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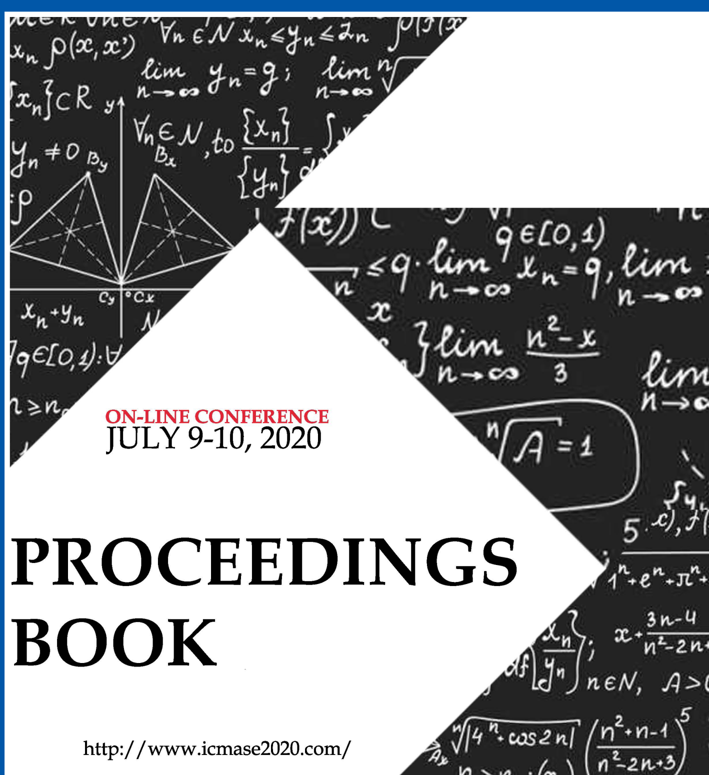
AND MELEK SOFYALIOĞLU (EDS.)

INTERNATIONAL CONFERENCE ON MATHEMATICS AND ITS APPLICATIONS IN SCIENCE AND ENGINEERING (ICMASE 2020)



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Preface

This proceedings book includes the papers that have been presented at International Conference on Mathematics and its Applications in Science and Engineering (ICMASE 2020) which was organized by Ankara Hacı Bayram Veli University, Turkey, between 9-10 July, 2020, via online because of the coronavirus pandemic COVID-19. Hopefully, this pandemic, which affects the whole world, will lose its effect as soon as possible and we will go back to the days before it.

The aim of this conference is to exchange ideas, discuss new developments in mathematics, promote collaborations and interact with professionals and researchers from all over the world in the following interesting topics: Functional Analysis, Approximation Theory, Real Analysis, Complex Analysis, Harmonic and non-Harmonic Analysis, Applied Analysis, Numerical Analysis, Geometry, Topology and Algebra, Modern Methods in Summability and Approximation, Operator Theory, Fixed Point Theory and Applications, Sequence Spaces and Matrix Transformation, Spectral Theory and Differential Operators, Boundary Value Problems, Ordinary and Partial Differential Equations, Discontinuous Differential Equations, Convex Analysis and its Applications, Optimization and its Application, Mathematics Education, Application on Variable Exponent Lebesgue Spaces, Applications on Differential Equations and Partial Differential Equations, Fourier Analysis, Wavelet and Harmonic Analysis Methods in Function Spaces, Applications on Computer Engineering, and Flow Dynamics.

Many thanks to all committee members.

We wish everyone a fruitful conference and pleasant memories from ICMASE 2020.

Fatih YILMAZ

Chairman ICMASE 2020

ON COMPUTING AN ARBITRARY SINGULAR VALUE OF A TENSOR SUM

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ABSTRACT

We consider computing a arbitrary singular value of a tensor sum: $T := I_n \otimes I_m \otimes A + I_n \otimes B \otimes I_\ell + C \otimes I_m \otimes I_\ell \in \mathbb{R}^{\ell mn \times \ell mn}$, where $A \in \mathbb{R}^{\ell \times \ell}$, $B \in \mathbb{R}^{m \times m}$, $C \in \mathbb{R}^{n \times n}$, I_n is the $n \times n$ identity matrix, and the symbol "$\otimes$" denotes the Kronecker product. The tensor sum T arises from a finite difference discretization of three-dimensional constant coefficient partial differential equations. The methods to compute the maximum/minimum singular values of T were provided.

To compute an arbitrary singular value of T, we focus on the shift-and-invert Lanczos method, which solves a shift-and-invert eigenvalue problem of $(T^{\mathrm{T}}T - \tilde{\sigma}^2 I_{\ell mn})^{-1}$, where $\tilde{\sigma}$ is set to the nearby the desired singular value. The desired singular value is computed by the maximum eigenvalue of the eigenvalue problem. This shift-and-invert Lanczos method needs to solve large-scale linear systems with the coefficient matrix $T^{\mathrm{T}}T - \tilde{\sigma}^2 I_{\ell mn}$. Since the direct methods cannot be applied due to the nonzero structure of the coefficient matrix, the preconditioned conjugate gradient (PCG) method is applied. However, it is difficult in terms of memory requirements to simply implement this shift-and-invert Lanczos method and the PCG method since the size of T grows rapidly by the sizes of A, B, and C.

In this paper, we present the following two techniques: 1) efficient implementations of the shift-and-invert Lanczos method for the eigenvalue problem of $T^{\mathrm{T}}T$ and the PCG method for $T^{\mathrm{T}}T - \tilde{\sigma}^2 I_{\ell mn}$ using three-dimensional arrays (third-order tensors) and the n-mode products; 2) a preconditioning matrix based on a structure of T for faster convergence of the PCG method. Finally, we show the effectiveness of the proposed method through numerical experiments.

Keywords Shift-and-invert Lanczos method · tensor sum · singular value

1 Introduction

We consider computing an arbitrary singular value of a tensor sum:

$$T := I_n \otimes I_m \otimes A + I_n \otimes B \otimes I_\ell + C \otimes I_m \otimes I_\ell \in \mathbb{R}^{\ell mn \times \ell mn}, \tag{1}$$

where $A \in \mathbb{R}^{\ell \times \ell}$, $B \in \mathbb{R}^{m \times m}$, $C \in \mathbb{R}^{n \times n}$, I_n is the $n \times n$ identity matrix, and the symbol "$\otimes$" denotes the Kronecker product. The tensor sum T arises from a finite difference discretization of

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three-dimensional constant coefficient partial differential equations. Matrix T tends to be too large even if A, B and C are not too large. Hence it is difficult to compute singular values of T with regard to the memory requirement.

In the previous work, the methods to compute the maximum and minimum singular values of T were provided in [3, 4]. These methods were based on the Lanczos bidiagonalization method (see, e.g., [1]), which computes the maximum and minimum singular values of a matrix. To reduce the memory requirement, the Lanczos bidiagonalization method for T was implemented using tensors and their operations.

The Lanczos method with the shift-and-invert technique, see, e.g., [1], is widely known for computing an arbitrary eigenvalue λ of a symmetric matrix $M \in \mathbb{R}^{n\times n}$. This method solves the shift-and-invert eigenvalue problem: $(M - \tilde{\sigma} I_n)^{-1}\boldsymbol{x} = (\lambda - \tilde{\sigma})^{-1}\boldsymbol{x}$, where $\boldsymbol{x}$ is the eigenvector of M corresponding to λ, and $\tilde{\sigma}$ is a shift point which is set to the nearby λ ($\tilde{\sigma} \neq \lambda$). Since the eigenvalue problem has the eigenvalue $(\lambda - \tilde{\sigma})^{-1}$ as the maximum eigenvalue, the method is effective for computing the desired eigenvalue λ near $\tilde{\sigma}$.

Therefore, we obtain a method of computing an arbitrary singular value of T based on the shift-and-invert Lanczos method. The method solves a shift-and-invert eigenvalue problem: $(T^{\mathrm{T}}T - \tilde{\sigma}^2 I_{\ell mn})^{-1}\boldsymbol{x} = (\sigma^2 - \tilde{\sigma}^2)^{-1}\boldsymbol{x}$, where σ is the desired singular value of T, $\boldsymbol{x}$ is the corresponding right-singular vector, and $\tilde{\sigma}$ is set to the nearby σ ($\tilde{\sigma} \neq \sigma$). This shift-and-invert Lanczos method needs to solve large-scale linear systems with the coefficient matrix $T^{\mathrm{T}}T - \tilde{\sigma}^2 I_{\ell mn}$. Since the direct methods cannot be applied due to the nonzero structure of the coefficient matrix, the preconditioned conjugate gradient (PCG) method, see, e.g., [1], is applied. However, it is difficult in terms of memory requirements to simply implement this shift-and-invert Lanczos method and the PCG method since the size of T grows rapidly by the sizes of A, B, and C.

In this paper, we present the following two techniques: 1) efficient implementations of the shift-and-invert Lanczos method for the eigenvalue problem of $T^{\mathrm{T}}T$ and the PCG method for $T^{\mathrm{T}}T - \tilde{\sigma}^2 I_{\ell mn}$ using three-dimensional arrays (third-order tensors) and the n-mode products, see, e.g., [2]; 2) a preconditioning matrix based on a structure of T for faster convergence of the PCG method. Finally, we show the effectiveness of the proposed method through numerical experiments.

2 Preliminaries of tensor operations

A tensor means a multidimensional array. Particularly, the third-order tensor $\boldsymbol{\mathcal{X}} \in \mathbb{R}^{I\times J\times K}$ plays an important role. In the rest of this section, the definitions of some tensor operations are shown. For more details, see, e.g., [2].

Firstly, a summation, a subtraction, an inner dot, and a norm for $\boldsymbol{\mathcal{X}}, \boldsymbol{\mathcal{Y}} \in \mathbb{R}^{I\times J\times K}$ are defined as follows:

$$(\boldsymbol{\mathcal{X}} \pm \boldsymbol{\mathcal{Y}})_{ijk} := \boldsymbol{\mathcal{X}}_{ijk} \pm \boldsymbol{\mathcal{Y}}_{ijk}, \quad (\boldsymbol{\mathcal{X}}, \boldsymbol{\mathcal{Y}}) := \sum_{i=1}^{I}\sum_{j=1}^{J}\sum_{k=1}^{K} \boldsymbol{\mathcal{X}}_{ijk}\boldsymbol{\mathcal{Y}}_{ijk}, \quad \|\boldsymbol{\mathcal{X}}\| = \sqrt{(\boldsymbol{\mathcal{X}}, \boldsymbol{\mathcal{X}})},$$

where $\boldsymbol{\mathcal{X}}_{ijk}$ denotes the (i, j, k) element of $\boldsymbol{\mathcal{X}}$. Secondly, the n-mode product of a tensor $\boldsymbol{\mathcal{X}} \in \mathbb{R}^{I_1\times I_2\times\cdots\times I_N}$ and a matrix $M \in \mathbb{R}^{J\times I_n}$ is defined as

$$(\boldsymbol{\mathcal{X}} \times_n M)_{i_1\ldots i_{n-1} j i_{n+1}\ldots i_N} = \sum_{i_n=1}^{I_n} \boldsymbol{\mathcal{X}}_{i_1 i_2\ldots i_N} M_{p i_n},$$

where $n \in \{1, 2, \dots N\}, i_k \in \{1, 2, \dots, I_k\}$ for $k = 1, 2, \dots N$, and $j \in \{1, 2, \dots, J\}$. Finally, vec and vec^{-1} operators are the following maps between a vector space $\mathbb{R}^{IJK}$ and a tensor space $\mathbb{R}^{I \times J \times K}$: $\mathrm{vec} : \mathbb{R}^{I \times J \times K} \to \mathbb{R}^{IJK}$ and $\mathrm{vec}^{-1} : \mathbb{R}^{IJK} \to \mathbb{R}^{I \times J \times K}$. vec operator can vectorize a tensor by combining all column vectors of the tensor into one long vector. vec^{-1} operator can reshape a tensor from one long vector.

3 Shift-and-invert Lanczos method for an arbitrary singular value over tensor space

This section gives an algorithm for computing an arbitrary singular value of the tensor sum T. Let σ and $\boldsymbol{x}$ be a desired singular value of T and the corresponding right singular vectors, respectively. Then, the eigenvalue problem of T is written by $T^{\mathrm{T}} T \boldsymbol{x} = \sigma^2 \boldsymbol{x}$. Here, introducing a shift $\tilde{\sigma} \approx \sigma$, the shift-invert eigenvalue problem is

$$(T^{\mathrm{T}} T - \tilde{\sigma}^2 I_{\ell mn})^{-1} \boldsymbol{x} = \frac{1}{\sigma^2 - \tilde{\sigma}^2} \boldsymbol{x}. \tag{2}$$

The shift-and-invert Lanczos method (see, e.g., [1]) computes the nearest singular value σ based on Eq. (2). Reconstructing this method over the $\ell \times m \times n$ tensor space, we obtain Algorithm 1 whose memory requirements is of $O(n^3)$ in the case of $n = m = \ell$.

Algorithm 1 Shift-and-invert Lanczos method for an arbitrary singular value over tensor space

1: Choose an initial tensor $\boldsymbol{\mathcal{Q}}_0 \in \mathbb{R}^{\ell \times m \times n}$;
2: $\boldsymbol{\mathcal{V}} := \boldsymbol{\mathcal{Q}}_0, \beta_0 := ||\boldsymbol{\mathcal{V}}||$;
3: **for** $k = 1, 2, \dots,$ until convergence **do**
4: $\quad \boldsymbol{\mathcal{Q}}_k := \boldsymbol{\mathcal{V}} / \beta_{k-1}$;
5: $\quad \boldsymbol{\mathcal{V}} := \mathrm{vec}^{-1} \left\{ \left(T^{\mathrm{T}} T - \tilde{\sigma} I_{\ell mn} \right)^{-1} \mathrm{vec}(\boldsymbol{\mathcal{Q}}_k) \right\}$;
6: $\quad \boldsymbol{\mathcal{V}} := \boldsymbol{\mathcal{V}} - \beta_{k-1} \boldsymbol{\mathcal{Q}}_{k-1}$;
7: $\quad \alpha_k := (\boldsymbol{\mathcal{Q}}_k, \boldsymbol{\mathcal{V}})$;
8: $\quad \boldsymbol{\mathcal{V}} := \boldsymbol{\mathcal{V}} - \alpha_k \boldsymbol{\mathcal{Q}}_k$;
9: $\quad \beta_k := ||\boldsymbol{\mathcal{V}}||$;
10: **end for**
11: Approximate singular value $\sigma = \sqrt{\tilde{\sigma}^2 + \frac{1}{\tilde{\lambda}^{(k)}}}$, where $\tilde{\lambda}^{(k)}$ is the maximum eigenvalue of $\tilde{T}_k$.

At step k, we have the following $\tilde{T}_k$ by Algorithm 1:

$$\tilde{T}_k := \begin{pmatrix} \alpha_1 & \beta_1 & & & \\ \beta_1 & \alpha_2 & \beta_2 & & \\ & \ddots & \ddots & \ddots & \\ & & \beta_{k-2} & \alpha_{k-1} & \beta_{k-1} \\ & & & \beta_{k-1} & \alpha_k \end{pmatrix} \in \mathbb{R}^{k \times k}.$$

To implement Algorithm 1, we need iteratively solve the system of linear equations

$$\boldsymbol{\mathcal{V}} := \mathrm{vec}^{-1} \left\{ \left(T^{\mathrm{T}} T - \tilde{\sigma} I_{\ell mn} \right)^{-1} \mathrm{vec}(\boldsymbol{\mathcal{Q}}_k) \right\},$$

whose coefficient matrix is $\ell mn \times \ell mn$, that is, the memory requirements is $O(n^6)$ in the case of $n = m = \ell$. We consider solving the system of linear equations in the next section.

4 Preconditioned conjugate gradient (PCG) method over tensor space

In this section, an efficient solver of $\boldsymbol{\mathcal{V}} := \mathrm{vec}^{-1}\left\{\left(T^{\mathrm{T}}T - \tilde{\sigma} I_{\ell mn}\right)^{-1} \mathrm{vec}(\boldsymbol{\mathcal{Q}}_k)\right\}$ using tensors is provided. This linear system is rewritten by $\boldsymbol{v} = \left(T^{\mathrm{T}}T - \tilde{\sigma}^2 I_{\ell mn}\right)^{-1} \boldsymbol{q}_k$, where $\boldsymbol{v} := \mathrm{vec}(\boldsymbol{\mathcal{V}})$ and $\boldsymbol{q}_k = \mathrm{vec}(\boldsymbol{\mathcal{Q}}_k)$. Then we solve $\left(T^{\mathrm{T}}T - \tilde{\sigma}^2 I_{\ell mn}\right)\boldsymbol{v} = \boldsymbol{q}_k$, where $\boldsymbol{v}$ and $\boldsymbol{q}_k$ are unknown and known vectors. Since the coefficient matrix is symmetric positive definite, we can use the preconditioned conjugate gradient method (PCG method, see, e.g., [5]), which is one of the widely used solvers. However, it is difficult to simply apply due to the nonzero structure of the coefficient matrix $T^{\mathrm{T}}T - \tilde{\sigma}^2 I_{\ell mn}$.

In order to apply the PCG method, we transform the linear system using the complex Schur decomposition of T as follows. Firstly, T is decomposed into $T := QRQ^{\mathrm{H}}$, where R and Q are upper triangular and unitary matrices, respectively. Then,

$$\begin{aligned}
\left(T^{\mathrm{T}}T - \tilde{\sigma}^2 I_{\ell mn}\right)\boldsymbol{v} = \boldsymbol{q}_k &\Leftrightarrow \left((QRQ^{\mathrm{H}})^{\mathrm{H}}(QRQ^{\mathrm{T}}) - \tilde{\sigma}^2 I_{\ell mn}\right)\boldsymbol{v} = \boldsymbol{q}_k \\
&\Leftrightarrow \left(QR^{\mathrm{H}}RQ^{\mathrm{H}} - \tilde{\sigma}^2 I_{\ell mn}\right)\boldsymbol{v} = \boldsymbol{q}_k \\
&\Leftrightarrow \left(R^{\mathrm{H}}R - \tilde{\sigma}^2 I_{\ell mn}\right)(Q^{\mathrm{H}}\boldsymbol{v}) = Q^{\mathrm{H}}\boldsymbol{q}_k.
\end{aligned}$$

Here, let R_A, R_B, R_C and Q_A, Q_B, Q_C are obtained from the complex Schur decomposition of A, B, C, respectively. Then we have $R = I_n \otimes I_m \otimes R_A + I_n \otimes R_B \otimes I_\ell + R_C \otimes I_m \otimes I_\ell$ and $Q = Q_C \otimes Q_B \otimes Q_A$ from the definition of T. We denote the linear system $\tilde{A}\tilde{\boldsymbol{y}} = \tilde{\boldsymbol{b}}$, where $\tilde{A} := R^{\mathrm{H}}R - \tilde{\sigma}^2 I_{\ell mn}$, $\tilde{\boldsymbol{y}} := Q^{\mathrm{H}}\boldsymbol{v}$, and $\tilde{\boldsymbol{b}} := Q^{\mathrm{H}}\boldsymbol{q}_k$.

4.1 Preconditioning matrix

When we solve $\tilde{A}\tilde{\boldsymbol{y}} = \tilde{\boldsymbol{b}}$ using the PCG method, the linear system is transformed into

$$\left(M^{-1}\tilde{A}M^{-\mathrm{H}}\right)\left(M^{-\mathrm{H}}\tilde{\boldsymbol{y}}\right) = M^{-1}\tilde{\boldsymbol{b}}, \tag{3}$$

where $M \in \mathbb{R}^{\ell mn \times \ell mn}$ is a preconditioning matrix. M must satisfy the following conditions: 1) a condition number of $(M^{-1}\tilde{A})$ is closed to 1; 2) the matrix-vector multiplication for M^{-1} is easily computed.

Therefore, we propose a preconditioning matrix

$$M := \bar{D}_R D_R - \tilde{\sigma}^2 I_{\ell mn},$$

where D_R is a diagonal matrix with diagonals of R. Since M is also the diagonal matrix, the above second conditions are satisfied. Moreover, if T is symmetric, R is a diagonal matrix, that is, $R = D_R$. Therefore $M = \tilde{A}$ in the case of the symmetric matrix T. From this, if T is not symmetric but almost symmetric, we expect that the preconditioning matrix M is effective.

