5]} = iC_5 \left(\alpha_2 \partial_{\alpha_1} + \alpha_1 \partial_{\alpha_2} - \alpha_2^\dagger \partial_{\alpha_1^\dagger} - \alpha_1^\dagger \partial_{\alpha_2^\dagger} + \rho \partial_\chi + \chi \partial_\rho - \rho^\dagger \partial_{\chi^\dagger} - \chi^\dagger \partial_{\rho^\dagger} \right),$$

$$Z_{\{\delta\}} = \delta \partial_m, \qquad \Lambda_{\{\lambda\}} = \psi \partial_\psi + \chi \partial_\chi + \rho \partial_\rho, \qquad \overline{\Lambda}_{\{\lambda^\dagger\}} = \psi^\dagger \partial_\psi^\dagger + \chi^\dagger \partial_{\chi^\dagger} + \rho^\dagger \partial_{\rho^\dagger},$$

where we have used the convention $\dot{} \equiv \frac{d}{dt}$, $' \equiv \frac{d}{dy}$. Then, the commutation relations among these operators may be performed, resulting in the following:

The Lie algebra associated to (6.156) decomposes as

$$\langle X^{[1]}_{\{K_1\}}, X^{[2]}_{\{K_2\}}, X^{[3]}_{\{K_3\}}, Y^{[1]}_{\{C_1\}}, Y^{[2]}_{\{C_2\}}, Y^{[3]}_{\{C_3\}}, Y^{[4]}_{\{C_4\}}, Y^{[5]}_{\{C_5\}}, Z_{\{\delta\}}\rangle \oplus \langle \Lambda_{\{\lambda\}}, \overline{\Lambda}_{\{\lambda\dagger\}}\rangle,$$

since the generators $\Lambda_{\{\lambda\}}$, $\overline{\Lambda}_{\{\lambda\dagger\}}$ commute with all other generators. Besides, $\langle \Lambda_{\{\lambda\}}, \overline{\Lambda}_{\{\lambda\dagger\}}\rangle$ is a two-dimensional Abelian Lie algebra. The remaining commutation relations are given by

	$X^{[1]}_{\{K_1\}}$	$X^{[2]}_{\{K_2\}}$	$X^{[3]}_{\{K_3\}}$	$Z_{\{\delta\}}$	$Y^{[1]}_{\{C_1\}}$	$Y^{[2]}_{\{C_2\}}$	$Y^{[3]}_{\{C_3\}}$	$Y^{[4]}_{\{C_4\}}$	$Y^{[5]}_{\{C_5\}}$
$X^{[1]}_{\{H_1\}}$	$X^{[1]}_{\{8H_1\check{K}_1-8K_1\check{H}_1\}}$	$X^{[2]}_{\{8H_1\check{K}_2-4K_2\check{H}_1\}}$	$X^{[3]}_{\{8H_1\check{K}_3\}}$	$Z_{\{8H_1\partial_t(\delta)+4\delta\check{H}_1\}}$	0	0	0	0	0
$X^{[2]}_{\{H_2\}}$	$-X^{[2]}_{\{8K_1\check{H}_2-4H_2\check{K}_1\}}$	$X^{[3]}_{\{2H_2\check{K}_2-2K_2\check{H}_2\}}$	$Z_{\{H_2\check{K}_3\}}$	0	0	0	0	0	0
$X^{[3]}_{\{H_3\}}$	$-X^{[3]}_{\{8K_1\check{H}_3\}}$	$-Z_{\{K_2\check{H}_3\}}$	0	0	0	0	0	0	0
$Z_{\{\gamma\}}$	$-Z_{\{8K_1\partial_t(\gamma)+4\gamma\check{K}_1\}}$	0	0	0	$-Z_{\{2C_1\partial_\theta(\gamma)\}}$	0	0	0	0
$Y^{[1]}_{\{J_1\}}$	0	0	0	$Z_{\{2J_1\partial_\theta(\delta)\}}$	$Y^{[1]}_{\{2J_1C_1'-2C_1J_1'\}}$	$Y^{[2]}_{\{2J_1C_2\}}$	$Y^{[3]}_{\{2J_1C_3\}}$	$Y^{[4]}_{\{2J_1C_4\}}$	$Y^{[5]}_{\{2J_1C_5\}}$
$Y^{[2]}_{\{J_2\}}$	0	0	0	0	$-Y^{[2]}_{\{2C_1J_2\}}$	0	0	$-Y^{[5]}_{\{J_2C_4\}}$	$Y^{[4]}_{\{J_2C_5\}}$
$Y^{[3]}_{\{J_3\}}$	0	0	0	0	$-Y^{[3]}_{\{2C_1J_3\}}$	0	0	$Y^{[5]}_{\{J_3C_4\}}$	$-Y^{[4]}_{\{J_3C_5\}}$
$Y^{[4]}_{\{J_4\}}$	0	0	0	0	$-Y^{[4]}_{\{2C_1J_4\}}$	$Y^{[5]}_{\{C_2J_4\}}$	$-Y^{[5]}_{\{C_3J_4\}}$	0	$-Y^{[2]}_{\{2J_4C_5\}}+Y^{[3]}_{\{2J_4C_5\}}$
$Y^{[5]}_{\{J_5\}}$	0	0	0	0	$-Y^{[5]}_{\{2C_1J_5\}}$	$-Y^{[4]}_{\{C_2J_5\}}$	$Y^{[4]}_{\{C_3J_5\}}$	$Y^{[2]}_{\{2C_4J_5\}}-Y^{[3]}_{\{2C_4J_5\}}$	0