4.2 Computations of $\mathrm{vec}^{-1}\left(\tilde{A}\mathrm{vec}(\boldsymbol{\mathcal{P}}_{k'})\right)$ and $\mathrm{vec}^{-1}\left(M^{-1}\mathrm{vec}(\boldsymbol{\mathcal{R}}_{k'})\right)$

We show the computations of $\mathrm{vec}^{-1}\left(\tilde{A}\mathrm{vec}(\boldsymbol{\mathcal{P}}_{k'})\right)$ and $\mathrm{vec}^{-1}\left(M^{-1}\mathrm{vec}(\boldsymbol{\mathcal{R}}_{k'})\right)$, which are required in the PCG method, using the 1, 2, and 3-mode products for tensors and the definition of T. First, from the definitions of $\tilde{A}$ and R, we have $\mathrm{vec}^{-1}\left(\tilde{A}\mathrm{vec}(\boldsymbol{\mathcal{P}}_{k'})\right) =$

$\mathrm{vec}^{-1}\left(R^{\mathrm{H}}\left(R\mathrm{vec}(\boldsymbol{\mathcal{P}}_{k'})\right) - \tilde{\sigma}^2\mathrm{vec}(\boldsymbol{\mathcal{P}}_{k'})\right)$. Therefore,

$$\begin{aligned}
\check{\boldsymbol{\mathcal{P}}} &= R\mathrm{vec}(\boldsymbol{\mathcal{P}}_{k'}) = \mathrm{vec}(\boldsymbol{\mathcal{P}}_{k'} \times_1 R_A + \boldsymbol{\mathcal{P}}_{k'} \times_2 R_B + \boldsymbol{\mathcal{P}}_{k'} \times_3 R_C),\\
R^{\mathrm{H}}\left(R\mathrm{vec}(\boldsymbol{\mathcal{P}}_{k'})\right) &= \mathrm{vec}(\check{\boldsymbol{\mathcal{P}}} \times_1 R_A + \check{\boldsymbol{\mathcal{P}}} \times_2 R_B + \check{\boldsymbol{\mathcal{P}}} \times_3 R_C),\\
\mathrm{vec}^{-1}\left(\tilde{A}\mathrm{vec}(\boldsymbol{\mathcal{P}}_{k'})\right) &= \check{\boldsymbol{\mathcal{P}}} \times_1 R_A + \check{\boldsymbol{\mathcal{P}}} \times_2 R_B + \check{\boldsymbol{\mathcal{P}}} \times_3 R_C - \tilde{\sigma}^2\boldsymbol{\mathcal{P}}_{k'}.
\end{aligned}$$

Next, from $M = \bar{D}_R D_R - \tilde{\sigma}^2 I_{\ell mn}$, we easily obtain

$$\left(M^{-1}\right)_{i,i} = \frac{1}{(\bar{D}_R)_{i,i}(D_R)_{i,i} - \tilde{\sigma}^2}, \qquad i = 1, 2, \dots, \ell mn.$$

Here, let $\boldsymbol{\mathcal{D}} = \mathrm{vec}^{-1}(\mathrm{diag}(D_R))$ and $\boldsymbol{\mathcal{M}} = \mathrm{vec}^{-1}(\mathrm{diag}(M^{-1}))$, where $\mathrm{diag}(D_R)$ returns an ℓmn-dimensional column vector with diagonals of D_R. Then $\boldsymbol{\mathcal{M}}_{ijk} = 1/(\bar{\boldsymbol{\mathcal{D}}}_{ijk}\boldsymbol{\mathcal{D}}_{ijk} - \tilde{\sigma}^2)$, where $\boldsymbol{\mathcal{D}}_{ijk} = (R_A)_{i,i} + (R_B)_{j,j} + (R_C)_{k,k}$. We compute $\mathrm{vec}^{-1}\left(M^{-1}\mathrm{vec}(\boldsymbol{\mathcal{R}}_{k'})\right)$ by

$$\mathrm{vec}^{-1}\left(M^{-1}\mathrm{vec}(\boldsymbol{\mathcal{R}}_{k'})\right) = \boldsymbol{\mathcal{M}} * \boldsymbol{\mathcal{R}}_{k'},$$

where "$*$" denotes elementwise product.

As shown in Algorithm 2, the PCG method can be implemented using the preconditioning matrix M and the aforementioned computations, where the linear system $\tilde{A}\tilde{y} = \tilde{b}$ is transformed into $\tilde{A}\ \mathrm{vec}(\tilde{\boldsymbol{\mathcal{Y}}}) = \mathrm{vec}(\tilde{\boldsymbol{\mathcal{B}}})$, where $\mathrm{vec}(\tilde{\boldsymbol{\mathcal{B}}}) := \tilde{b} = \mathrm{vec}(\boldsymbol{\mathcal{Q}}_k \times_1 Q_A + \boldsymbol{\mathcal{Q}}_k \times_2 Q_B + \boldsymbol{\mathcal{Q}}_k \times_3 Q_C)$ and $\mathrm{vec}(\tilde{\boldsymbol{\mathcal{Y}}}) := \tilde{y}$. Algorithm 2 requires only small matrices A, B, and C and $\ell \times m \times n$ tensors $\boldsymbol{\mathcal{X}}_{k'}, \boldsymbol{\mathcal{R}}_{k'}, \boldsymbol{\mathcal{P}}_{k'}$, and $\boldsymbol{\mathcal{Z}}_{k'}$. Therefore the memory requirements is of $O(n^3)$ in the case of $n = \ell = m$.

Algorithm 2 PCG method over tensor space for the 5-th line of Algorithm 1 [Proposed inner algorithm]

Choose an initial tensor $\boldsymbol{\mathcal{X}}_0 \in \mathbb{R}^{\ell\times m\times n}$ and $\boldsymbol{\mathcal{P}}_0 = O \in \mathbb{R}^{\ell\times m\times n}$, and an initial scalar $\beta_0 = 0$;
$\boldsymbol{\mathcal{R}}_0 = (\boldsymbol{\mathcal{Q}}_k \times_1 Q_A + \boldsymbol{\mathcal{Q}}_k \times_2 Q_B + \boldsymbol{\mathcal{Q}}_k \times_3 Q_C) - \mathrm{vec}^{-1}\left(\tilde{A}\mathrm{vec}(\boldsymbol{\mathcal{X}}_0)\right)$;
$\boldsymbol{\mathcal{Z}}_0 = \mathrm{vec}^{-1}\left(M^{-1}\mathrm{vec}(\boldsymbol{\mathcal{R}}_0)\right)$;
for $k' = 1, 2, \dots,$ until convergence **do**
  $\boldsymbol{\mathcal{P}}_{k'} = \boldsymbol{\mathcal{Z}}_{k'-1} + \beta_{k'-1}\boldsymbol{\mathcal{P}}_{k'-1}$;
  $\hat{\boldsymbol{\mathcal{P}}}_{k'} = \mathrm{vec}^{-1}\left(\tilde{A}\mathrm{vec}(\boldsymbol{\mathcal{P}}_{k'})\right)$;
  $\alpha_{k'} = (\boldsymbol{\mathcal{Z}}_{k'-1}, \boldsymbol{\mathcal{R}}_{k'-1}) / \left(\boldsymbol{\mathcal{P}}_{k'-1}, \hat{\boldsymbol{\mathcal{P}}}_{k'}\right)$;
  $\boldsymbol{\mathcal{X}}_{k'} = \boldsymbol{\mathcal{X}}_{k'-1} + \alpha_{k'}\boldsymbol{\mathcal{P}}_{k'}$;
  $\boldsymbol{\mathcal{R}}_{k'} = \boldsymbol{\mathcal{R}}_{k'-1} - \alpha_{k'}\hat{\boldsymbol{\mathcal{P}}}_{k'}$;
  $\boldsymbol{\mathcal{Z}}_{k'} = \mathrm{vec}^{-1}\left(M^{-1}\mathrm{vec}(\boldsymbol{\mathcal{R}}_{k'})\right)$;
  $\beta_{k'} = (\boldsymbol{\mathcal{Z}}_{k'}, \boldsymbol{\mathcal{R}}_{k'}) / (\boldsymbol{\mathcal{Z}}_{k'-1}, \boldsymbol{\mathcal{R}}_{k'-1})$;
end for
Approximate solution $\boldsymbol{\mathcal{V}} = \mathrm{vec}^{-1}(Q\mathrm{vec}(\boldsymbol{\mathcal{X}}_{k'}))$;

5 Numerical experiments

This section gives results of numerical experiments using Algorithms 1 and 2. For comparison, the results of those using Algorithm 1 with Algorithm 2 in the case of $M = I$ are also given. All computations were carried out using MATLAB R2018a version on a workstation with two Xeon processors (2.4GHz).

In the next subsection, all the initial guesses are tensors with random numbers. The stopping criteria of Algorithms 1 and 2 we used was $\beta_k||\boldsymbol{e}_k^{\mathrm{T}}\boldsymbol{s}_{\mathrm{MAX}}^k|| < 10^{-8}$ and $||\boldsymbol{\mathcal{R}}_{k'}||/||\tilde{\boldsymbol{\mathcal{B}}}|| < 10^{-12}$, where $\boldsymbol{s}^k_{\mathrm{MAX}}$ is eigenvector corresponding to the maximum eigenvalue of $\tilde{T}_k$. Test matrices T in (1) are obtained from a seven-point central difference discretization of the PDE in over an $(n+1)\times(n+1)\times(n+1)$ grid. The test matrices T in (1) are generated from

$$A = B = C, \quad A := \frac{1}{h^2}aM_1 + \frac{1}{2h}bM_2 + \frac{1}{3}cI_n, \tag{4}$$

where $h = 1/(n+1)$, M_1 and M_2 are symmetric and skew-symmetric matrices given below.

$$M_1 = \begin{pmatrix} -2 & 1 & & & \\ 1 & -2 & 1 & & \\ & \ddots & \ddots & \ddots & \\ & & 1 & -2 & 1 \\ & & & 1 & -2 \end{pmatrix} \in \mathbb{R}^{n\times n}, M_2 = \begin{pmatrix} 0 & 1 & & & \\ -1 & 0 & 1 & & \\ & \ddots & \ddots & \ddots & \\ & & -1 & 0 & 1 \\ & & & -1 & 0 \end{pmatrix} \in \mathbb{R}^{n\times n}.$$

We set $\tilde{\sigma} = \sigma - 10^{-2}$, where σ are singular values computed by the `svd` function in MATLAB.

5.1 Numerical results

In tables, the number of iterations of Lanczos method and the average of the number of iterations of CG or PCG method are summarized.

We show the first results in the case of almost symmetric case ($a = c = 1$ and $b = 0.01$). From Tables 1-3, the numbers of iterations of the PCG method were less than the number of iterations of the CG method. Particularly, in the case of $n = 15, 25$, the numbers of iterations of the PCG method decreased to less than 10% of the number of iterations of the CG method. It seems that the preconditioning matrix M is effective in the case of almost symmetric matrix T.

Table 1: Number of iterations in the case of the 5-th max. singular value of almost symmetric matrix T

Method		Comparison		Proposed	
		Lanczos	CG	Lanczos	PCG
n	5	6	55	6	15
	15	6	175	6	17
	25	6	318	6	17

Table 2: Number of iterations in the case of the median of singular value of almost symmetric matrix T

Method		Comparison		Proposed	
		Lanczos	CG	Lanczos	PCG
n	5	15	55	15	15
	15	39	175	39	17
	25	(CG did not converge)		48	17

Table 3: Number of iterations in the case of the 5-th min. singular value of almost symmetric matrix T

Method		Comparison		Proposed	
		Lanczos	CG	Lanczos	PCG
n	5	6	94	6	15
	15	6	3711	6	17
	25	6	2043	6	16

Next, we show the second results in the case of slightly symmetric case ($a = c = 1$ and $b = 0.1$). From Tables 4-6, when the PCG method converged, the numbers of iterations of the PCG method were less than the number of iterations of the CG method. However, there were cases not the PCG method to converge. Namely, it seems that the preconditioning matrix M can be effective in the case of a slightly symmetric matrix T.

Table 4: Number of iterations in the case of the 5-th max. singular value of slightly symmetric matrix T

Method		Comparison		Proposed	
		Lanczos	CG	Lanczos	PCG
n	5	5	63	(PCG did not converge)	
	15	5	203	(PCG did not converge)	
	25	5	347	(PCG did not converge)	

Table 5: Number of iterations in the case of the median of singular value of slightly symmetric matrix T

Method		Comparison		Proposed	
		Lanczos	CG	Lanczos	PCG
n	5	13	174	(PCG did not converge)	
	15	30	7960	32	77
	25	(CG did not converge)		42	89

Table 6: Number of iterations in the case of the 5-th min. singular value of slightly symmetric matrix T

Method		Comparison		Proposed	
		Lanczos	CG	Lanczos	PCG
n	5	5	1141	5	117
	15	5	6784	5	80
	25	5	2697	5	89

6 Conclusion

We considered computing an arbitrary singular value of a tensor sum. The shift-and-invert Lanczos method and the PCG method reconstructed over tensor space. We proposed the preconditioning matrix which is a diagonal matrix with diagonals of the upper diagonal matrix by the Schur decomposition. From numerical results, we confirmed that the proposed method reduces memory requirements under any conditions and the number of iterations of the PCG method by the proposed preconditioning matrix in the case of almost or slightly symmetric matrix.

For future work, we will consider a robust preconditioning matrix for symmetric tensor sum and a suitable preconditioning matrix for non-symmetric tensor sum.

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Cyclic Codes over the Ring $\mathbb{Z}_8 + u\mathbb{Z}_8 + v\mathbb{Z}_8$

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Abstract

In this paper, we study the cyclic codes over the ring $R = \mathbb{Z}_8 + u\mathbb{Z}_8 + v\mathbb{Z}_8$, where $u^2 = u$, $v^2 = v$, $uv = vu = 0$. We construct the generator polynomials of cyclic codes over R. Also, we introduce a Gray map from R^n to $\mathbb{Z}_8^{3n}$ and show that the Gray image of the cyclic code with odd length n is a quasi-cyclic code of index 3 and length $3n$ over $\mathbb{Z}_8$.

Keywords Cyclic codes · Quasi-cyclic code · Gray map

1 Introduction

The study of codes over finite rings is a topic of growing interest in coding theory. Linear codes over various finite rings have been studied in the last two decades [7, 8] . Cyclic codes form an important subclass of linear codes and they are important and useful in both practical and theoretical point of view. Because of such reasons lastly and recently many of researchers are interested to work on cyclic codes.

In [1], the authors discussed the construction of cyclic codes over the ring $\mathbb{Z}_2 + u\mathbb{Z}_2$. In [9] the authors studied the constacyclic codes over $\mathbb{Z}_4 + u\mathbb{Z}_4$ and constructed good binary codes from such codes. In [4] the linear codes over the ring $\mathbb{Z}_4 + u\mathbb{Z}_4 + v\mathbb{Z}_4$, where $u^2 = u$, $v^2 = v$, $uv = vu = 0$ were introduced. Also the Gray images of the cyclic, constacyclic and quasi-cyclic codes over $\mathbb{Z}_4 + u\mathbb{Z}_4 + v\mathbb{Z}_4$ were determined. In [3], the linear codes over the ring $\mathbb{Z}_8 + u\mathbb{Z}_8 + v\mathbb{Z}_8$, where $u^2 = u$, $v^2 = v$, $uv = vu = 0$ were introduced. The Lee weight of an element of R was defined.

In this paper, cyclic codes over the ring $R = \mathbb{Z}_8 + u\mathbb{Z}_8 + v\mathbb{Z}_8$, where $u^2 = u$, $v^2 = v$, $uv = vu = 0$ are studied and the Gray images of the cyclic codes over R are determined.

2 Linear Codes over $\mathbb{Z}_8 + u\mathbb{Z}_8 + v\mathbb{Z}_8$

The ring $R = \mathbb{Z}_8 + u\mathbb{Z}_8 + v\mathbb{Z}_8$ is a commutative, characteristic 8 ring with $u^2 = u$, $v^2 = v$, $uv = vu = 0$. It can be also viewed as the quotient ring $\mathbb{Z}_8[u, v]/\langle u^2 - u, v^2 - v, uv = vu\rangle$. Let d be any element of R, which can be expressed uniquely as $d = a + ub + vc$, where $a, b, c \in \mathbb{Z}_8$ [3].

Definition 2.1. *A linear code C of length n over the ring R is a R-submodule of R^n. A codeword is denoted as $\boldsymbol{d} = (d_0, d_1, \ldots, d_{n-1})$[3].*

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We define a cyclic shift operator as:

$$\sigma\left(d_0, d_1, \ldots, d_{n-1}\right) \quad = \quad \left(d_{n-1}, d_0, \ldots, d_{n-2}\right).$$

Let C be linear code of length n over R, then C is called cyclic if $\sigma(C) = C$.

In the case of $\mathbb{Z}_4 + u\mathbb{Z}_4 + v\mathbb{Z}_4$, a Gray map from $\mathbb{Z}_4 + u\mathbb{Z}_4 + v\mathbb{Z}_4$ to $\mathbb{Z}_4^3$ was defined by sending $a + bu + cv$ to $(a, a + b, a + c)$ with $a, b, c \in \mathbb{Z}_4$. A similar technique is adopted here.

We define the Gray map as follows

$$\Phi : R \to \mathbb{Z}_8^3$$
$$a + ub + vc \mapsto (a, a + b, a + c).$$

This map is extended componentwise to

$$\Phi : R^n \to \mathbb{Z}_8^{3n}$$

$$\left(d_0, d_1, \ldots, d_{n-1}\right) \mapsto \left(a_0, \ldots, a_{n-1}, a_0 + b_0, \ldots, a_{n-1} + b_{n-1}, a_0 + c_0, \ldots, a_{n-1} + c_{n-1}\right).$$

where $d_i = a_i + ub_i + vc_i$ with $i = 0, 1, \ldots, n-1$. Φ is a $\mathbb{Z}_8$-module isomorphism.

Theorem 2.1. *The Gray map $\Phi : R^n \to \mathbb{Z}_8^{3n}$ is a distance preserving linear isometry.*

Proof.
For every $\mathbf{x} = \left(x_0, x_1, \ldots, x_{n-1}\right),\ \mathbf{y} = \left(y_0, y_1, \ldots, y_{n-1}\right) \in R^n$, let $x_i = a_{x_i} + b_{x_i}u + c_{x_i}v$ and $y_i = a_{y_i} + b_{y_i}u + c_{y_i}v$, where $a_{x_i}, a_{y_i}, b_{x_i}, b_{y_i}, c_{x_i}$ and $c_{y_i} \in \mathbb{Z}_8$. Then

$$\begin{aligned}
d_L\left(\mathbf{x}, \mathbf{y}\right) &= w_L\left(\mathbf{x} - \mathbf{y}\right)\\
&= w_L\left(x_0 - y_0, \ldots, x_{n-1} - y_{n-1}\right)\\
&= w_L\left(a_{x_0} - a_{y_0} + (b_{x_0} - b_{y_0})u + (c_{x_0} - c_{y_0})v, \ldots\right.\\
&\quad \left., a_{x_{n-1}} - a_{y_{n-1}} + (b_{x_{n-1}} - b_{y_{n-1}})u + (c_{x_{n-1}} - c_{y_{n-1}})v\right)\\
&= w_L\left(a_{x_0} - a_{y_0}, \ldots, a_{x_{n-1}} - a_{y_{n-1}}\right.\\
&\quad , (a_{x_0} - a_{y_0}) + (b_{x_0} - b_{y_0}), \ldots, (a_{x_{n-1}} - a_{y_{n-1}}) + (b_{x_{n-1}} - b_{y_{n-1}})\\
&\quad \left., (a_{x_0} - a_{y_0}) + (c_{x_0} - c_{y_0}), \ldots, (a_{x_{n-1}} - a_{y_{n-1}}) + (c_{x_{n-1}} - c_{y_{n-1}})\right)
\end{aligned}$$

$$\begin{aligned}
d_L\left(\Phi(\mathbf{x}), \Phi(\mathbf{y})\right) &= w_L\left(\Phi(\mathbf{x}) - \Phi(\mathbf{y})\right)\\
&= w_L\left(\left(a_{x_0}, \ldots, a_{x_{n-1}}, a_{x_0} + b_{x_0}, \ldots, a_{x_{n-1}} + b_{x_{n-1}}, a_{x_0} + c_{x_0}, \ldots, a_{x_{n-1}} + c_{x_{n-1}}\right)\right.\\
&\quad \left. - \left(a_{y_0}, \ldots, a_{y_0}, a_{y_0} + b_{y_0}, \ldots, a_{y_{n-1}} + b_{y_{n-1}}, a_{y_0} + c_{y_0}, \ldots, a_{y_{n-1}} + c_{y_{n-1}}\right)\right.\\
&= w_L\left(a_{x_0} - a_{y_0}, \ldots, a_{x_{n-1}} - a_{y_{n-1}}\right.\\
&\quad , (a_{x_0} - a_{y_0}) + (b_{x_0} - b_{y_0}), \ldots, (a_{x_{n-1}} - a_{y_{n-1}}) + (b_{x_{n-1}} - b_{y_{n-1}})\\
&\quad \left., (a_{x_0} - a_{y_0}) + (c_{x_0} - c_{y_0}), \ldots, (a_{x_{n-1}} - a_{y_{n-1}}) + (c_{x_{n-1}} - c_{y_{n-1}})\right)
\end{aligned}$$

Therefore $d_L\left(\mathbf{x}, \mathbf{y}\right) = d_L\left(\Phi(\mathbf{x}), \Phi(\mathbf{y})\right)$.

Let $\mathbf{x} = \left(x_0, x_1, \ldots, x_{n-1}\right),\ \mathbf{y} = \left(y_0, y_1, \ldots, y_{n-1}\right)$ be two vectors in R^n. The inner product between $\mathbf{x}$ and $\mathbf{y}$ is defined as

$$\langle \mathbf{x}, \mathbf{y} \rangle = x_0y_0 + x_1y_1 + \ldots + x_{n-1}y_{n-1}$$

where the operation are performed in the ring R.

Definition 2.2. *Let C be a linear code over the ring R of length n, then we define the dual of C as*

$$C^{\perp} = \{\boldsymbol{y} \in R^n | \langle \boldsymbol{x}, \boldsymbol{y} \rangle = 0,\ forall\ \boldsymbol{x} \in C\}$$

Note that from the definition of inner product, it is clear that $C^{\perp}$ is also a linear code over R^n. A code C is said to be self-orthogonal if $C \subseteq C^{\perp}$, and self-dual if $C = C^{\perp}$.

Let C be a linear code of length n over R, we denote C_i $(1 \leqslant i \leqslant 3)$ as:

$$\begin{array}{rcl} C_1 & = & \{a \mid \exists\, b, c \in \mathbb{Z}_8^n, a + ub + vc \in C\} \\ C_2 & = & \{a + b \mid \exists\, c \in \mathbb{Z}_8^n, a + ub + vc \in C\} \\ C_3 & = & \{a + c \mid \exists\, b \in \mathbb{Z}_8^n, a + ub + vc \in C\} \end{array}$$

where C_1, C_2 and C_3 are linear codes over $\mathbb{Z}_8$ of length n. And C can be uniquely expressed as

$$C = (1 - u - v)C_1 + uC_2 + vC_3.$$

According to the direct sum decomposition in above, we have $|C| = |C_1|\,|C_2|\,|C_3|$ [3].

Let $M,\ N,\ K$ be three linear codes of length n over $\mathbb{Z}_8$, we define

$$M \otimes N \otimes K = \{(c_1, c_2, c_3) : c_1 \in M, c_2 \in N, c_3 \in K\}\,.$$

Thus we have the following.

Theorem 2.2. *Let $C = (1 - u - v)C_1 + uC_2 + vC_3$ be a linear code of length n over R. Then*

$$\Phi(C) = C_1 \otimes C_2 \otimes C_3\ and\ \Phi(C)^{\perp} = \Phi(C^{\perp}).$$

If C is a self-dual code, then $\Phi(C)$ is also a self-dual code.

Proof.
Clearly $C_1 \otimes C_2 \otimes C_3 \subseteq \Phi(C)$ and $|C_1 \otimes C_2 \otimes C_3| = |C_1|\,|C_2|\,|C_3|$. Then we have $\Phi(C) = C_1 \otimes C_2 \otimes C_3$. From [3, Theorem 1 (2)], we have $\Phi(C^{\perp}) = C_1^{\perp} \otimes C_2^{\perp} \otimes C_3^{\perp}$, hence $\left|\Phi(C^{\perp})\right| = \frac{8^n}{|C_1|}\frac{8^n}{|C_2|}\frac{8^n}{|C_3|} = \frac{8^{3n}}{|C|}$. Let $\mathbf{d} = (1 - u - v)\mathbf{a} + u\mathbf{b} + v\mathbf{c} \in C$ and $\mathbf{d}' = (1 - u - v)\mathbf{a}' + u\mathbf{b}' + v\mathbf{c}' \in C^{\perp}$, where $a \in C_1,\ b \in C_2,\ c \in C_3$ and $a' \in C_1^{\perp},\ b' \in C_2^{\perp},\ c' \in C_3^{\perp}$, then

$$\begin{array}{rcl} \Phi(d)\Phi(d') & = & (a, a + b, a + c)\,(a', a' + b', a' + c') \\ & = & 0 \end{array}$$

which means $\Phi(C)^{\perp} \supseteq \Phi(C^{\perp})$. Moreover, $\left|\Phi(C)^{\perp}\right| = \frac{8^{3n}}{|\Phi(C^{\perp})|}$, hence $\Phi(C)^{\perp} = \Phi(C^{\perp})$.

3 Cyclic Codes over $\mathbb{Z}_8 + u\mathbb{Z}_8 + v\mathbb{Z}_8$

Cyclic codes are a significant family of linear codes because of their natural encoding and decoding algorithm. In this section, we discuss cyclic codes over the ring R.

Theorem 3.1. *Let $C = (1 - u - v)C_1 + uC_2 + vC_3$ be a linear code of length n over R. Then C is a cyclic code over R if and only if one of following three conditions is satisfied:*

1. *C_i $(1 \leqslant i \leqslant 3)$ is a cyclic code over $\mathbb{Z}_8$.*
2. *$C_i^{\perp}$ $(1 \leqslant i \leqslant 3)$ is a cyclic code over $\mathbb{Z}_8$.*
3. *$C^{\perp}$ is a cyclic code over R.*

Proof.
For any $d_i = (d_{i0}, d_{i1}, \ldots, d_{i,n-1}) \in C_i$ $(1 \leqslant i \leqslant 3)$, then

$$\begin{aligned} d &= (1-u-v)d_1 + ud_2 + vd_3 \\ &= ((1-u-v)d_{10}, (1-u-v)d_{11}, \ldots, (1-u-v)d_{1,n-1}) \\ &\quad + (ud_{20}, ud_{21}, \ldots, ud_{2,n-1}) \\ &\quad + (vd_{30}, vd_{31}, \ldots, vd_{3,n-1}) \in C. \end{aligned}$$

Since C is a cyclic code, we have

$$\begin{aligned} \sigma(d) &= ((1-u-v)d_{1,n-1}, (1-u-v)d_{10}, \ldots, (1-u-v)d_{1,n-2}) \\ &\quad + (ud_{2,n-1}, ud_{20}, \ldots, ud_{2,n-2}) \\ &\quad + (vd_{3,n-1}, vd_{30}, \ldots, vd_{3,n-2}) \in C. \end{aligned}$$

Thus, C_i is a cyclic code over $\mathbb{Z}_8$. Vice versa. Since C_i is a cyclic code over $\mathbb{Z}_8$, we have $C_i^\perp$ is a cyclic code over $\mathbb{Z}_8$. From [3, Theorem 1], we have $C^\perp$ is a cyclic code over R. Moreover, C is a cyclic code over R.

Let **d** $= (d_0, \ldots, d_{n-1}) \in C$ which is equivalent to $d(x) = \sum_{i=0}^{n-1} d_i x^i$ under an isomorphic map, then C is a cyclic code if and only if C is an ideal of $R_n = R[x]/\langle x^n - 1\rangle$. When n is an odd integer, from [2] an ideal of the ring $R_n = R[x]/\langle x^n - 1\rangle$ can be written

$$\langle p(x) + 2q(x) + 4r(x)\rangle$$

where $r(x)|q(x)|p(x)|(x^n - 1)$.

Let $\tau : \mathbb{Z}_8[u] \to \mathbb{Z}_2[u]$ be the map which sends 0,2,4,6 to 0; 1,3,5,7 to 1, and x to x. A polynomial $f(x)$ in $\mathbb{Z}_8[x]$ is basic irreducible if $\tau(f(x))$ is irreducible in $\mathbb{Z}_2[x]$. When n is odd, then $x^n - 1$ is square-free over $\mathbb{Z}_8$. Therefore from [5, Proposition 2.7], $x^n - 1$ factors over $\mathbb{Z}_8$ uniquely as a product of monic basic irreducible pairwise coprime polynomials.

We will use the generator polynomial of C_1, C_2 and C_3 over $\mathbb{Z}_8$ to construct the generator polynomials of cyclic codes over R.

Theorem 3.2. *Let $C = (1-u-v)C_1 + uC_2 + vC_3$ be a cyclic code of odd length n over R, there exist $p_i(x), q_i(x), r_i(x) \in \mathbb{Z}_8[x]$ for $i = 1, 2, 3$ such that $C_i = \langle p_i(x) + 2q_i(x) + 4r_i(x)\rangle$, then*

$$\begin{aligned} C &= \langle (1-u-v)\left(p_1(x) + 2q_1(x) + 4r_1(x)\right) \\ &\quad + u\left(p_2(x) + 2q_2(x) + 4r_2(x)\right) + v\left(p_3(x) + 2q_3(x) + 4r_3(x)\right)\rangle. \end{aligned}$$

Proof.
Let

$$\begin{aligned} D &= \langle (1-u-v)\left(p_1(x) + 2q_1(x) + 4r_1(x)\right) \\ &\quad + u\left(p_2(x) + 2q_2(x) + 4r_2(x)\right) + v\left(p_3(x) + 2q_3(x) + 4r_3(x)\right)\rangle. \end{aligned}$$

For any $d(x) \in C$, there exist $k_i(x) \in \mathbb{Z}_8[x]$ $(1 \leqslant i \leqslant 3)$ such that

$$\begin{aligned} c(x) &= (1-u-v)\left(p_1(x)+2q_1(x)+4r_1(x)\right)k_1(x) \\ &\quad +u\left(p_2(x)+2q_2(x)+4r_2(x)\right)k_2(x)+v\left(p_3(x)+2q_3(x)+4r_3(x)\right)k_3(x). \\ &= \left[(1-u-v)k_1(x)+uk_2(x)+vk_3(x)\right]\left((1-u-v)\left(p_1(x)+2q_1(x)+4r_1(x)\right)\right. \\ &\quad \left.+u\left(p_2(x)+2q_2(x)+4r_2(x)\right)+v\left(p_3(x)+2q_3(x)+4r_3(x)\right)\right). \end{aligned}$$

then $C \subseteq D$. Obviously, $D \subseteq C$. Thus $C = D$.

4 Quasi-cyclic Codes over $\mathbb{Z}_8 + u\mathbb{Z}_8 + v\mathbb{Z}_8$

Quasi-cyclic codes are a natural generalization of cyclic codes. Many (record breaking) new codes are found from these codes. Many researchers worked on quasi-cyclic codes and have been able to discover new record breaking codes.

Definition 4.1. *Let σ be the cyclic shift over $\mathbb{Z}_8$. For any positive integer s, let σ_s be the quasi-shift given by $\sigma_s\left(e^{(1)}|e^{(2)}|\cdots|e^{(s)}\right) = \sigma(e^{(1)})|\sigma(e^{(2)})|\cdots|\sigma(e^{(s)})$, where $e^{(1)}, e^{(2)}, \ldots, e^{(s)} \in \mathbb{Z}_8^n$ and "$|$" denotes the usual vector concatenation. An octal quasi-cyclic code C of index s and length ns is a subset of $(\mathbb{Z}_8^n)^s$ such that $\sigma_s(C) = C$.*

Theorem 4.1. *Let $C = (1-u-v)C_1 + uC_2 + vC_3$ be a cyclic code of odd length n over R. Then $\Phi(C)$ is a quasi-cyclic code of index 3 and length $3n$ over $\mathbb{Z}_8$.*

Proof.
Let $e = (e_0, e_1, \ldots, e_{n-1}) \in C$ with $e_i = (1-u-v)a_i + ub_i + vc_i$ where $0 \leqslant i \leqslant n-1$.

$$\begin{aligned} \Phi(e) &= \Phi\left(e_0, e_1, \ldots, e_{n-1}\right) \\ &= \left(a_0, \ldots, a_{n-1}, a_0+b_0, \ldots, a_{n-1}+b_{n-1}, a_0+c_0, \ldots, a_{n-1}+c_{n-1}\right) \end{aligned}$$

Since C_1, C_2 and C_3 are cyclic codes. Then we have

$$\begin{aligned} &\Gamma\left(a_0, \ldots, a_{n-1}|a_0+b_0, \ldots, a_{n-1}+b_{n-1}|a_0+c_0, \ldots, a_{n-1}+c_{n-1}\right) \\ =\ &\left(\mu(a_0, \ldots, a_{n-1})|\mu(a_0+b_0, \ldots, a_{n-1}+b_{n-1})|\mu(a_0+c_0, \ldots, a_{n-1}+c_{n-1})\right) \in \Phi(C). \end{aligned}$$

Thus $\Phi(C)$ is a quasi-cyclic code of index 3 and length $3n$ over $\mathbb{Z}_8$.

An octa-linear code is a subgroup of $\mathbb{Z}_8^n$, and its order is a power of 2. So we can say the type of a octa-linear code generated by

$$G = \begin{pmatrix} I_{k_0} & A & B & T \\ 0 & 2I_{k_1} & 2D & 2E \\ 0 & 0 & 4I_{k_2} & 4F \end{pmatrix}.$$

is $8^{k_0}4^{k_1}2^{k_2}$ [6]. Here $\Phi(C)$ is a $\mathbb{Z}_8$-linear codes, then we have the following discussions.

Theorem 4.2. *Let C_i $(1 \leqslant i \leqslant 3)$ be a a cyclic code of n (n is odd) over $\mathbb{Z}_8$ and $C_i = \langle p_i(x) + 2q_i(x) + 4r_i(x)\rangle$, where $p_i(x), q_i(x)$ and $r_i(x)$ are the monic factors of $x^n - 1$ over $\mathbb{Z}_8$ and $r_i(x)|q_i(x)|p_i(x)|(x^n-1)$. Then the type of C_i $(1 \leqslant i \leqslant 3)$ is $8^{n-deg(p_i(x))}4^{deg(p_i(x))-deg(q_i(x))}2^{deg(q_i(x))-deg(r_i(x))}$.*

Proof.
For $C_i = \langle p_i(x) + 2q_i(x) + 4r_i(x)\rangle$ the type of C_i $(1 \leqslant i \leqslant 3)$ is $8^{k_{0,i}}4^{k_{1,i}}2^{k_{2,i}}$, we define a map

" $-$ " as $\mathbb{Z}_8 \to \mathbb{Z}_2$. Thus $k_{0,i} = dim(C_i) = dim\left(\overline{\langle p_i(x)\rangle}\right) = n - deg(p_i(x))$. We denote

$$\overline{(C_i:2)} = \left\{\overline{q_i(x)} \in \mathbb{Z}_2[x] : 2q_i(x) \in C_i\right\},$$

which is equal to $\overline{\langle q_i(x)\rangle}$, thus

$$k_{0,i} + k_{1,i} = n - deg(q_i(x)).$$

So

$$k_{1,i} = deg(p_i(x)) - deg(q_i(x)).$$

Also,

$$\overline{(C_i:4)} = \left\{\overline{r_i(x)} \in \mathbb{Z}_2[x] : 4r_i(x) \in C_i\right\},$$

which is equal to $\overline{\langle r_i(x)\rangle}$, thus

$$k_{0,i} + k_{1,i} + k_{2,i} = n - deg(r_i(x)).$$

So

$$k_{2,i} = deg(q_i(x)) - deg(r_i(x)).$$

Then the type of C_i $(1 \leqslant i \leqslant 3)$ $8^{n-deg(p_i(x))}4^{deg(p_i(x))-deg(q_i(x))}2^{deg(q_i(x))-deg(r_i(x))}$.

Corollary 4.1. *Let $\Phi(C) = C_1 \otimes C_2 \otimes C_3$ be a linear code of length $3n$ (n is odd) over $\mathbb{Z}_8$, where C_i $(1 \leqslant i \leqslant 3)$ is a cyclic code of n over $\mathbb{Z}_8$. Then the type of $\Phi(C)$ is*

$$8^{\sum_{i=1}^{3}(n-deg(p_i(x)))}4^{\sum_{i=1}^{3}(deg(p_i(x))-deg(q_i(x)))}2^{\sum_{i=1}^{3}(deg(q_i(x))-deg(r_i(x)))}.$$

Let $\Phi(C) = C_1 \otimes C_2 \otimes C_3$ be a linear code of length $3n$ (n is odd) over $\mathbb{Z}_8$ and d be the Lee distance of $\Phi(C)$. Then $d = min\{d_1, d_2, d_3\}$, where d_i $(1 \leqslant i \leqslant 3)$ is the Lee distance of C_i.

Example 4.1. *Consider a cyclic code over R of length 7. In $\mathbb{Z}_8[x]$, $x^7 - 1 = W_1(x)W_2(x)W_3(x)$, where*

$$\begin{aligned} W_1(x) &= x + 7 \\ W_2(x) &= x^3 + 3x^2 + 2x + 7 \\ W_3(x) &= x^3 + 6x^2 + 5x + 7 \end{aligned}$$

Let $C_1 = C_2 = \langle W_1(x)W_2(x) + 2W_2(x) + 4\rangle$ and $C_3 = \langle W_1(x)W_3(x) + 2W_3(x) + 4\rangle$ over $\mathbb{Z}_8$. By Theorem 3.2, we have C is a cyclic code and

$$C = \langle (1-v)\left(W_1(x)W_2(x) + 2W_2(x) + 4\right) + v\left(W_1(x)W_3(x) + 2W_3(x) + 4\right)\rangle$$

over R. Further, by Corollary 4.1 we have $\Phi(C)$ is a $(21, 8^9 4^3 2^9, 4)$ octa quasi-cyclic code. $|C_1|\left|C_1^{\perp}\right| = 8^n$.

5 Conclusion

In this paper, we study some properties of cyclic codes over the ring $R = \mathbb{Z}_8 + u\mathbb{Z}_8 + v\mathbb{Z}_8$, where $u^2 = u$, $v^2 = v$, $uv = vu = 0$. We define a Gray map from R^n to $\mathbb{Z}_8^{3n}$, which is a distance preserving map. It is shown that the Gray image of a cyclic code over R is a linear code over $\mathbb{Z}_8$.

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ASSESSMENT WITH MATHEMATICS COMPETENCIES-BASED RULES_MATH'S GUIDES FOR CALCULUS I

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ABSTRACT

In their professional routines, teachers perform a complex and demanding task in relation to being able to evaluate their students in all 8 mathematical competencies. These requires some training, counseling and help. In the RULES_MATH project we theoretically conceptualized and validated specific models for teachers' counseling in using diagnostic mathematical competence for the domain of student learning behavior, and constructed several instruments for their assessment. Subsequently, we developed specific training programs on counseling and diagnostics of the mathematical competencies that are or are not present in the evaluations of engineering students based on the "Guide for a problem" and evaluated by means of those specified instruments. In this paper, we describe the results of some applications of mathematical competencies in the assessment of Calculus I students and discuss future prospects for their inclusion throughout the teaching and learning process of mathematics in engineering.

Keywords Assessment · Significant Learning · Competencies · Mathematics · Engineering

1 Introduction

According to theoretical and applied evaluation of the literature, Competence-based learning is becoming much more widely used in engineering education, and there is evidence to support its effectiveness in improving learning outcomes, meeting the needs of diverse student populations, and responding to industry's demands for competent engineers [14]. The purpose of this paper is to illustrate that, when evaluating students, it is possible to capture a broader set of skills, attitudes, which go beyond the usual test scores. In the context of the RULES_MATH group we have been implementing a new approach to assess Engineering students based on the assessment of their mathematical competencies [9]. In this sense we intended to integrate the 8 mathematical competencies presented by Niss [2, 7] and developed according to Alpers in the Framework for Mathematical Curricula in Engineering Education [3], in the academic evaluation models. These 8 mathematical competencies are: (C1) thinking mathematically; (C2) reasoning mathematically; (C3) posing and solving mathematical problems; (C4) modelling mathematically; (C5) representing mathematical entities; (C6) handling mathematical symbols and formalism;

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(C7) communicating in, with, and about mathematics and (C8) make use of aids and tools for mathematical activity.

Assessment for learning, or formative assessment, is concerned with how to evoke information about learning and use it to modify teaching and learning activities [1, 8]. Assessment can be used both to evaluate student outcomes and to support student learning. Previous research has demonstrated that the new paradigma in evaluating engineering competencies proposed by RULES_MATH project [10, 11, 12, 13] is a methodology able to formative assessment and support the engineer students in their learning process. The questions proposed by RULES_MATH guides, covers most of the mathematical competencies, and the students acquire the competencies and knowledge that we intended to [5], and despite two particular questions, all students obtained positive grades. Thus, we can promote students' academic performance and well-being effectively and efficiently.

2 Methodology

This work pretended to study the mathematical competencies in assessment of Engineering students regarding the curricular unit of Calculus I. So, we used the "RULES_MATH guide for a problem" [4], a guide developed for different mathematical areas with examples/tests and competence-based activities proposed for all project partners. In our case we used the AC7 guide "Analysis and Calculus" developed by Ion Mierlus and Stefanie Constantinescu from Technical University of Civil Engineering in Bucharest (UTCB).

2.1 Participants

The participants of this study were students of Calculus I, in the first semester of 2019/2020 academic year. Eighty students from first year of Electrotechnical Engineering, fifty-nine regular students (only two females) and 21 student workers. Many of these students are native from higher technical courses in engineering where mathematics is taught in less depth and for that reason they have few mathematical bases.

2.2 Instruments

The instruments used are some exercises of AC7 guide [6]. Four questions about primitives techniques, represented in questions' type (1).

$$\begin{array}{ll} 1a) & \displaystyle\int e^x \sin x \, dx \\ 1e) & \displaystyle\int \cot x \, dx \\ 1g) & \displaystyle\int \tan x + \tan^3 x \, dx \\ 1h) & \displaystyle\int x^2 e^{x^3} \, dx \end{array} \tag{1}$$

These questions are associated with the 8 mathematical competencies (C1 to C8) described in Figure 1, where green represents that in that question to the evaluation of that competence is given high importance, yellow a medium importance and red is given less importance to the evaluation of that competence.

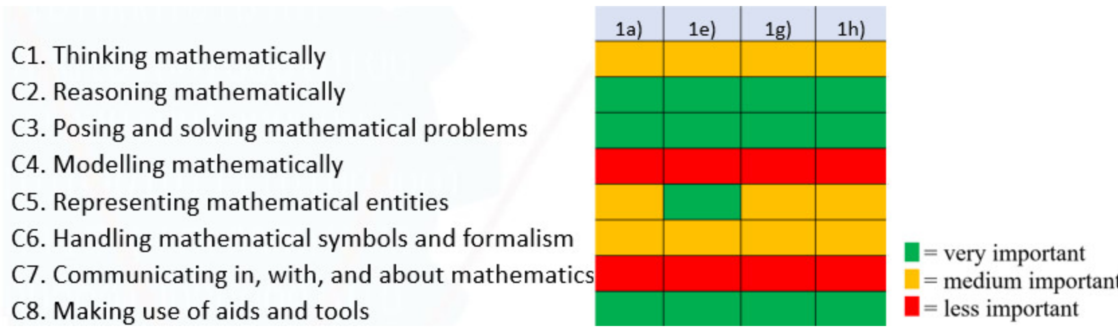

Figure 1: Mathematical competencies versus primitive techniques questions.

For example, to solve exercise $1e)$, the competencies C2, C3, C5 and C8 are very important, the competencies C1 and C6 are medium importance and the competencies C4 and C7 are the less important.

Two other questions about improper integrals, represented in questions' type (2), were also used in this competencies assessment.

$$\begin{array}{ll} 1o) & \displaystyle\int_0^{+\infty} \frac{1}{x^2+1}\,dx \\ 1q) & \displaystyle\int_1^{+\infty} \frac{1}{x(x^2+1)}\,dx \end{array} \tag{2}$$

In Figure 2 the mathematical competencies are represented for each of the questions about improper integrals in greater or lesser importance. These two questions are similar in relation to the mathematical competencies that the student must have and their order of importance. Thus C2, C3 and C8 competencies are the most important to be evaluated, C1, C5 and C6 of medium importance and C4 and C7 of less importance.

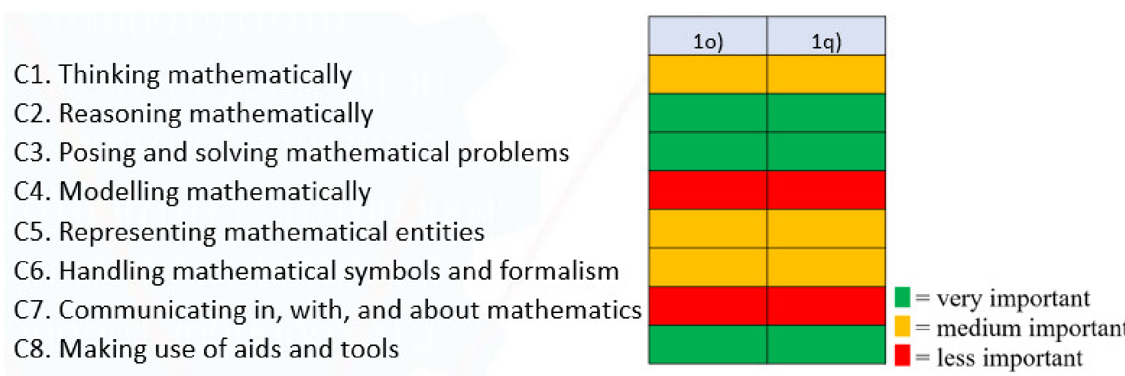

Figure 2: Mathematical competencies versus improper integrals questions.