Table B.5: *Commutation relations for (6.156)*

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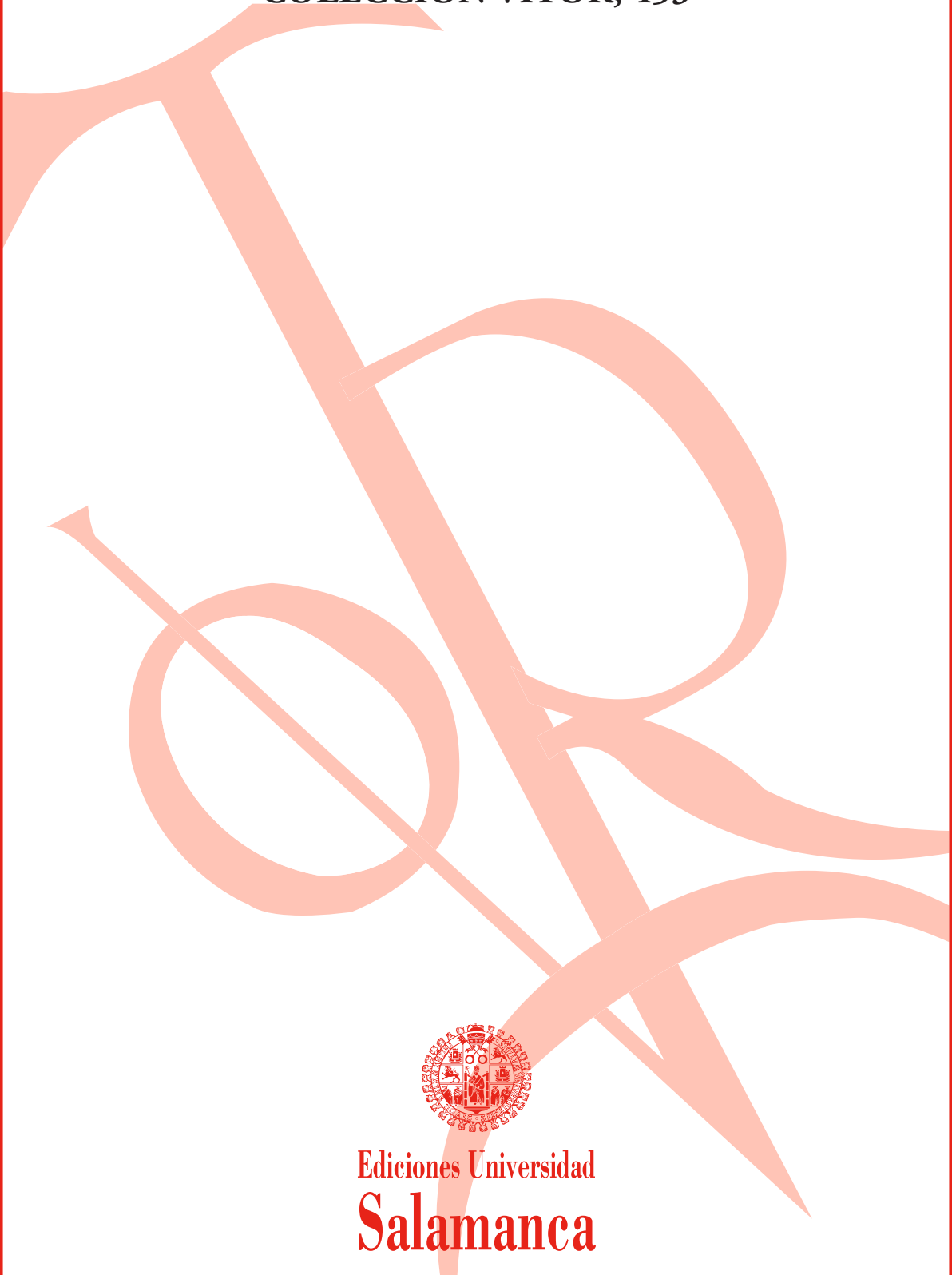
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