The last 3 questions in the RULES_MATH guide that were used in this evaluation are about the application of the defined integral and are represented in questions' type (3). In this set of question, question 2) is about the application of integrals to the calculation of velocity, acceleration and space traveled by a vehicle, the question 3) is about the calculation of areas of flat figures and the question 4) is about the application of Simpson's rule in integral calculus.

2) Let $v(t)$be the velocity of the vehicle and $v(0) = 72km/h$.
When breaks are applied, $a = -5m/s^2$. How far did it travel?

(3)

3) Compute the area between the lines $y = 5x, x = 2$ and Ox axis.

5) Using one third Simpson's rule, approximate $\int_0^1 2x\,dx$ for $n = 6$.

The mathematical competencies associated with the last 3 questions are defined in the table of the Figure 3.

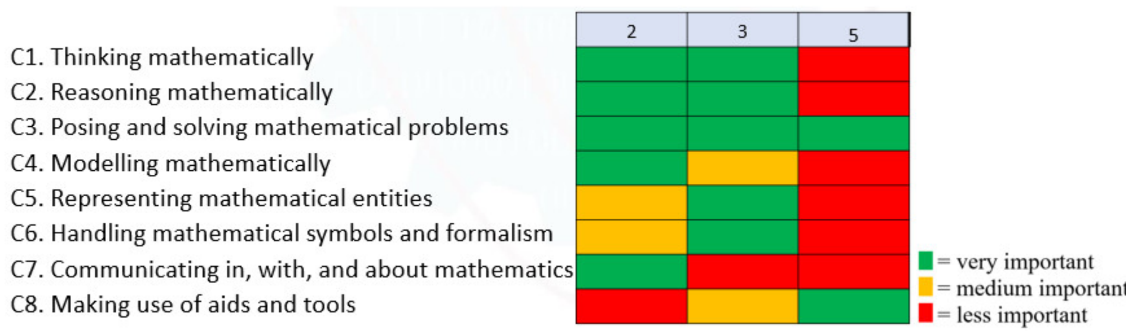

Figure 3: Mathematical competencies versus integral applications questions.

Question 2 of questions' type (3) is the only question in this assessment that has the C4 (Modelling mathematically) and C8 (Make use of aids and tools) competencies with maximum importance, in the remaining questions these competencies are defined as minimum important (except in question 3 where C4 is of medium importance). Another important note is that for the resolution of the last question (5) only the mathematical competencies of C3 and C8 are important, the remaining competencies are of minimal importance.

2.3 Data analysis

One important characteristic of this study is to know if the students acquired the mathematical competencies that we associate to the questions. That is why it is important to relate the result obtained by the student on a given question in relation to the competencies associated with the same question. Table 4 shows the means and standard deviation obtained from the students' answers to the 9 questions. Only in 3 questions (in 9) the results were positive (averages over 50%). In the remaining 6 questions, the results were negative, with 3 questions with averages below 30% $(1e), 1h)$ and 2).

For example, in the question $1e)$ (Figure 1) about primitive techniques, the students obtained a mean less than 25%, which means that the results were very bad on that question. If we analyse the resolution of the students (Figure 4), we can see that the students did not recognise the "cotangent" function $(\cot x)$ as the fraction of two trigonometric functions "sine" by "cosine" $\left(\frac{\sin x}{\cos x}\right)$, so they don't use the correct primitive rule. In this case the C5 (Representing mathematical entities) competence is missing.

	N	Minimum	Maximum	Mean	Std. Deviation
$Q1a)$	77	0	100	55.84	39.382
$Q1e)$	69	0	60	24.93	28.780
$Q1g)$	57	0	100	74.39	32.897
$Q1h)$	76	0	100	29.58	29.748
$Q1o)$	62	0	40	32.58	14.135
$Q1q)$	66	0	100	59.58	41.475
$Q2$	61	0	80	29.97	21.573
$Q3$	62	0	60	30.97	20.702
$Q5$	54	0	80	36.30	26.656
Valid N (listwise)	28				

Table 1: Mean and Standard deviation of the 9 questions.

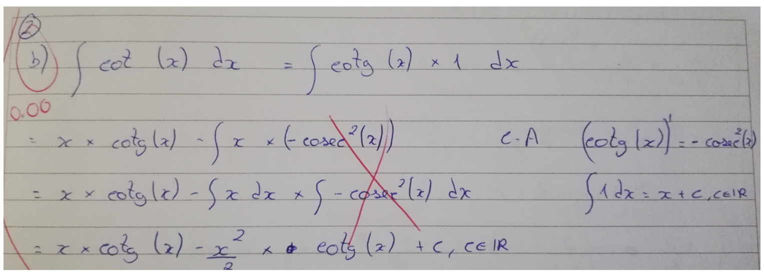

Figure 4: Student resolution of question $1e)$ about primitive techniques.

In question $1o)$ of improper integrals, the students have a mean of about 33%, a little more than the previous exercise, but still a negative value. In student resolution represented in Figure 5, we detected that student recognised correctly the formula to apply but they are not able to identify the limit of arc tangent ($\arctan x$) when x goes to minus infinity. The competence C8 (Make use of aids and tools) is missing.

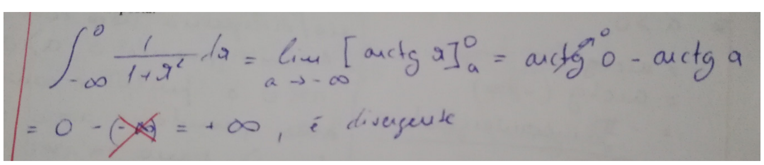

Figure 5: Student resolution of question $1o)$ about improper integrals.

In integral applications to Simpson's rule, the students have a mean of 36%, but only 54 students answered the question, this is 32% of the students don't solve the question. We verify that the students couldn't define the asked partition of the interval (Figure 6), so they fail in application of the Simpson's rule.

In question 3), students were asked to solve an application of the integral to the area calculation. This kind of question is of relevant importance in an Engineer course, however only 62 students of 80 answered this question, with an average of 31% and no student obtains the maximum score. When we analyse the students answers we verify the absence of two different competencies according to their resolution. In Figure 7 left, the students don't use the C2 competence (Reasoning mathematically). They do not understand the limits of the region that defines the integration area.

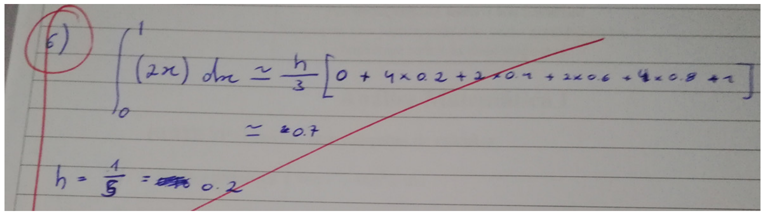

Figure 6: Student resolution of question $5)$ about integral applications of Simpson's rule.

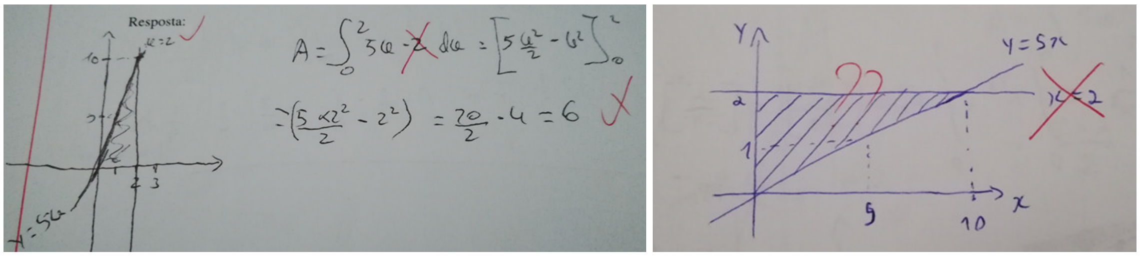

Figure 7: Student resolution of question $3)$ about integral applications to areas calculation.

On the other hand, in the resolution shown in the Figure 7 on the right, students do not use the C5 competence about representing mathematical entities. They do not know how to represent a vertical line with equation $x = 2$ or a oblique line $y = 5x$.

From the analyses of the performance obtained by students in the 9 proposed questions, we can verify from Figure 8 that the highest performance was obtained in questions $1a), 1g)$ and $1q)$ but the mean is highest in $1g)$ and in question 1o) almost all students obtained 40% of the question value.

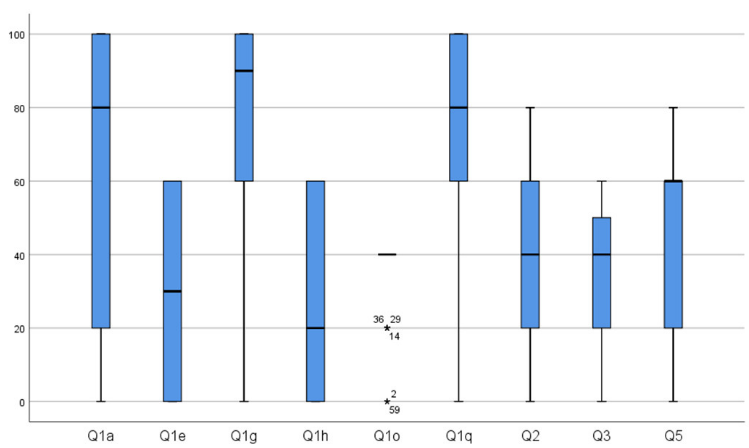

Figure 8: Box-plots comparison of grades questions.

Regarding questions 1e), 2), 3) and 5) 75% percent of the students obtained less that 60% of the question value. Therefore, we decided to study two different groups of questions: "Antimean" with questions about primitive techniques and improper integrals and the other group “Aplicmean” with questions about the integral application. When, we analyse this two groups separately, we see that the results were worse in "Aplicmean", this is when students were asked to apply there knowledge. The Figure 9 shows those results. Here we must reflect on our way to teach and how we are reaching the students. They are going to be engineers therefore they must be able to apply there knowledge to resolve problems. Future engineers need to understand the world of mathematics as well as how engineers fit into a process of fundamental-research-turned-into-applied-science [15].

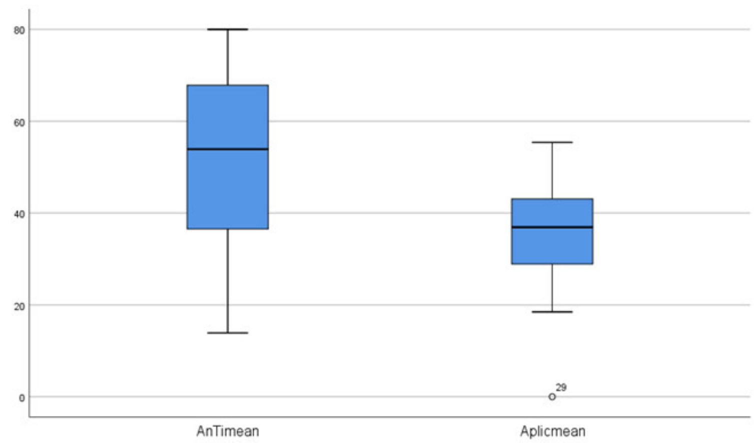

Figure 9: Box-plots comparison of groups of grades questions: "Antimean" and "Aplicmean".

3 Discussion

Our interest in this study was to understand how the competence assessment in mathematic can be applied to Engineer students and how students demonstrate the mathematical competencies were or were not acquired during the assessment. What competencies should be strengthened for students to obtain better results in assessments. The results presented before, indicate that the highest result was obtain the question $1g)$, a simple question to resolve where the most used competencies were C3 (posing and solving mathematical problems) and C8 (make use of aids and tools). In this question, the students had a good grade with a mean of 74%, and many of them solve the question correctly. In the opposite direction, the question $1e)$ was the worst, with less than 25%. This question had very low values because one of the competencies that was not acquired by many students was the C5 competence (Representing mathematical entities). In 5 out of 9 questions the students had a negative performance, the mean was less than 50%. This critical situation has been happening in recent years, in the curricular unit of Calculus I, where very few students obtain the necessary competencies and knowledge to obtain approval and many students who do not approve have very poor grades.

In relation to the competencies assessment in question, we conclude that:

- Negative overall assessment in some competencies (C1, C5, C8);
- The best evaluated competencies were those related to reasoning, posing, handling and solving mathematical problems (C2, C3);
- The least evaluated competencies were modelling and communicating in, with, and about mathematical (C4, C7).

In general next year, we have to work harder on the competencies that had the worst results, we have to make a continuous assessment of the competencies so that we can better assess which competencies are being less (or not) acquired by the students and we still have to pay more attention to certain students who do not have basic mathematical competencies.

4 Conclusions

In conclusion we want to reflect about "Student's comprehension and competencies achievement of the contents taught". In this case, we conclude that students did not acquire all the competencies and knowledge that we intended to. In general students obtained negative grades (less than 50%) due to the lack of bases in mathematics, not being motivated and having great difficulties in mathematics. In this sense, many students need a personal attention (office hours, extra work, etc.) because their results were clearly negative. Another reflection that we also want to make is regarding "The questions itself and its usefulness in assessing competencies". We conclude that, the questions proposed in RULES_MATH guide covers most of the competencies that we need to evaluate. The questions difficulty is adequate to our students, however many fail to obtain positive grades. Therefore, in next year, we will utilize similar questions. We will also choose questions that allow students to test all 8 mathematical skills in a balanced way and collect throughout the year the feedback on the skills least acquired by students so that we can obtain a formative assessment. To assess communication-competence we will use other activities such as students' videos, oral presentations, etc.

Acknowledgement

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Assessment with mathematics competencies-based RULES_MATH's guides for Linear Algebra

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Abstract

The purpose of this study is to investigate the effectiveness of using mathematical competencies as a formative assessment tool in undergraduate engineering and to identify association between learning outcomes and mathematical competencies inside assessment questions. For this study, 8 questions from the RULES_MATH guide for Linear Algebra LA3, LA4 (5 of multiple choice and 3 of development) were used, discriminated by the learning outcomes and associated with the 8 mathematical competencies. These questions were applied to 143 engineering students (Biomedical, Mechanical and Electromechanical) from the Calculus 1 curricular unit. The results were quite satisfactory and it was possible to identify the competencies less and more evaluated; the competencies in which students had greater and lesser difficulties and the learning results that reflect more or less competencies. As future research we intend to design new activities (questions) structured to evaluate mathematical competencies versus learning outcomes in a formative assessment way.

Keywords Assessment · Significant Learning · Competencies · Mathematics · Engineering

1 Introduction

The formative assessment (assessment for learning) is increasingly being emphasised in the academic world [1]. In order to improve students' learning on subject contents, the formative assessment should be seen as an important element to facilitate the learning process [7].

The study presented here follows on from the whole process already developed to assess the mathematical competencies in assessment of engineering students [5] which is the mainly objective of the RULES_MATH project [4]. The purpose of this paper is to illustrate that, when evaluating students, it is possible to capture a broader set of skills, attitudes, which go beyond the test scores. In the context of the RULES_MATH group we have been implementing a new approach to assess Engineering students based on the assessment of their mathematical competencies [9]. In this sense we intend to integrate the 8 mathematical competencies presented by Niss [2, 10] and developing according to Alpers in the Framework for Mathematical Curricula in Engineering Education [3], in the academic evaluation models. These 8 mathematical competencies are: (C1) thinking mathematically; (C2) reasoning mathematically; (C3) posing and solving mathematical problems; (C4) modelling mathematically; (C5) representing mathematical entities; (C6) handling mathematical symbols and formalism; (C7) communicating in, with, and about mathematics and (C8) make use of aids and tools for mathematical activity and knowledge.

2 Methodology

Good assessment follows an intentional and reflective process of design, implementation, evaluation, and revision. The "RULES_MATH project intend to follow this reflective process in the evaluation of mathematical competencies in engineering students. One study line of RULES_MATH project is to evaluated the mathematical competencies in assessment. Previous research has demonstrated that the new paradigm in evaluating engineering competencies proposed by RULES_MATH project [16, 11, 12, 13] is a methodology able to formative assessment and support the engineering students in their learning process. Using "RULES_MATH guide for a problem" [6], a guide developed for different mathematical areas with examples/tests and competence-based activities proposed for all project partners, we will assess whether the identified mathematical competencies for each question were or were not acquired by the students and in what quantity. In this study we used the LA3, LA4 guide “Linear Algebra” developed by the University of Salamanca (USAL) partner.

2.1 Participants

In the first semester of 2019/2020, Linear Algebra students of the first year of Engineer in Coimbra Institute of Engineering, answered some questions defined in the LA3 and LA4 guide during their examination. One hundred and forty-three students: twenty students from Biomedical Engineering (only 9 were female students); thirty-nine from Eletromechanic Engineering (only 3 were female students) an eighty-four from Mechanical Engineering (only 2 were female students) participated in this study.

2.2 Instruments

The questions chosen from the LA3 and LA4 guide for our study were: 5 multiple choice questions and 3 development questions. In Figure 1 is presented the multiple choice questions proposed to the students.

One of these questions allows the choice of more than one option (first question), the remaining questions allow only one choice option (only one option is correct). The fourth question (Q4) is about calculate the range of a matrix. In this question there are 3 learning outcomes LA32, "Recall the basic terms associated with matrices", LA37 – "Calculate the determinant" and LA313 "Calculate the rank of a matrix" (Figure 2).

The mathematical competencies related to them are represented in the table where green represents the competencies that are considered to be present, and therefore very important, yellow medium and red competencies that are considered has being evaluated in a less important way. Thus, in this question the competencies related to the determinant calculation (LA37) are: C6 (Handling mathematical symbols and formalism) and C8 (Making use of aids and tools) with very importance and the other 6 competencies with less importance.

5.1 MULTIPLE-CHOICE QUESTIONS

Solve the following test

1. Given the following matrix, detemine from wich of them you can calculate its determinant:
   i) $\begin{pmatrix} 1 & 1 \\ 0 & 2 \end{pmatrix}$
   ii) $\begin{pmatrix} 1 & 3 \\ 2 & 1 \\ 0 & 4 \end{pmatrix}$
   iii) $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$
   iv) $\begin{pmatrix} -1 & 3 & 0 \\ 3 & 1 & 5 \\ 2 & 1 & 1 \end{pmatrix}$
   v) $\begin{pmatrix} 1 & 2 & 0 & -3 \\ 5 & -3 & 4 & 1 \\ 3 & 0 & -2 & 6 \end{pmatrix}$
2. The determinant of de matrix $\begin{pmatrix} 1 & 4 & 2 \\ 3 & 1 & 5 \\ 2 & 1 & 1 \end{pmatrix}$ is
   i) -13
   ii) 26
   iii) 0
   iv) You can't calculate
   v) None of the above
3. The range of de matrix $\begin{pmatrix} -1 & 3 & 0 \\ 3 & 1 & 5 \\ 2 & 1 & 1 \end{pmatrix}$ is
   i) 1
   ii) 2
   iii) 3
   iv) 4
   v) You can't calculate
4. The range of de matrix $\begin{pmatrix} 1 & 2 & 0 & -3 \\ 5 & -3 & 4 & 1 \\ 3 & 0 & -2 & 6 \end{pmatrix}$ is
   i) 2
   ii) 3
   iii) 4
   iv) You can't calculate
   v) None of the above
5. Let A X = B be a system of equations with three unknowns. If |A| = 0, then
   i) Dependent System
   ii) Inconsistent System
   iii) Independent System
   iv) You can't know
   v) None of the above

Figure 1: Multiple choice questions.

The range of de matrix $\begin{pmatrix} 1 & 2 & 0 & -3 \\ 5 & -3 & 4 & 1 \\ 3 & 0 & -2 & 6 \end{pmatrix}$ is

i) 2
ii) 3
iii) 4
iv) You can't calculate
v) None of the above

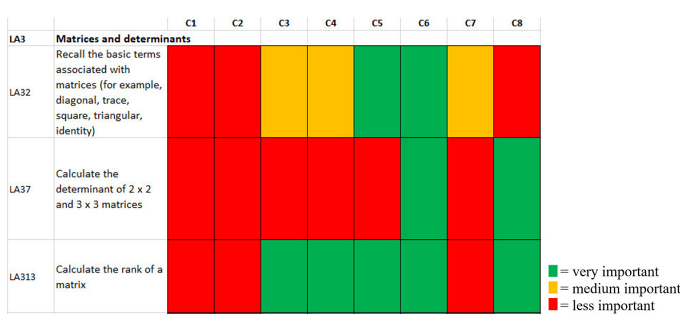

Figure 2: Multiple choice question Q4 (left), the learning outcomes and mathematical competencies associated (right).

One of the 3 development questions is Q5.2 represented in Figure 3. This question, about matrices and their operations, has 5 items and 10 learning outcomes related with mathematical competencies described in the table.

Given the matrix A= $\begin{pmatrix} 1 & 2 & 0 & -3 \\ 5 & -3 & 4 & 1 \\ 3 & 0 & -2 & 6 \end{pmatrix}$ and the matrix B= $\begin{pmatrix} 1 & 4 & 2 \\ 3 & 1 & 5 \\ 2 & 1 & 1 \end{pmatrix}$. Detemine:

i) Transpose A
ii) Adjugate B
iii) A+B, 2 B
iv) A.B and B.A
v) A^{-1} and B^{-1}

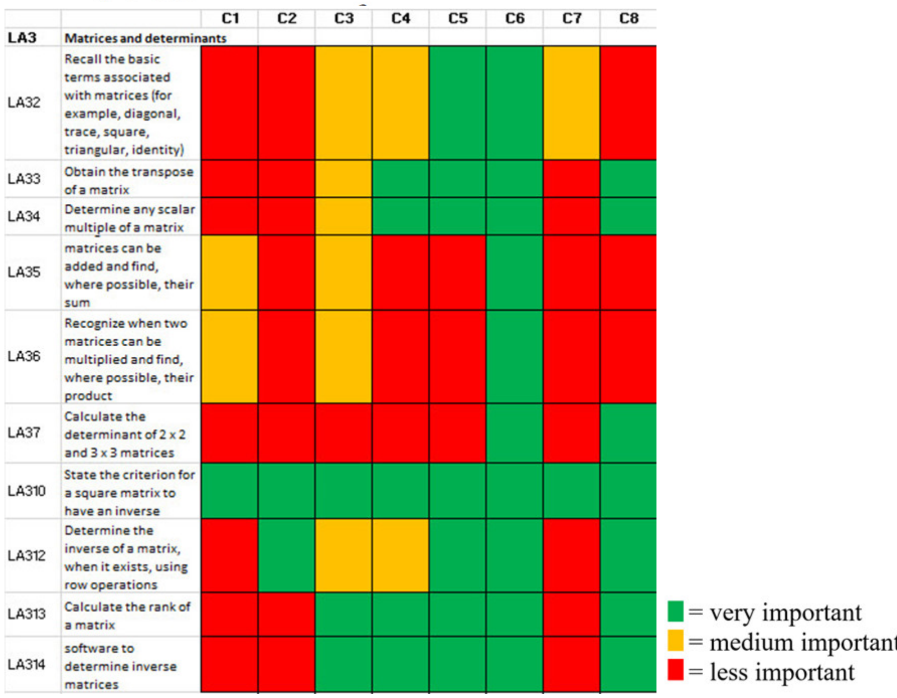

Figure 3: Development exercises Q5.2 (up), the learning outcomes and mathematical competencies associated (down)

The learning outcome LA310 "States the criterion for a square matrix to have an inverse" involves the 8 mathematical competencies in a very important way, finds the learning outcome LA37 "Calculate the determinant of 2×2 and 3×3 matrices" has only 2 very important competencies (C6 and C8) and the remaining minor ones. In the others, the competencies evaluation varies from the less important to the very important.

Finally, the other 2 development questions Q44 and Q55 explore many learning outcomes and for each learning outcome, the mathematical competencies associated with these. Figure 4 shows the relations between learning outcomes and competencies. In this case, it is important to observe that these 2 questions use some learning outcomes that are related in a very important way (green) to all the eight mathematical competencies (for example the last 3 lines of the table). Thus, we can say that they perform a complete formative assessment.

2.3 Data analysis

For the analysis of the mathematical competencies that are asked to perform during the questions and that are acquired by the students, we calculate the mean and the standard deviation for all

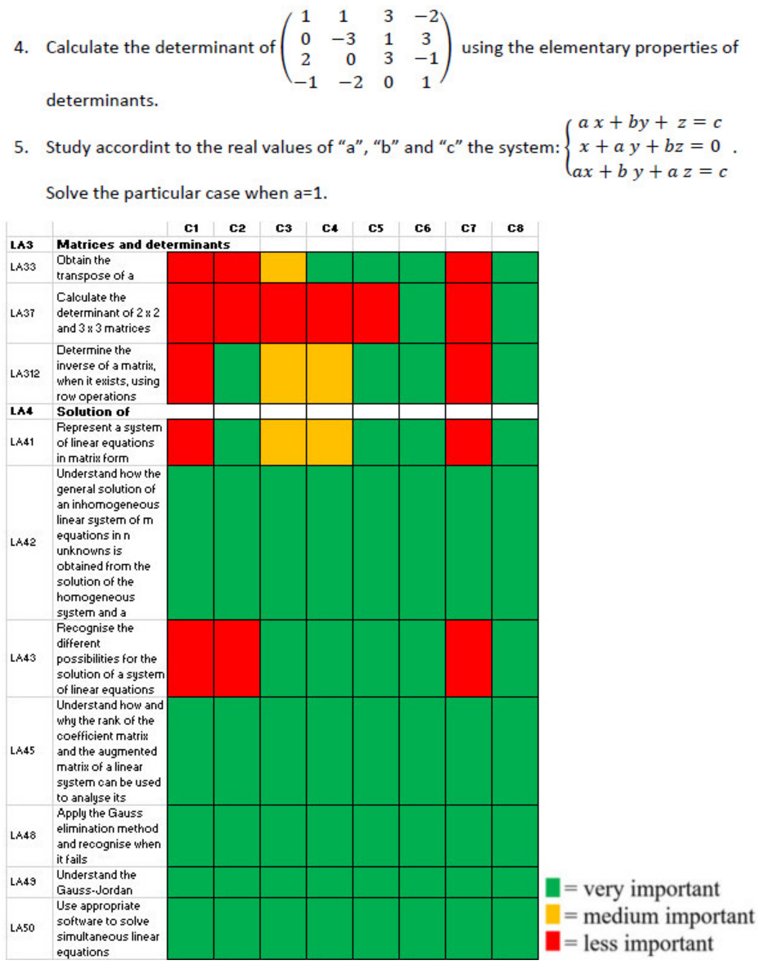

Figure 4: Development questions Q44 and Q55 (up), the learning outcomes and mathematical competencies associated (down)

grades obtained by the students in these questions. When, we analyse the data in Table 4, we verify that in questions Q5 and Q55 the students had a negative performance, therefore the mean is lower than 25% (marked in red). Only 8 and 4 students answered correctly, getting the total quote in questions Q5 and Q55 respectively. The competence C2 (Reasoning mathematically) is the less achieved.

In the remaining questions, the students obtained averages above 62% and in the answer Q5.2_iii the average was 94%, being answered by all students. An overall performance is satisfactory since in 11 questions only 3 had a negative average and the remaining 8 answers had high averages.

The reaction of the students to all the 8 questions will be observed in the Figure 5 where we have ploted the data boxplots of all questions. Has one can see, questions Q5 and Q55 were the ones with clear worse results (at least 75% of the students had less than 30% of the grade). Observe also that in questions Q5.1, Q2, Q5.2_iii and Q5.2_iv students had a very good performance (almost all achieved 100% of the grade). From the all boxplots picture we may conclude that, if all questions had the same value, a huge percentage of students would approve to this Algebra test.

The students reaction to the questions related to the different course are represented in Figure 6. The highest performance was from Biomedical Engineering; and the highest standard deviation

Descriptive Statistics	N	Minimum	Maximum	Mean	Std. Deviation
Q5.1	140	0	100	83.25	31.855
Q2	141	0	100	80.67	35.673
Q3	143	0	100	77.30	36.957
Q4	142	0	100	62.67	45.113
Q5	136	0	100	15.10	27.272
Q5.2_i_ii	140	0	100	64.55	33.350
Q5.2_iii	143	0	100	93.53	16.782
Q5.2_iv	142	0	100	89.23	24.794
Q5.2_v	137	0	100	64.18	37.675
Q44	98	0	100	44.56	36.937
Q55	126	0	100	21.43	27.772
Valid N (listwise)	85				

Table 1: Descriptive statistics for questions.

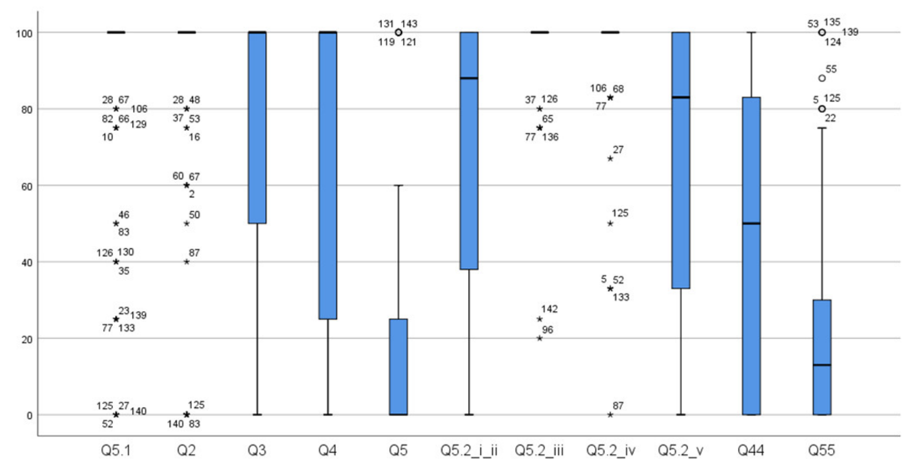

Figure 5: The student reaction to the questions.

was in question Q4 from Biomedical Engineering and the question Q5 from Eletromechanical Engineering.

In Figure 7 the reaction to the Q5.1, Q2, Q3, Q4, Q5, the multiple choice questions in relation to the courses is presented. The highest performance is from Biomedical Engineering. However, Biomedical and Mechanical Engineering have also the highest standard deviation. The results of the Q5.1 are bigger that 50% for Biomedical Engineering and the results of Q2 are bigger that 50% for Mechanical Engineering. In question Q2 the values obtained were bigger than 75% for the 3 graduation courses studied (with some exceptions). On the other hand, in question Q4, the results obtained were very different in all graduation courses studied and was the question less answered.

The reaction to the Q5.2 for all the 5 items related to the courses is presented in Figure 9. The highest performance was obtained in question 5.2_i_ii where the highest grade ($> 60\%$) was from Biomedical Engineering students. In item v, the standard deviation is great for Eletromechanical and Mechanical Engineering (from 33% to 100%). The grades are positive ($> 50\%$) in i and ii for Biomedical and Eletromechanical, in item iii for Eletromechanical and in item v for Biomedical Engineering.

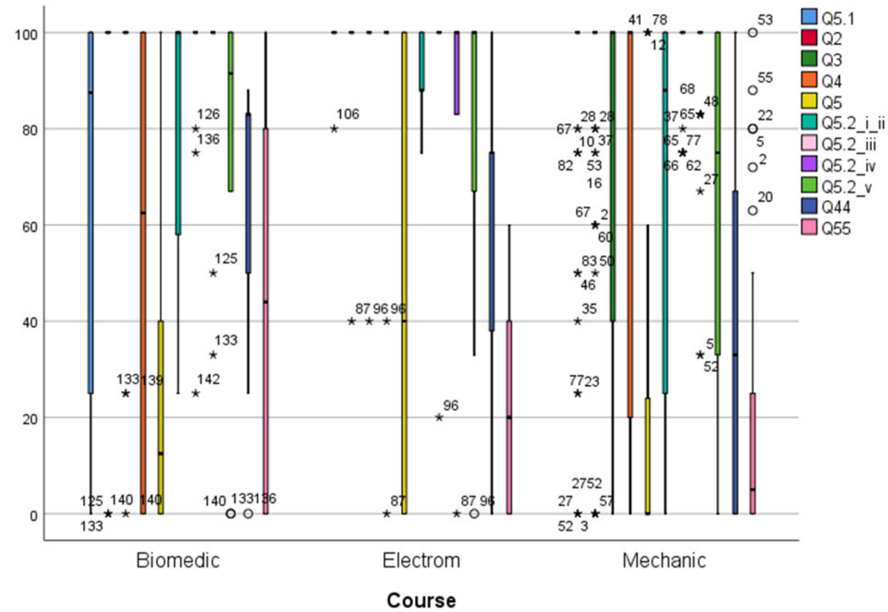

Figure 6: The student reaction to the questions by the courses.

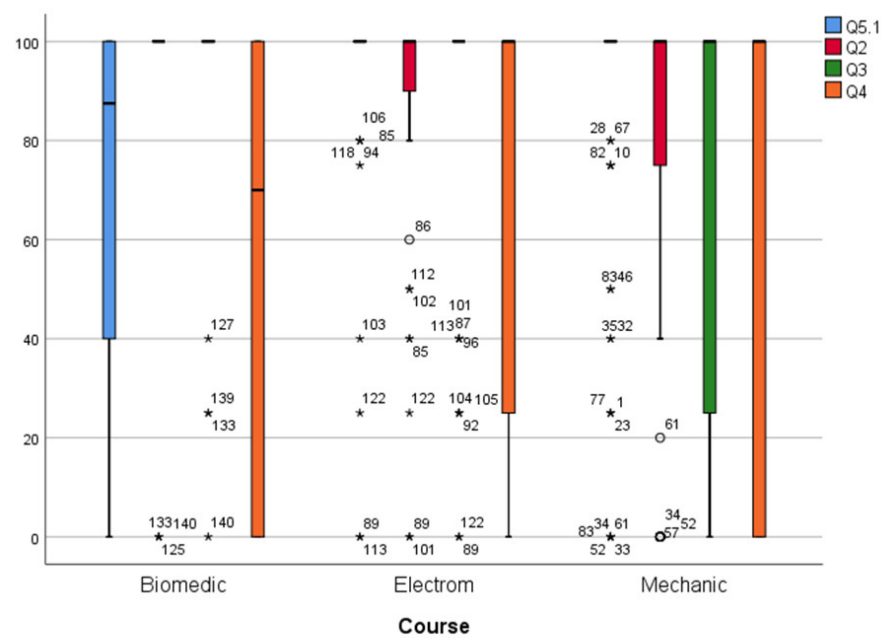

Figure 7: The students reaction to the Q5.1, Q2, Q3, Q4, Q5 questions by the courses.

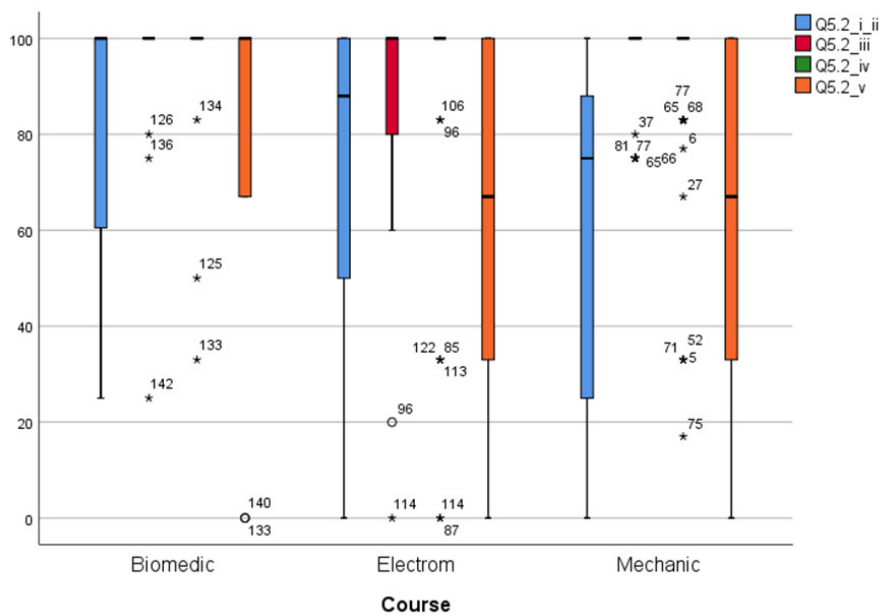

Figure 8: The student reaction to the Q5.2_i_ii, Q5.2_iii, Q5.2_iv, Q5.2_v questions by the courses.

The reaction to the development question Q5, Q44, Q55 related with courses (Figure 9). The highest performance is in question Q44 for Biomedical Engineering students. The question 5 have a highest standard deviation for Biomedical Engineering, with mean of 40%, and valuers between 100% and 0%. All the grades for question Q5 in Biomedical and Mechanical engineering and for question Q55 in Eletromechanical and Mechanical engineering are less than 50%. The results are very poor and to overcome this it will be necessary more theoretical questions or questions that compel the student to elaborate and write an answer in a coherent reasoning way.

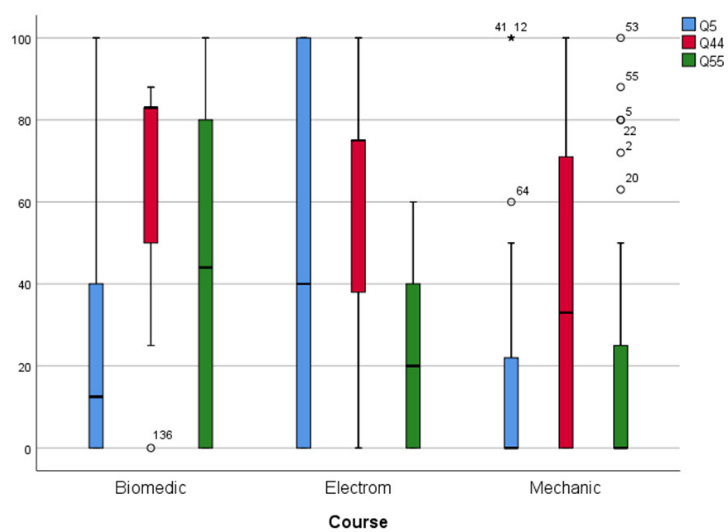

Figure 9: The student reaction to the Q5, Q44, Q55 questions by the courses.

When, we interpret the results obtained in question Q5 (with the worst results) related with courses (Figure 10), we confirm that the highest performance and the highest standard deviation were from Biomedical Engineering. We think this results are because the 2 learning outcomes LA43 "Recognise the different possibilities for the solution of a system of linear equations" and LA45 "Understand how and why the rank of the coefficient matrix and the augment matrix of linear system can be used to analyse its solution" are based on all 8 mathematical competencies. Thus, question 5 involves complete learning by students, hence it is a more complex and demanding question.

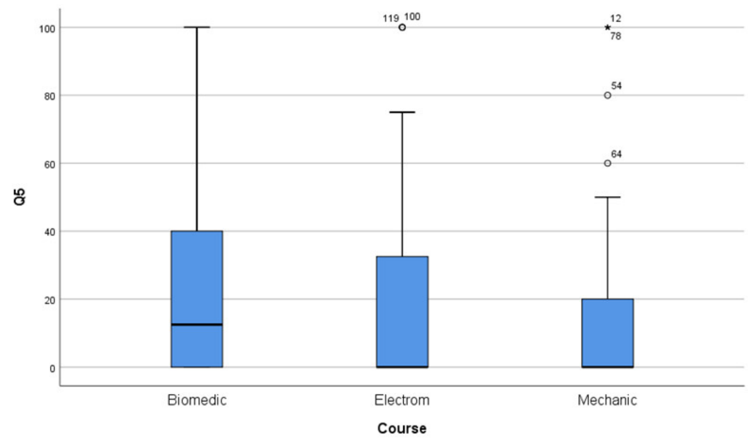

Figure 10: The student reaction to the Q5 question by the courses.

3 Discussion

How can and should we evaluate the existence or not of a mathematical competence in the evaluation of students. What are the most important competencies in student assessment? Should we evaluate all 8 competencies? These questions have led us to conduct some research, analysis and reflection on how we evaluate our students in mathematics. From our experience in relation to the mathematical competencies acquired by the students, we found that:

- Positive overall assessment in some competencies;
- The best evaluated competencies were those related to representing, mathematical entities, handling mathematical symbols and formalism (C5, C6);
- The least evaluated competencies were thinking and reasoning mathematically (C1, C2);
- In multiple choice questions the competencies most evaluated were C5, C6 and C8 and less evaluated were C1 and C2;
- In development questions the competencies less evaluated were C1, C2 and C7 and the others were most evaluated.

In relation to the mathematical competencies associated with the existent learning outcomes in the questions, we found that:

- The questions Q3, Q5.2_iii have highest results, because is about simple matrix operations;
- Q5 have the worst results for all courses, since it is a question with 5 items and involves all competencies of greater or lesser importance and where the C6 competence is very important in all learning outcomes.

- Q5, Q55 have results above 50% (in all courses), because they are related to competence C1 and C2;
- The Biomedical Engineering have a better results because to these students is demanded an high grade to enter the graduate course.

4 Conclusions

Although the assessment cycle focuses on self-assessment, peer assessment can help learners develop a variety of skills, including collaboration, communication, conceptual understanding and problem-solving skills [8]. Likewise, providing feedback can help improve students' communication more than just analyzing their work, because students practice communicating their ideas to other students. In this way, learning outcomes can be improved and the mathematical competence C8 will be more evaluated in assessment. Thus, first it is necessary to reflect on the mathematical competencies that questions must present to be useful in the evaluation of students and second to reflet on the mathematical competencies that the students acquired about the contents taught. Although the questions proposed by RULES_MATH guide covers most of the competencies that we need to evaluate and the difficulty is adequate to our students, a given assessment structure may include all mathematical competencies at different scales according to the learning outcomes that the student should acquired. Most mathematical competencies and knowledge were acquired by students, obtaining positive grades except for questions Q5 and Q55. The developed questions Q5 and Q55 are closely related to the C1 and C2 competencies, the competencies in which students have greater difficulty in acquiring. It was also found that some of the students need a personal attention in some learning outcome and the competencies associated with it.

A next step for our research is to use the mathematical competencies to design new activities structures, and use it as a guide for future research studies to test and validate the model.

Acknowledgement

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MATHEMATICAL COMPETENCY ORIENTED ASSESSMENT – RULES_MATH GUIDES ON COMPLEX NUMBERS

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ABSTRACT

Assessment makes an integral part in education. Generally, it is considered to be the representative of a level of mastering some specification, and gives a promise for future that the person will be able to use the achieved knowledge and skills in further study, occupation or common life. It comprises several very important roles. It provides the feedback of study process and its results for the person oneself, and for his or her educator as well. It reveals strengths and weaknesses, gives motivation for further study, or decides about admission to following grades of study or to future employment. Speaking about assessment of a student within a course, it gives the information about the level of student's knowledge and skills on some curricula or a curricula unit. On one hand, it reflects the level of mastering the content, what is in general took into consideration predominantly, and on another hand, it is usually automatically supposed to reflect also the level of corresponding competency. Regard to competency education becomes very important in these days and the international team of Rules_Math Erasmus + project developed the guides focussed especially for training and assessment of mathematical competency for students of technical tertiary education. With respect to the SEFI MWG Group document "A Framework for Mathematics Curricula in Engineering Education" [1], we deal with Core 1 contents – mathematics curricula units studied mostly at bachelor technical degree, and distinguish eight main mathematical competencies. previously proposed in the KOM project [5].

In the paper, we deal with the Guide on Complex Numbers prepared in the Department of Mathematics and Physics at the Faculty of Mechanical Engineering STU in Bratislava, the partner of the project, and introduce a competencies evaluation methodology. The results of testing, and the applied teaching methods for competency development are presented too.

Keywords Mathematical competency · Assessment · Technical university · Complex numbers

1 Introduction

The international team around Rules_Math Erasmus Plus Project: 2017-1-ES01-KA203-038491 publishes a book "New Rules for Assessing Mathematical Competencies – User Guide", where a reader can find collections of problems which aim to serve as models for competence oriented assessment in mathematics for bachelor studies at tertiary technical schools. The Rules math project is coordinated by University of Salamanca, and unites eight partners, STU Slovak University of Technology in Bratislava, Slovakia; HBV Ankara Haci Bayram Veli University, Turkey; CVUT The Czech Technical University in Prague, Czech Republic; PU University of Plovdiv, Bulgaria; CSIC Spanish National Research Council, Spain; IPC Polytechnic Institute of Coimbra, Portugal; TU Dublin, Technological University Dublin, Ireland; and UTCB Technical University of Civil Engineering Bucharest, Romania.
The book followed Core I learning outputs defined by SEFI Mathematics Working Group in the document "A Framework for Mathematics Curricula in Engineering Education" ([1], hereinafter as the Framework), third edition, where the learning outputs are formulated in active verbs, what allows to associate them with mathematical competence previously introduced in KOM project as

"the ability to understand, judge, do, and use mathematics in a variety of intra- and extra-mathematical contexts and situations in which mathematics plays or could play a role" [5]

consisting of eight overlapping partial competencies: C1: Thinking mathematically, C2: Reasoning mathematically, C3: Posing and solving mathematical problems, C4: Modelling mathematically, C5: Representing mathematical entities, C6: Handling mathematical symbols and formalism, C7: Communicating in, with, and about mathematics, C8: Making use of aids and tools; and three dimensions for specifying and measuring progress: 1. Degree of coverage (the 'reproduction' level, the 'connections' level, the 'reflection' level (corresponding to taxonomies of education: Bloom, Nemierko), 2. Radius of action, 3. Technical level.
In the following, we will introduce the Guide on Complex numbers, the methodology for evaluation and results obtained by testing the material within training courses at the Faculty of Mechanical Engineering, STU in Bratislava.

2 Guide on Complex Numbers

Guidelines on Complex Numbers (See Appendix) were designed at FME STU. They consist of three parts. Multiple Choice Questions, Questions 1 comprising open answer problems, and Questions 2 – small project, imitating real problem in practice. The questions were proposed with aim to cover general learning outputs defined in the Framework and above all, to provide adequate space for performance of all competencies C1 – C8. We enriched traditional concept of tasks exhibiting mostly mastering computation techniques with tasks of type to find the most efficient way, to defence one's opinion, to give reason, etc. For instance, to find the most efficient way requests to try or to think about all available techniques or approaches, to compare them, to realize their limitations, to discover pro et con, and to give reasons why the chosen technique is the most efficient taking into account various points of views. Such tasks stimulate diverse multisolution way of thinking, noticing similar and different items. From the point of evaluation, they provide explicit performance unit even for assessment of C2 or C7.
Moreover, our multiple choice test operates on questions with various number of correct answers from 0 to all, breaking the traditional multiple choice tests with just one correct answer and activating to consider properties of various types, join them and to form a comprehensive idea,

concept of the matter.

		C1	C2	C3	C4	C5	C6	C7	C8
ANALYSIS AND CALCULUS									
AC3	3.Complex numbers								
AC31	State and use Euler's formula								
AC32	State and understand De Moivre's theorem for a rational index								
AC33	Find the roots of a complex number								
AC34	Link trigonometric and hyperbolic functions								
AC35	Describe regions in the plane by restricting the modulus and / or the argument of a complex								

Figure 1: *
Table 1: Learning outputs with levels of competencies importance

Although the proposed questions are primarily targeted on written forms of exams, they can be successfully used in various teaching and learning forms such as group work, educational games, etc. provoking to discuss problems. To vary questions, interested party can find some suggestions in the key together with questioned learning output. Besides this, each curricula unit is armed with a table of involved learning outputs and competencies importance. (see Table 1) Partial competencies C1 – C8 are ranked by one of three colour values (green= very important, yellow = medium important, red = less important), following their importance level in a LO. The table can serve to all associates entering competence assessment. The proposed tasks satisfy both, the competencies as well as contents requirements. The model tasks could be then applied in standard courses and used also for assessment of traditional requirements on knowledge and skills for passing the course. The level of importance was rated by experienced university teachers of mathematics; nevertheless we do not consider it to be fixed. It can vary with respect to needs of engineering specification, study program or particular course, and mainly on entering level of students.

3 Competence assessment

Assessment makes an integral part in education. Generally, it is considered to be the representative of a level of mastering some specification, and gives a promise for future that the person will be able to use the achieved knowledge and skills in further study, occupation or common life. It comprises several very important roles. It provides the feedback of study process and its results for the person oneself, and for his or her educator as well. It reveals strengths and weaknesses, gives motivation for further study, or decides about admission to following grades of study or to future employment. Speaking about assessment of a student within a course, it gives the information about the level of student's knowledge and skills on some curricula or a curricula unit. On one hand, it reflects

the level of mastering the content, what is in general took into consideration predominantly, and on another hand, it is usually automatically supposed to reflect also the level of corresponding competency.
Besides traditional contents assessment, the ambition to transfer more importance to competence evaluation is noticeable. To educe working competence assessment model, it is worth to comprehend three main aspects.

- First, competence is inner capability of a person. It refers to knowledge and skills, the person has learned in past, to abilities, the person is able to do in present time, and also to aptitudes, what person is able to learn, and do in future.
- Second, detection of competence is possible only through some performance.
- Third, a competence is related to some content. The competence cannot be monitored without content. Only through particular situations, the observation can be generalized and we can speak about person's competence in general.

Expressing learning outputs via active verbs, in the form like "a student should be able to: carry out, express, calculate, evaluate, represent, manipulate, obtain, distinguish, recognise, interpret, plot, understand, etc." [1], enables to put content into relation with action and ability, and gives possibility to evaluate competencies via content models.
The cornerstone of competence assessment lies in question *what we want to exam, evaluate*. One can distinguish between two main approaches

1. Competence related content, where content is evaluated, or
2. Content related competencies, where competencies are evaluated

The approach, we choose for assessment, implies the selection of assessment forms, which can be

- Oral or written (oral: speech, answers, discussion, etc.; written: test with open or closed answers, project, essay
- Summative or formative
- Open book or closed book
- Paper or electronic, etc.

The selection of forms will be also influenced by local variables such as number of students, time possibilities, to assess during or outside teaching hours, at school or as a homework, etc. The assessment is usually executed by means of scores, the scores can be single or weighted and weights can be assigned with respect to importance, quantity, demands, level of complication, etc.

Approach 1: Competence related content
Scoring category: content determined learning outputs
In competence related content, content is evaluated. This approach can be recognized more frequently since it is similar and combinable with classical way of assessment. Aiming at assessment of competencies, one can follow the procedure:

- Assign points to content items in a task. Scoring category is content determined learning outputs.

- Determine relevant competencies and their values (simple or weighted) in a task.
- Presume the quality of involved competencies with respect to the reached score of a task. We can do so only qualitatively, in words (see e.g. [2]); or it can be quantified by conversion with respect to competence occurrence and weights (see [3]).

Approach 2: Content related competencies

Scoring category: competencies determined learning outputs

On contrary to above, we developed and trailed the methodology based on direct evaluation of competencies through performance elements. The procedure could be as follows:

- Define the performance element representing quality of examined competence.
- Assign points to the elements. Scoring category is competence determined learning outputs.
- Calculate scores (simple or weighted) of competencies across all tasks.
- If needed, calculate scores of tasks given by the sum of values of involved performance elements.

Comparing the approaches, one can see that different items in tasks are identified and valuated. While the first approach only presumes the acquired level of assumed competencies through solved problem units, the second one directly figured out the competence level through performed elements, previously carefully defined to represent the competence. On other hand , it expects open answer questions, where the performance can be explicitly followed. In quizzes with ticked answers, the first approach or its modification may be the only possible. We see two risks in implementation of the second approach. First, schools still have greater demands on contents than on competencies; and the second one, we have to be aware of the fact that person's level of competence depends on the level of mental capability or/and drill, and it can be developed only gradually, what puts demands on adequate learning objectives, their degree of coverage (how high aims of taxonomy we choose) and radius of action. The possible starting point could be the technique, when the assessment tasks are constructed in a way, they contain parts dealing with all desired items to be evaluated, comprising both, content and competence point of view. Since students consider the worth what is graded, we recommend precisely select competencies which take part in overall grading. The rest of them should be assessed by formative assessment. Then appropriate assessment and training forms could be selected, through which students will be able to cultivate all desired content related competencies.

4 Complex Numbers at FME STU

Complex Numbers have special position in the FME STU mathematics curricula, since the unit is not a part of summative assessment. It is taught only first two hours of the semester as a repetition of high school content. In the school year 2019/2020, due to Rules_Math project we provided special class on learning outputs in Guides supplemented by one teaching hour test within the Additional Exercises Course. We have provided also activities for some groups of students comprising heuristic methods, discussions, group work, tasks with diverse solutions or with diverse methods of solutions since 2018/2019. The classes were focused on building mathematical competencies, especially developing competencies C7 and C2, coming together with C3, C4, and C1.

4.1 Competence assessment in Analysis and Calculus AC3 Complex Numbers

Guides problems on complex numbers were tested at FME STU via

- written tests in amount of three problems picked up from question II part, and question I part with open answer (see the example in appendix)
- multiple choice test in formative assessment joined with discussion and reasoning answers
- voluntary home work on Questions 3

Twenty five written tests were evaluated using approach 2. At first we identified performance elements to represent particular involved mathematical competencies (see Table 2).

problem		LO	Performance / Competence						max sum
			C1 think	C2 reason	C3 probl	C4 mod	C5 repr	C6 handl	
multiplication	1a	AC31	choice of repr.	reason			repr.	calcul.	4
arg, abs	1b	AC31	concept				repr.	calcul.	3
gon. form, power	1c	AC32 AC34			graph interpret		repr.	calcul.	3
roots cubic eq	2	AC33	how to grasp	reason	find the way	way	repr.		5
graph of a set	3	AC35	concept				repr.		2
max sum			4	2	2	1	5	3	17

Figure 2: *
Table 2: Identification of competencies performance units and their maximal values (1/unit)

Student

problem		LO	Competence						task score	relative task score
			C1 think	C2 reason	C3 probl	C4 mod	C5 repr	C6 handl		
multiplication	1a	AC31	1	0			0.5	0	1.5	0.38
arg, abs	1b	AC31	0				0	0	0	0.00
gon. form, power	1c	AC32 AC34			1		1	1	3	1.00
roots cubic eq.	2	AC33	0.25	0	0.5	1	0.5		2.25	0.45
graph of a set	3	AC35	1				1		2	1.00
competence score			2.25	0	1.5	1	3	1	8.75	
relative competence score			0.56	0.00	0.75	1.00	0.60	0.33		**0.51**

Figure 3: *
Table 3: Assessment of student's competencies through performance units defined in the Table 2

In the test, the first six competencies were assessed. The achieved competence level was evaluated by five degrees from 0 – fail to 1- excellent (0, 0.25, 0.5, 0.75, 1). In the table 3, we demonstrate the competence assessment results of one of the students. The student's scores of involved competencies are as follows: C1 – 0.56, C2 – 0, C3 – 0.75, C4 – 1, C5 – 0.6, C6 – 1. His or her overall mathematical competence related to complex numbers is evaluated by 0.51. Looking at the results one can easily recognize that the student did not perform any valuated item of reasoning, and he or she had considerable gaps in computations – handling with mathematical symbols and formalism.

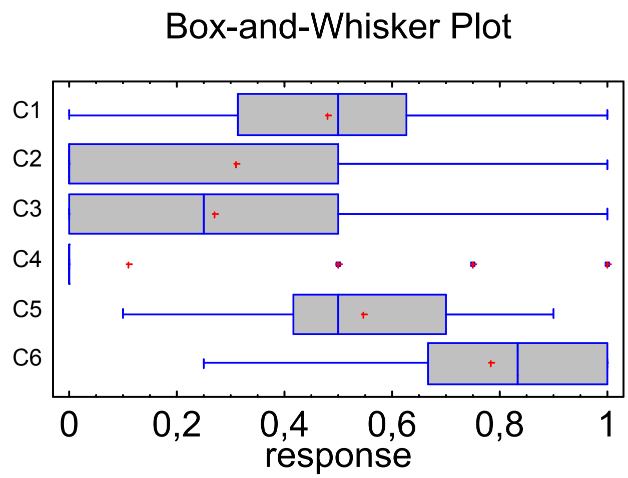

Figure 4: *
Figure 1: Box plot of results in the test on Complex Numbers

The best results students achieved in C6 and then in C5 (see Figure 1), competencies which are the most trained in this topic, and students are the most familiar with them. Here, all students were valuated with a score greater than 0 (see Figure 2: C6 values $\geqslant 0.25$; C5 values $\geqslant 0.1$, and they also have the highest values of central tendency). Good results were achieved also in C1 competency where the students showed they understood the concept of complex numbers and their representations. (The values of central tendency were similar to C5.) The worst results students achieved in C4, since it was valuated only through one performance unit involved in the most demanding sub problem. Here only 16% of students have better scores than 0 (0.5, 0.75 and 1 – seen as distinct points in the box plot.). C3 and C4 competencies require the high level of highest learning objectives as analysis, synthesis and evaluation (aims in the Bloom's taxonomy). Bad result were achieved also in C2 competence (median was 0, and only 40% of students have better values than 0). All competencies showed large variability, what points out on large differences between students. It is in line with the fact, that the Complex numbers is the first topic, the newcomers from different high schools meet with, at their bachelor study, and their performance reflects the level of competencies the students have from their high schools.
From statistical point of view, only C1, C5, and C6 have normal distribution, that's why we look more at medians as representants of central value, and variability. In case we would like to have better image on less involved competencies, we recommend to create the test with higher involvement of desired competence performance units, or to assess them across more curricula units.

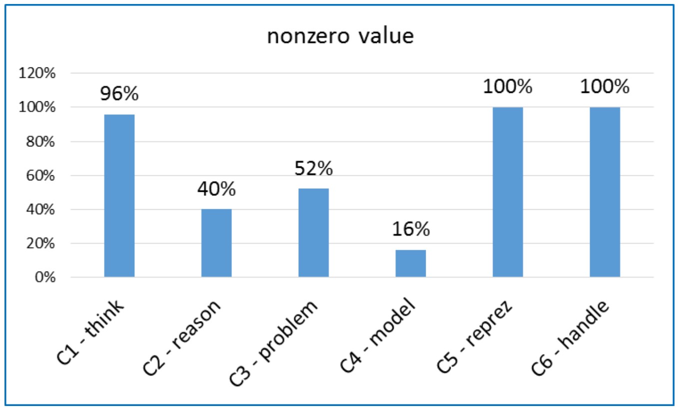

Figure 5: *
Figure 2: Percentage of non zero values achieved in competencies assessment

4.2 Formative assessment, training methods and opinion survey

Since, complex numbers are taught at very beginning of the first semester, students gained here their first experience with learning methods like guided discussion or revealing mistakes. They formed pairs or small groups for work in groups. The observation showed that there were very big differences between students coming from various high schools. A lot of them were not interested in mathematics, and almost all of them identified it just with calculation. Only here, a lot of them learned to make their first own mathematical statements, discuss about them, argue and justify them. Seeing more ways of process, a correct result expressed by different expressions, or a task with more correct solutions was confusing for many of them. They were surprised by more correct answers in the multiple choice test, and much more, that they had to reason their opinion. On other hand, the small group of clever students in mathematics helped to overcome embarrassing period and to draw classmates for better performance. Nevertheless at the end of semester, in survey, 82% of students claimed the discussion helped them to understand mathematical concept; to learn efficaciously, 66% of students consider group work to be suitable, and 68% of students preferred combination of methods. Students appreciated group work, possibility to ask questions and make discussions, the heuristic approach, comparison of various approaches and methods used in solutions. Above all, they praised the friendly atmosphere, willingness of a teacher and enjoyment. The quick tempo was the most disliked (one of main differences and problems in transition from secondary to tertiary education). Complex numbers were also among topics, which addressed students the most. Although they had problems with solution of applied problems formulated with respect to practical usage in life, students appreciated to see the instant example of practice. Concurrently it reveals that students had not been used to join information from different fields. Although 23% students in a group finished the course before exam, only 17% evaluated their work at practicals with E or FX, and even 70% evaluated themselves on A–C, comparing to 39% who reached those marks in real. It indicates lack of self criticism, and at the same time belief they felt well at practicals holding methods oriented on developing mathematical competencies, mainly the higher aims of Bloom's taxonomy, requiring posing and answering questions, arguments, tasks with diverse solutions, etc. Despite training, only 9% said they liked to chain arguments; students were not familiar with justification and not at all with proofs. It clarifies not well results in reasoning. The students realized their lack of quality especially in higher aims. On a scale from

1 (the worst) to 10 (the best), students evaluated themselves math competencies. At the level 7 and higher, 54% of students can identify math concept, 58% of students are aware of possibilities of using mathematical methods and their limitations, 40% can express themselves mathematically, 46% can formulate a mathematical question, 38% manage to give reasons for their math opinion, 44% can abstract and generalize results, 54% can apply general knowledge and skills to specific task, 85% can apply learned techniques, 46% manage to come up with their own procedure, 73% can understand mathematical notation (symbolic, graphic, verbal ..), 58% can write their statement mathematically (symbolically, graphically, verbally), 54% can discuss on mathematical problems, and 73% understand a classmate and they are able to explain a math problem or a schoolwork to him/her.

5 Conclusion

The guidelines on Complex numbers, we created for Rules_Math guides, aim to cover general learning outputs defined in the Framework and to provide adequate performance for all competencies C1 – C8. Although the proposed questions are primarily targeted on written forms of exams, they can be successfully used in various teaching and learning methods and forms such as group work, educational games, quizzes, etc., evoking to discuss problems, pose mathematical statements, or defence one's opinion. In order to evaluate quantitavely the level of competencies, we introduce the special methodology. Its main idea lies in an identification of such performance elements, which represent examined level of partial mathematical competence. The presented approach allows directly to figure out the competence level through performed elements. Competence addressed performance could much better reveal the lacks and strong sides of learners' mathematical competence, as it could be seen on examined group of students. The survey on training courses showed that asked students praised applied problems, group work, possibility to ask questions and make discussions, the heuristic approach, comparison of various approaches and methods used in solutions. Above all, they appreciated the friendly atmosphere, willingness of a teacher and enjoyment. "I had no idea math could be so fun" was one of the comments appeared in the survey.

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Appendix

New Rules for Assessing Mathematical Competencies

5.1 MULTIPLE CHOICE QUESTIONS

Instruction: The number of correct answers vary between 0 and 4. The sum of correct answers is 10.

1. Imaginary part in trigonometric form of a complex number is expressed through
   a) Cosine of the argument
   b) Sine of the argument
   c) Tangent of the argument
   d) Euler number

2. Real part in trigonometric form of a complex number is expressed through
   a) Cosine of the argument
   b) Sine of the argument
   c) Tangent of the argument
   d) Euler number

3. Which of the pairs are considered to be complex conjugates?
   a) $ci+d$, $-ci+d$
   b) $a+bi$, $-(a+bi)$
   c) $\cos\varphi - i\sin\varphi$, $\cos\varphi + i\sin\varphi$,
   d) $e^{-i\varphi}$, $e^{i\varphi}$

4. Images of complex conjugates in Gauss plane are
   a) symmetric about horizontal axis
   b) symmetric about imaginary axis
   c) symmetric about origin of axes
   d) not symmetric

5. The solution of equation $x^4 - 16 = 0$ is
   a) {2}
   b) {2, -2}
   c) {2, 2*i*, $\sqrt{2}$, -2}
   d) {2, 2*i*, -2*i*, -2}

6. Which of the triads $\{ {}_1, x_2, x_3$ are possible solutions of the cubic equation ${}^3 = z$? Give the reason. Sketch the numbers.

   a), $\{2\,,\ 2(\cos\frac{2}{3}\ \ + i\sin\frac{2}{3}\pi)\,,$
   $2(\cos\left(\frac{2}{3}\ \ \right) - i\sin\left(\frac{2}{3}\pi\right))\}$,

   b) $\{2\,,\ 2(\cos\frac{2}{3}\ \ + i\sin\frac{2}{3}\pi)\,,$
   $2(\cos\left(-\frac{2}{3}\ \ \right) + i\sin\left(-\frac{2}{3}\pi\right))\}$,

   c) $\{e^{i3}\,,\ e^{i(3+\frac{2}{3}\ \ )}\,, e^{i(3+\frac{4}{3}\ \ )}\}$

5.2 QUESTIONS 1

1. Given two complex numbers z_1 and z_2 : $\mathrm{Re}(z_1)=1$, $\mathrm{Im}(z_1)=\sqrt{3}$, the argument of z_2 is $-\dfrac{\pi}{6}$ and it's absolute value is 2
   a) expres z_1 and z_2 in all forms (algebraic, trigonometric, exponential). Determine their cartesian and polar coordinates.
   b) first, think over a strategy of calculation, and calculate
      i) the division $z = z_1 / z_2$ (product, sum or subtraction)
      ii) Give the reason for your strategy. Was it the most efficient way?
      iii) interpret calculation of the argument in graphical way
      iv) which kind of complex numbers is the result *z* from? Give the reason.
      v) display the result *z* in complex plane.

2. Given the complex number z_1 : $\mathrm{Re}(z_1)=1$, $\mathrm{Im}(z_1)=\sqrt{3}$
   a) calculate z_1^4 in trigonometric form. Interpret calculation of the argument in graphical way. Write the result in exponential form.
   b) calculate z_1^{70} in exponential form.
   c) calculate $\sqrt[3]{z}$. Interpret calculation of the argument in graphical way. What shape is created by the roots of z?
   d) how $z^{p/q}$ will be calculated? Express both in trigonometric and exponential form.

3. Given z_2 , the argument of z_2 is $-\dfrac{\pi}{6}$ and it's absolute value is 2. Calculate in trigonometric form
   a) the sum of complex conjucates z_2 and $\overline{z}_2$
   b) the subtraction of complex conjucates z_2 and $\overline{z}_2$
   c) express Cos (2) using an operation of complex conjugate numbers suggested by you. Write the expression via the exponential form of complex numbers.
   d) express Sin (2) using an operation of complex conjugate numbers suggested by you. Write the expression via the exponential form of complex numbers.

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5.3 Questions 2

A wall clock is designed in the shape of square with the horizontal edge of the length 32 cm.

a) Determine the vertices of the square in terms of complex numbers in all forms: algebraic, goniometrical and exponentional. The middle of the clock lies in the origin of complex plane.

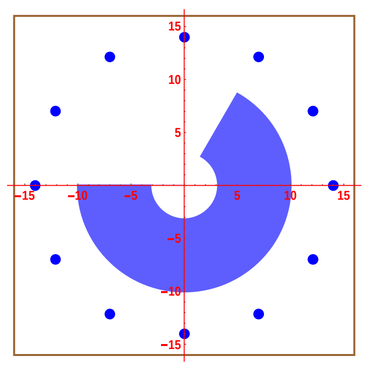
An engraving machine engraves the clock numerals images on the clock face with radius 14 cm. The middle of the clock lies in the origin of complex plane, the places of images are determined as complex numbers and they are engraved gradually one by one by turning the tool on reference position of the image.

b) Propose the computation formula for the algorithm to engrave

   i) all 12 numerals images

   ii) the images of 1, 9, and 4

c) Propose the region description in means of ranges for modulus and arguments to colour the annullus part between 1 and 9, in the distance from 3 cm to 10 cm from the middle of the clock (see the figure).



Modification of the Test on Complex Numbers

1) Given two complex numbers $z_1 = i - \sqrt{3}$, $z_2 = 2e^{-\frac{i\pi}{6}}$.

   a) Transform numbers to appropriate form and calculate the division $z = \frac{z_1}{z_2}$. Interpret calculation of the argument in graphical way. Which kind of complex numbers is the result *z* from? Give the reason. Display the result *z* in complex plane.

   b) Determine and sketch in a complex plane: the argument and the absolute value of $z_1 = i - \sqrt{3}$.

   c) Express $z_1 = i - \sqrt{3}$ in goniometric form and calculate z_1^{12}. Interpret the calculation of the argument of z_1^{12} in graphical way. Transform result into exponential form and display it.

2) Which of the triads $\{x_1, x_2, x_3\}$ are possible solutions of the cubic equation $x^3 = z$? Give the reason. Sketch the numbers.

   a) $\{2,\ 2(\cos\frac{2}{3}\pi + i\sin\frac{2}{3}\pi),\ 2(\cos\left(\frac{2}{3}\pi\right) - i\sin\left(\frac{2}{3}\pi\right))\}$,

   b) $\{2,\ 2(\cos\frac{2}{3}\pi + i\sin\frac{2}{3}\pi),\ 2(\cos\left(-\frac{2}{3}\pi\right) + i\sin\left(-\frac{2}{3}\pi\right))\}$,

   c) $\{e^{i3},\ e^{i(3+\frac{2}{3}\pi)}, e^{i(3+\frac{4}{3}\pi)}\}$

3) Sketch the set of complex numbers z in complex plane for which $|z| \le 3$ and $\varphi \in \left\langle -\frac{\pi}{4}, \frac{\pi}{2} \right\rangle$

PROBABILITY AND STATISTICAL METHODS: ASSESSING KNOWLEDGE AND COMPETENCIES – CASE STUDY AT ISEC

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ABSTRACT

The concepts taught during a Statistical Methods course make use of different mathematical skills and competencies. The idea of presenting a real problem to students and expect them to solve it from beginning to end is, for them, a harder task than just to obtain the value of a probability given a known distribution. Much has been said about teaching mathematics related to day life problems. In fact, we all seem to agree that this is the way for students to get acquainted of the importance of the contents that are taught and how they may be applied in the real world. The definition of mathematical competence as was given by Niss ([2]) means the ability to understand, judge, do, and use mathematics in a variety of intra– and extra – mathematical contexts and situations in which mathematics plays or could play a role. Necessarily, but certainly not sufficient, prerequisites for mathematical competence are lots of factual knowledge and technical skills, in the same way as vocabulary, orthography, and grammar are necessary but not sufficient prerequisites for literacy. In the OEDC PISA document (OECD, 2009), it can be found other possibility of understanding competence which is: reproduction, i.e, the ability to reproduce activities that were trained before; connections, i.e, to combine known knowledge from different contexts and apply them do different situations; and reflection, i.e, to be able to look at a problem in all sorts of fields and relate it to known theories that will help to solve it. The competencies that were identified in the KOM project ([2, 4]) together with the three "clusters" described in the OECD document referred above were considered and adopted will slightly modifications by the SEFI MWG (European Society for Engineering Education), in the Report of the Mathematics Working Group ([1]). At Statistical Methods courses often, students say that assessment questions or exercises performed during classes have a major difficulty that is to understand what is asked meaning the ability to read and comprehend the problem and to translate it into mathematical language. The study presented in this paper reflects an experience performed with second year students of Mechanical Engineering graduation of Coimbra Institute of Engineering, where the authors assessed Statistical Methods contents taught during the first semesters

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of the academic years 2017/2018 through 2019/2020. The questions assessment tests that were on the base of this study make part of RULES_MATH guide for Statistical Methods (one of the learning outcomes of the project) where questions are separated into two types: ones that are referred only to problem comprehension and its translation into what needed to be calculated and others where students need only to apply mathematical techniques in order to obtain the results. This paper is one of the results of RULES_MATH project which aims to develop tools to assess mathematical competencies. Eight mathematical competencies identified are recognized in the assessment made to students in what concerns learning probability theory concepts. Since 2017 a study was carried out with Mechanical Engineering students at Coimbra Institute of Engineering. The results obtained cover the test as a tool to assess competencies and, also its fitness to our students.

Keywords Assessment · Significant Learning · Competencies · Mathematics · Engineering

1 Introduction

Felder's Index of Learning Styles (ILS) has been used in several studies to characterize engineering students in general ([9, 10, 11]). Many other studies report the use of ILS in engineering students, concluding that they are mostly active, sensory, visual, and sequential individuals ([12, 13, 14]). The answers obtained from ISEC students in the academic year 2011/2012, ([8]) show that these students have a preference for an active learning style (78.4%), sensory (78.4%), visual (91.2%) and sequential (70.2%). The strong preference for visual style can be seen in Figure 1.

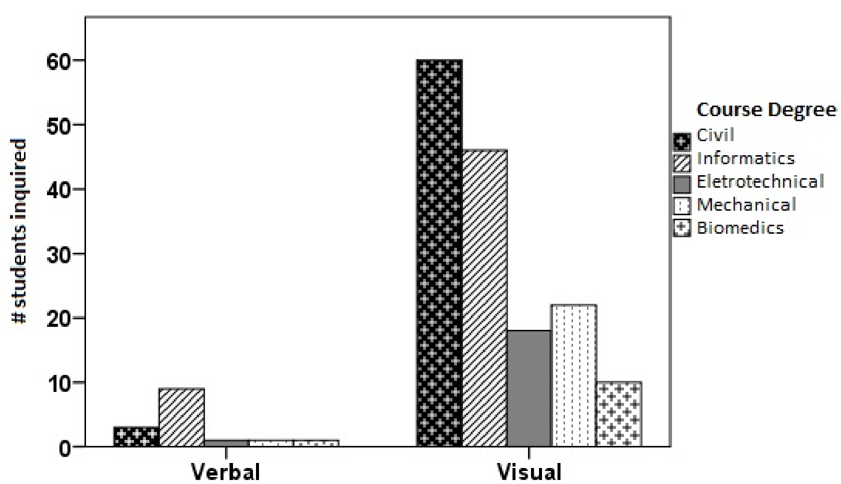


Figure 1: ISEC students'learning styles.

As the authors stated in [15], often the concept of mastering a subject does not have the same definition for students and math teachers. Regarding students we, as teachers, also should make a difference to which students we are teaching. Mathematics is, of course, the same but the usage that will be given to their math knowledge and math competence is different if they are going to be mathematicians or engineers or else. The authors are math teachers at Coimbra Engineering Institute and for them to teach math is much more than to transmit concepts and resolution methods. It also involves the ability of looking at a real-life problem and to be able of selecting, among all the variety of mathematical tools and concepts, the ones that may be applied to solve the problem in hand. The main objective of the RULES_MATH project is to develop assessment standards for a competencies-based teaching-learning system for mathematics in engineering education. The aims of the project can be summarizing as:

1. To develop a collaborative, comprehensive and accessible competencies-based assessment model for mathematics in engineering context.
2. To elaborate and collect the resources and materials needed to devise competencies-based assessment courses.
3. To disseminate the model to European Higher Education Institutions through the partner networks and promote the dissemination all over Europe.

The innovative idea of this project was to build the curriculum on the concept of mathematics competence. Niss and his colleagues developed a framework where eight clear and distinct mathematics competencies were distinguished: thinking mathematically, reasoning mathematically, posing and solving mathematical problems, modelling mathematically, representing mathematical entities, handling mathematical symbols and formalism, communicating in, with, and about mathematics and making use of aids and tools. Although the referred competencies may overlap a bit in terms of required abilities, each competence maintains a unique major focus, a distinct "centre of gravity" (Niss, [2], p. 9). The RULES_MATH project partners' working groups have developed a set of "Guide for a Problem" in the different areas of Mathematics that are intended to provide some examples of proposed forms of assessment and competence-based activities. The materials are available at https://rules-math.com/ and all project partners applied them to different students from different courses at their institutions. The "Guide for a Problem" presented in this paper is the one developed at IPC/ISEC for Statistical Methods. Its development, appliance and improvement were performed during the academic years 2017/2018, 2018/2019 and 2019/2020. The students with whom the study was carried out were from the second-year degree of Mechanical Engineering engaged in Probability and Statistical Methods curricular unit. The paper is organized as follows: In the section 2. the methodology, the participants and the instruments that were used is presented and in section 3. the authors conclusions and perspectives of future work are discussed.

2 Methodology, Target students and Research Instruments

The students that participated in the study were from the second-year degree of Mechanical Engineering engaged in Probability and Statistical Methods curricular unit. Each year a mean of 120 students engaged in this curricular unit, most of them male students. The concepts taught during a Statistical Methods course make use of different mathematical skills and competencies. Even though the authors made an effort to integrate real-life situations into the exercises proposed to Statistical Methods' students what they usually commented was:

1. “We have a lot of trouble trying to understand the problem and what teachers want us to calculate!”
2. “It is a different type of Mathematics . . . ”
3. “We need to know a large amount of real-life situations in order to solve the presented problems”

In order to motivate students to really engage themselves in the learning process, actual newspaper news, as shown in Figure 2. and 3., were discussed with them in an effort to show that contents were necessary to be an effective news reader and a critic person towards some articles that are object of public reading. Other real-life problems directly related with Mechanical Engineering passive to be solved using Probability and Statistical Methods were presented and discussed with students which were also invited to search and bring world situations to class and discuss them with their peers. In this way the authors believe that the third comment (2) posed by students was answered and solved.

Regarding the second comment (2) above, indeed the Mathematics that is discussed in this type of course is clearly different from Calculus or Algebra although it may use it the initial approach to the subject needs a comprehensive attitude from the subject/student and has its own literacy. A language and symbols dictionary needs to be constructed with students at the same time the contents are being explored, and the thinking and modelling mathematically are imperative competencies that this subject teachers need to take good care with and explore exhaustively.

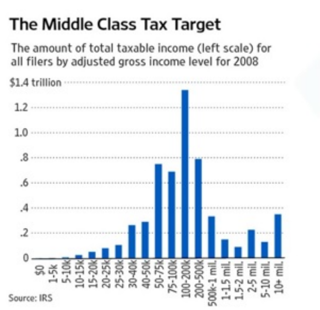


Figure 2: Newspaper news - Tax target

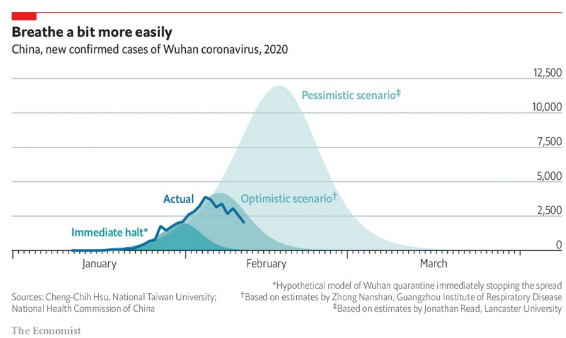
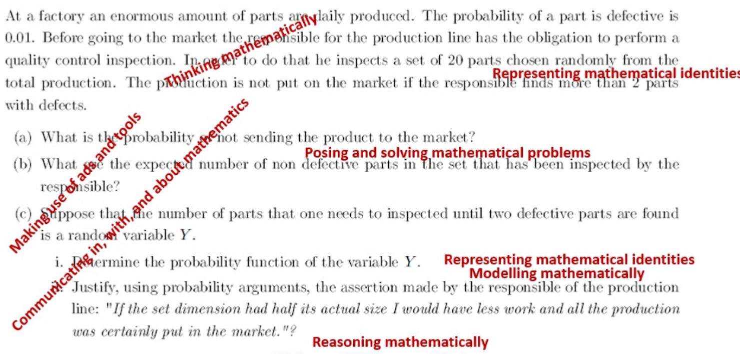


Figure 3: Covid-19 distribution - Internet information during Covid-19 pandemic

According to theoretical and applied evaluation of the literature, competence-based learning is becoming much more widely used in engineering education, and there is evidence to support its effectiveness in improving learning outcomes, meeting the needs of diverse student populations, and responding to industry’s demands for competent engineers [16].

The exercises proposed before 2017 to these students had the structure presented in Figure 4.



At a factory an enormous amount of parts are daily produced. The probability of a part is defective is 0.01. Before going to the market the responsible for the production line has the obligation to perform a quality control inspection. In order to do that he inspects a set of 20 parts chosen randomly from the total production. The production is not put on the market if the responsible finds more than 2 parts with defects.

(a) What is the probability of not sending the product to the market?

(b) What are the expected number of non defective parts in the set that has been inspected by the responsible?

(c) Suppose that the number of parts that one needs to inspected until two defective parts are found is a random variable Y.

i. Determine the probability function of the variable Y.

ii. Justify, using probability arguments, the assertion made by the responsible of the production line: *"If the set dimension had half its actual size I would have less work and all the production was certainly put in the market."*?

Figure 4: Old type of questions.

As one may observe from Figure 4. some of the competencies enumerated by Niss ([2]) and later on worked by the SEFI Mathematic Working Group ([1]) were not separated and, at the same question, they were assessed simultaneously what the authors consider that might be the reason for the first comment (2) made by students.

As a result of the investigation made during RULES_MATH project, a new approach to assessment was performed and the questions were, as much as possible, separated into calculus items and models identification and deduction items and inside each one of those questions'groups the competencies to be assessed were as much as possible also separated. The RULES_MATH project working groups have developed a set of “Guide for a Problem” files in the different areas of Mathematics that are intended to provide some examples of proposed forms of assessment and competence-based activities. The files were compiled in a book with ISBN: 978-619-202-575-5 and will be published during the year 2020. One of such activities is “Guide for a Problem. Statistics and Probability: SP3, SP4, SP5, SP6” which aims to evaluate students about probability and statistical methods. The tests designed were composed by three different parts: 5 essay direct questions identified as development questions, 2 essay questions identified as comprehension questions and 4 multiple choice questions.

To each question a matching table similar to the one presented at Figure 5. was constructed in order to evaluate which competencies were more and less acquired by students and where more work should be done, together by students and teachers, in order to clearly acquire statistical methods competence.

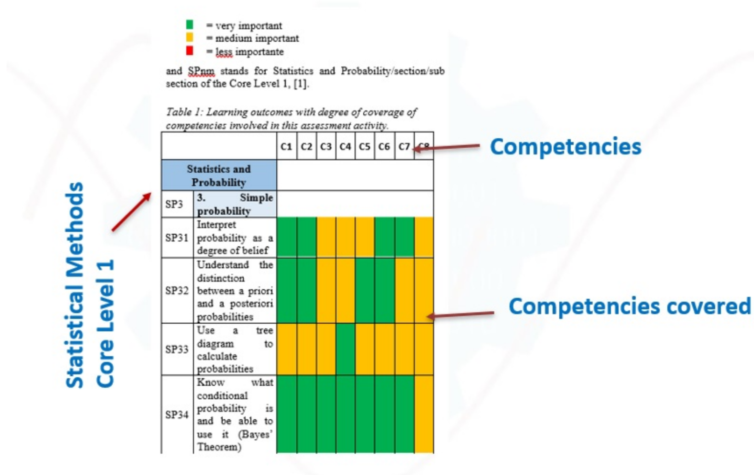


Figure 5: Learning outcomes with degree of coverage of competencies involved in this assessment activity (Question vs Competence assessed).

One example of the proposed questions is the one in Figure 6., and an example of a problem based learning used is shown in Figure 7.

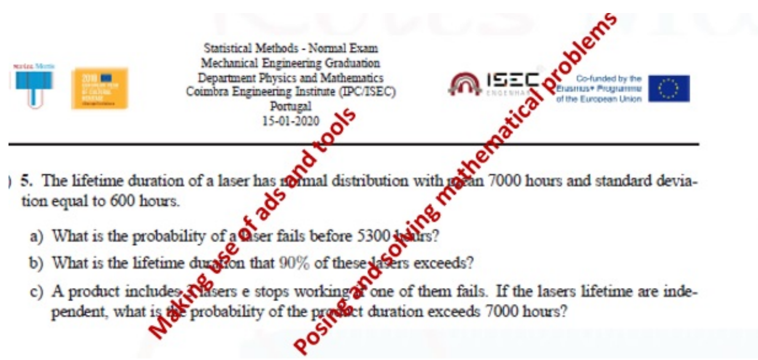


Figure 6: New type of questions.

Suppose that you are playing a computer game where the objective is to destroy a section of a railway line piloting an airplane (Figure 1). You are an element of the airplane crew that receives the order to destroy and you are the one that has to do the calculations and give the instructions to your fellow colleagues.

Figure 2. Railway section to be destroyed

2. To the crew of an airplane is assigned the task of destroying a section of railway line. The crew's aim statistics reveal that the variable X, distance, in meters, from the point of impact of a bomb to the targeted line, follows a Normal distribution with standard deviation $6m$. It is considered equally probable that the bomb falls to one side (for which it is agreed to take $X > 0$) or to the other (for which it is agreed to take $X < 0$) of the line. The mission is considered fulfilled if at least one bomb hits the target, that is, if it "falls to less than $1m$ of the line".
   a. Justify that X has mean value equal to zero.
   b. What is the probability of the bomb hits the target?
   c. Characterize the distribution function of the random variable Y = "Number of bombs that hit the target, supposing that n bombs are launched ($n \in \mathbb{N}$)".
   d. Suppose that $n > 20$ and determine, using the approximate distribution of Y, the minimum number of bombs that is necessary to launch in order to have a successful mission with probability greater than 0.95.

3. Suppose the weight, W, of each bomb is a random variable with Normal distribution with mean 65 kg and standard deviation $3kg$ and that the maximum load supported by the mission plane is 3 $tones$.

Figure 7: Problem-based learning example.

As the reader may observe only two competencies are present in this question and one of them (making use of aids and tools) regards the proper utilisation of a calculator which is present in almost questions where calculus have to be performed. Still in Statistical Methods the knowledge of distributions functions and procedures that are programmed in the calculator is also something that needs to be assessed and valued.

The students'grades were obtained from activities and from the two tests, Figure 8., performed with them. In order to understand if the students attendance to classes were or were not important a class attendance register was also performed and data analysis between classes attendance and grades obtained was also done.

Figure 8: Students at one of the tests performed.

3 Data Analysis, Conclusions and Perspectives of Future Work

From data analysis collected, grades from different groups of questions through the three academic years from 2017 to 2020 and classes attendance, the authors raised the conclusions presented bellow.

The students that passed Statistical Methods course were the ones that attended more than 50% of the classes as we can see in Figure 9.

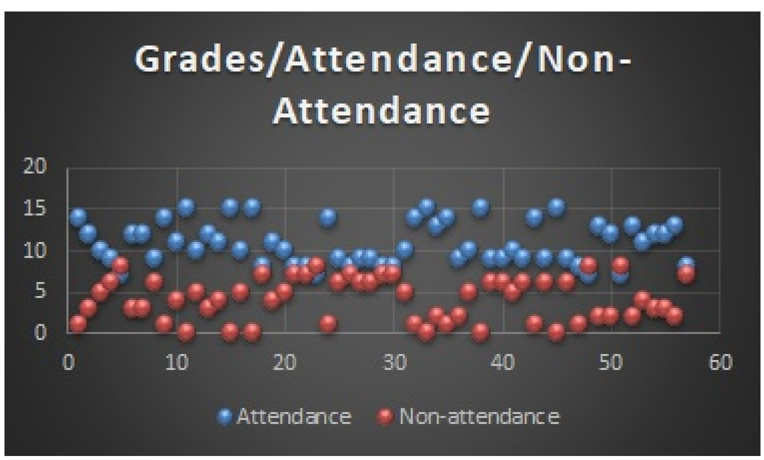


Figure 9: Grades versus classes attendance.

Considering the different questions groups explained above and afterwards dividing them into 2 major groups of questions, one regarding calculus and other regarding modelling questions we observed, as expected, that students have more difficulties on the modelling questions. The calculus questions are certainly more easily worked based on mimic and they do not involve the needed literacy referred above. Figure 10. confirms these conclusions

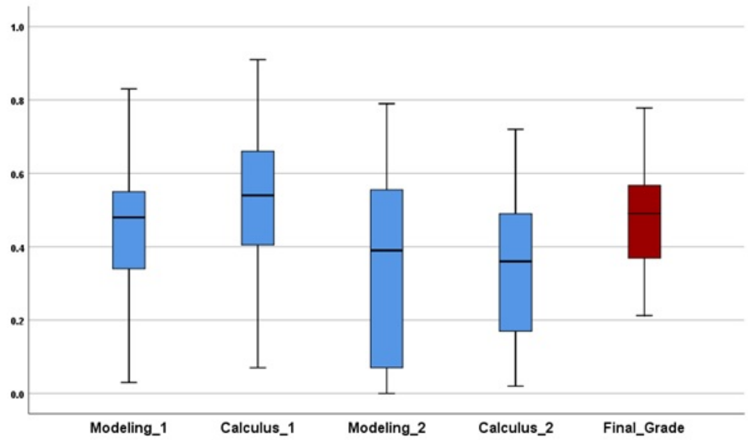


Figure 10: Calculus and Modelling questions comparison.

Notice that a very good students percentage grade, 76.3% of students present at the assessment moments, is above 10 out 20, which ensures students course approval as we can observe in Figure 11.

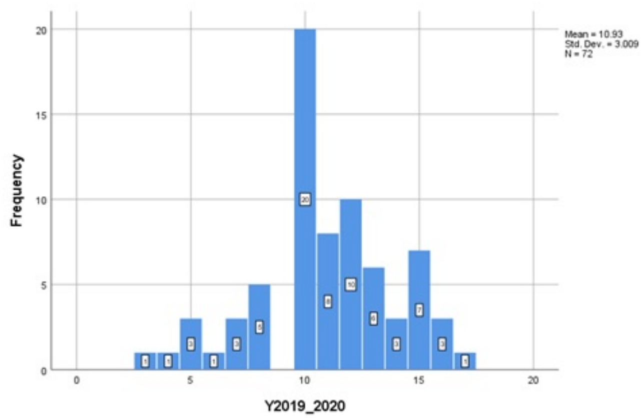


Figure 11: 2019/2020 Grades Histogram.

Comparing grades obtained through the 3 consecutive years (from 2017 to 2020), Figure 12., we may observe that the final grades globally are not that different although the maximum grade is higher than it used to be. In spite the referred just before, the authors believe that competence-methodology empowers students with a better preparation to face and deal with real-life problems, students became more critics and analysed questions and solutions in a more professional way.

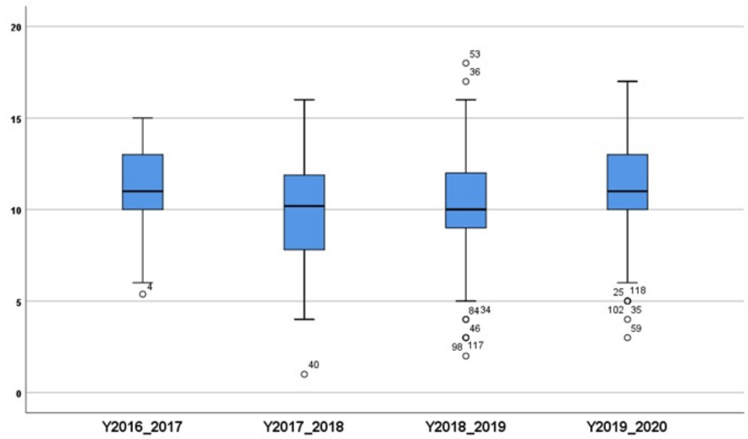


Figure 12: Grades comparison 2017/2018/2019/2020.

Another conclusion that brought great satisfaction to the authors, see Table 1., was that students are more engaged in the exams. Throughout the past decade we observed that students were increasing their absence to assessment moments and that fact was seriously a preoccupation. With competence-methodology we observe a generous setback on this situation (the last year is not an example for this observation since we faced the Covid-19 pandemic which spoil the second test calm performance).

	Valid N	Valid percent	Missing N	Missing Percent	Total N	Total Percent
Y2016_2017	53	44.2%	67	55.8%	120	100%
Y2017_2018	57	47.5%	63	52.5%	120	100%
Y2018_2019	85	70.8%	35	29.2%	120	100%
Y2019_2020	72	60.0%	48	40.0%	120	100%

Table 1: Students presence at exams through years.

From overall investigation made on this subject in the last 3 years RULES_MATH project duration we noticed that what used to be done, previous 2017, were assessment tests and a small amount of activities. Throughout the project the amount of activities increased and are leveled with the assessment tests and the results are very promising.

In general next year as future work, we have to make a continuous assessment probably during classes in order to know which competencies are less acquired by students and immediately solve those issues. Other preoccupation that the authors are going to deal with, is the initial students level up. Students at IPC/ISEC arrive from different secondary schools and different contents programs: some of them are "regular" students and others come from professional studies where the mathematical basic concepts are not dealt with the same depth as "regular" students do. Although students arriving from professional studies are, in general, more able to discuss and criticize results, problem solving techniques and even some concepts and notation are missing. As an ideal each student should be regarded as a person with his learning style and his objectives to achieve. Nevertheless the impossibility of individual attention, the student center teaching and learning process together with competence-based methodology and a previous Index of Learning Styles procedure to the course students will certainly increase students assessment results and their in market performance.

Acknowledgement

The authors would like to acknowledge the financial support of Project Erasmus+ 2017-1-ES01-KA203-038491 "New Rules for Assessing Mathematical Competencies".

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Assessing Knowledge and Competencies: RULES_MATH Project's effects at ISEC

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Abstract

In higher education, mathematics has an important role in engineering courses. From any engineering course curriculum there are, mostly in the first course year, curricular units (CU) in the area of mathematics that are fundamental for students to acquire the necessary basic knowledge and competence to the most specific CU of their course. Without this well-established mathematical foundation, success in the engineering CUs is seriously compromised. During an engineering course, students learn and consolidate the basic principles of mathematics to solve practical problems, reinforcing their mathematical concepts knowledge and competence. However, although mathematics is a basic discipline in admission to engineering courses, difficulties are identified by Engineering students in CUs related to Mathematics basic core. In this context, it seems pertinent to identify the mathematics competencies attained by engineering students so that they can use these skills in other CU and take them as a tool support to their professional activities. In this paper we present some results obtained while participating in project RULES_MATH, namely ways to assess competencies in Calculus, Algebra and Statistical Methods curricular units. The study presented in this paper reflects an experience performed with second year students of Mechanical Engineering graduation and with first year students of Electrotechnical Engineering, Biomedical Engineering, Electromechanics and Mechanics of Coimbra Institute of Engineering, where the authors assessed curricular units contents taught during the first and second semesters from 2017/2018 to 2019/2020 academic years. This paper is one of the results of RULES_MATH project which aims to develop tools to assess mathematical competencies. The eight mathematical competencies already identified (thinking mathematically; reasoning mathematically; posing and solving mathematical problems; modelling mathematically; representing mathematical entities; handling mathematical symbols and formalism; communicating in, with, and about mathematics and make use of aids and tools for mathematical activity) are recognized in the assessment made to students in what concerns learning concepts. The authors will present the changes performed in their curricular units regarding teaching and assessment methodologies as a result of the RULES_MATH project development and show how students react to them.

***K*eywords** Assessment · Significant Learning · Competencies · Mathematics · Engineering

1 Introduction

Learning takes place in students' heads where it is invisible to us as teachers. This means that learning must be assessed through performance: what students can do with their learning and how do they learn.

Felder's Index of Learning Styles (ILS) has been used in several studies to characterize engineering students in general ([9, 10, 11]). Many other studies report the use of ILS in engineering students, concluding that they are mostly active, sensory, visual, and sequential individuals ([12, 13, 14]). The answers obtained from IPC/ISEC students in the academic year 2011/2012, ([8]) show that these students have a preference for an active learning style (78.4%), sensory (78.4%), visual (91.2%) and sequential (70.2%). The strong preference for visual style can be seen in Figure 1.

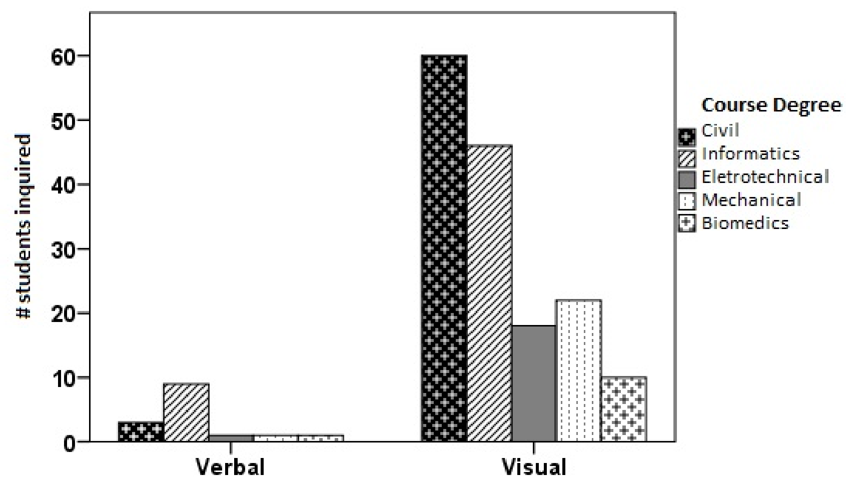


Figure 1: ISEC students'learning styles.

Assessing students' performance can involve assessments that are individual or collective. The educational assessment process presents interdependence between knowledge, learning and competencies development. In this perspective, knowledge and competencies are processes that are articulated, but not confused. In fact, the use of knowledge is a strong requirement in the process of building competencies. However, the more human actions require the deepening or organization of knowledge, the more time is required for the development of competencies. Thus, one of the major future challenges facing evaluation processes is to privilege, among the aspects to be evaluated, the development of competencies. According to B. Alpers et al. in the Framework for Mathematical Curricula in Engineering Education [1], previously proposed in the Danish KOM project [2], the mathematical competencies are: (C1) thinking mathematically; (C2) reasoning mathematically; (C3) posing and solving mathematical problems; (C4) modelling mathematically; (C5) representing mathematical entities; (C6) handling mathematical symbols and formalism;

(C7) communicating in, with, and about mathematics and (C8) make use of aids and tools for mathematical activity. Although the referred competencies may overlap a bit in terms of required abilities, each competence maintains a unique major focus, a distinct "centre of gravity" (Niss, [2], p. 9). Thinking and reasoning about this subject, the "New Rules for assessing Mathematical Competencies"was proposed. This project intends to change the educational paradigm and to get a common European teaching and learning system based on mathematical competencies rather than contents [https://rules-math.com/], [3].

The main objective of the RULES_MATH project was to develop assessment standards for a competencies-based teaching-learning system for Mathematics in Engineering education. The aims of the project can be summarizing as:

1. To develop a collaborative, comprehensive and accessible competencies-based assessment model for mathematics in engineering context.
2. To elaborate and collect the resources and materials needed to devise competencies-based assessment courses.
3. To disseminate the model to European Higher Education Institutions through the partner networks and promote the dissemination all over Europe.

Often the concept of mastering a subject does not have the same definition for students and math teachers. Regarding students we, as teachers, also should make a difference to which students we are teaching. Mathematics is, of course, the same but the usage that will be given to their math knowledge is different if they are going to be mathematicians or engineers or else.

During an engineering course, students learn and consolidate the basic principles of mathematics and learn other advanced ones to solve practical problems, reinforcing their conceptual mathematical knowledge. However, although mathematics is a basic discipline regarding the admission to any engineering degree, difficulties related to Mathematics' basic core are identified by almost all engineering students at each curricular unit. In this context, it seems relevant to identify the mathematics competencies attained by engineering students so that they can use these skills in their professional activities. Mathematical competence is the ability to apply mathematical concepts and procedures in relevant contexts which is the essential goal of Mathematics in Engineering Education. Thus, the fundamental aim is to help students to work with engineering models and solve engineering problems [1].

The teaching and learning process in Mathematics during an engineering course academic year at University or Polytechnic Institute has special characteristics. First-year students come from diverse high schools with different levels of knowledge and skills and varied personal attitudes toward mathematical education. Overcoming these differences is crucial to achieve the objectives of higher education-training highly qualified specialists for the labour market. In addition to knowledge and skills, students need to possess various abilities to apply their knowledge and skills in solving practical problems, too. Gaps were detected between engineers' required mathematic competencies and acquired mathematics competencies of engineering students under the current engineering mathematics curriculum. To revise the mathematics curriculum of engineering education making the achievement of the mathematics competencies more explicit in order to bridge this gap and prepare students to acquire enough mathematical competencies (RULES_MATH Project, [5, 6]). Hence an important aspect in mathematics education for engineers is to identify mathematical competencies explicitly and to recognize them as an essential aspect in teaching and learning in higher education. It is the fundamental that all mathematics teaching must aim at promoting the development of pupils' into student and develop their mathematical competencies and (different forms of) overview and judgment [1, 3, 6].

This paper presents the RULES_MATH impact at IPC/ISEC Mathematics curricular units of Calculus I, Algebra and Statistical Methods and to evaluate and recognize what are the competencies that Engineering students must have or, acquire, when algebra, mathematical analysis and statistical methods contents are taught to them.

The innovative idea of this project was to build the curriculum assessment on the concept of mathematics competence.

The RULES_MATH project partners' working groups have developed a set of "Guide for a Problem" in the different areas of Mathematics that are intended to provide some examples of proposed forms of assessment and competence-based activities. The materials are available at https://rules-math.com/ and all project partners applied them to different students from different courses at their institutions. The "Guide for a Problem" regarding the above referred curricular units are presented in this paper as they were the ones applied at IPC/ISEC. Its development, appliance and improvement were performed during the academic years 2017/2018, 2018/2019 and 2019/2020. The students with whom the study was performed were second year students of Mechanical Engineering graduation and with first year students of Electrotechnical Engineering, Biomedical Engineering, Electromechanics and Mechanics of Coimbra Institute of Engineering. The paper is organized as follows: In the section 2. the methodology, the participants and the instruments that were used is presented and in section 3. the authors conclusions and perspectives of future work are discussed.

2 Methodology, Target students and Research Instruments

According to theoretical and applied evaluation of the literature, competence-based learning is becoming much more widely used in engineering education, and there is evidence to support its effectiveness in improving learning outcomes, meeting the needs of diverse student populations, and responding to industry's demands for competent engineers [20].

A variety of techniques were experimented with students at IPC/ISEC. Students were motivated to engage in the mathematical learning process with real-life problems, [6, 11, 13], invited to look up at news, as shown in Figure 2., 3. and 4., where data conclusions (some of them wrongly taken) were presented to readers, as a sort of awakening process to the necessity of learning statistical methods properly and were also invited to produce videos about subject contents that reflected their own learning process and were evaluated by their peers (Figure 4).

Figure 2: Determination of the Iberian Peninsula Perimeter.

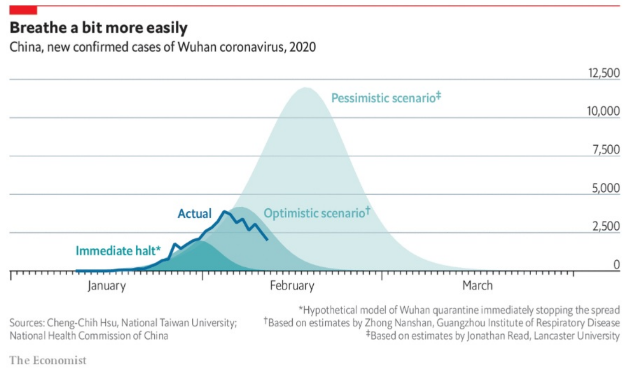


Figure 3: Covid-19 distribution - Internet information during Covid-19 pandemic.

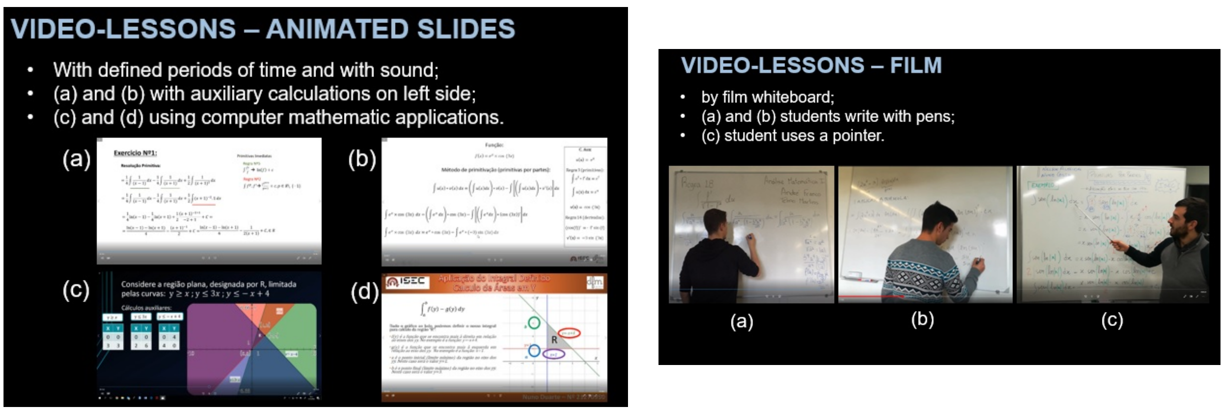


Figure 4: Students videos examples.

Researchers involved in RULES_MATH consortium presented and publish from 2017 to 2020 several papers, [[5] - [14]] where specific examples focusing the main aspects of the assessment standards and activities to be implemented as an innovative pedagogical approach in engineering and science degrees. The approach is grounded on a competence-based methodology changing some of the traditional teaching methods, learning, and assessment and assigns an active role to student in the educational process, [7].

From the investigation made during the last 3 years (2017 to 2020) the researchers belonging to RULES_MATH consortium developed a set of files designated "Guide for Problem" applied to different mathematics course subjects namely: Analysis and Calculus, Linear Algebra, Discrete Mathematics, Geometry and Statistical Methods. The files were compiled in a book with ISBN: 978-619-202-575-5 and will be published during the year 2020. The authors applied on their students at IPC/ISEC the guides regarding Analysis and Calculus, Linear Algebra and Statistical Methods. As referred in this paper introduction, the students with whom the study was performed were second year students of Mechanical Engineering graduation and with first year students of Electrotechnical Engineering, Biomedical Engineering, Electromechanics and Mechanics of Coimbra Institute of Engineering. The study was conducted from 2017 until 2020 being each academic year evaluated and adjusted by the authors.

On each guide a matching between competencies and questions, see as an example Figure 5., is provided in order to better understand which competencies are more or less acquired by students. The colors code is the following: green represents that in that question to the evaluation of that competence is given high importance, yellow a medium importance and red is given less importance to the evaluation of that competence.

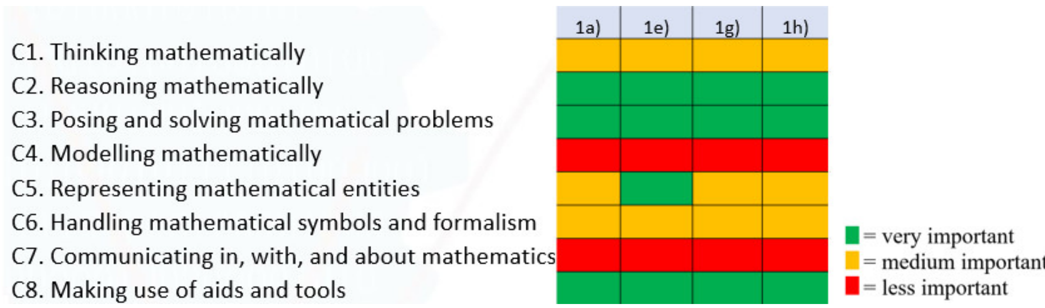


Figure 5: Matching competencies and questions.

The subjects covered are the ones stated at SEFI mathematics working group document, [1], regarding Mathematics core level 1. As an example for Probability and Statistical Methods Figure 6. shows the matching between those subjects and the competencies assessed in the guide for problem proposed.

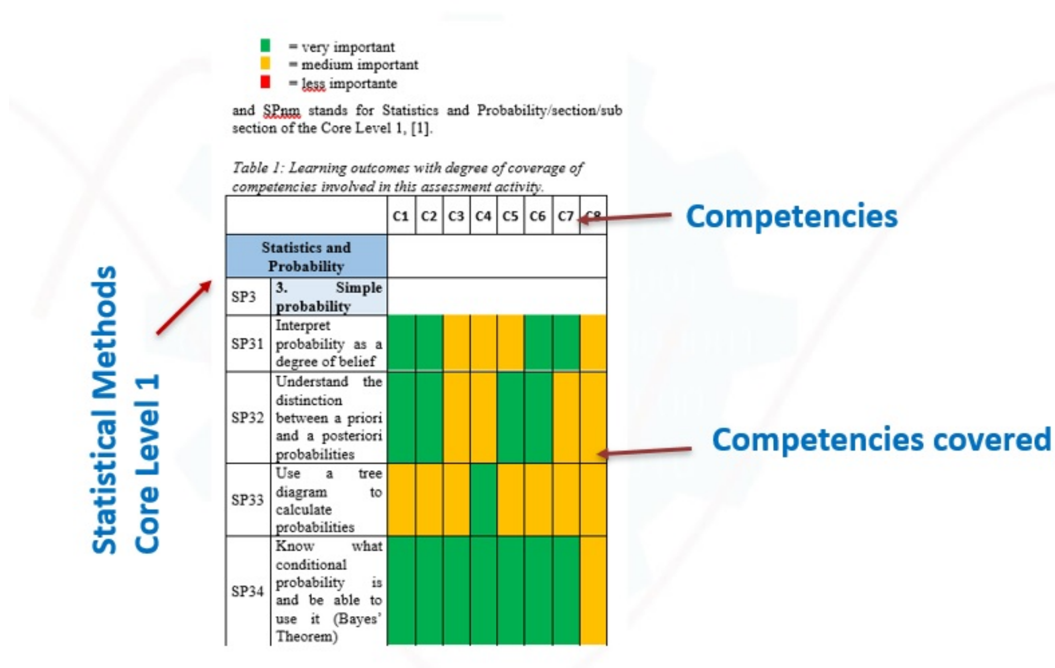
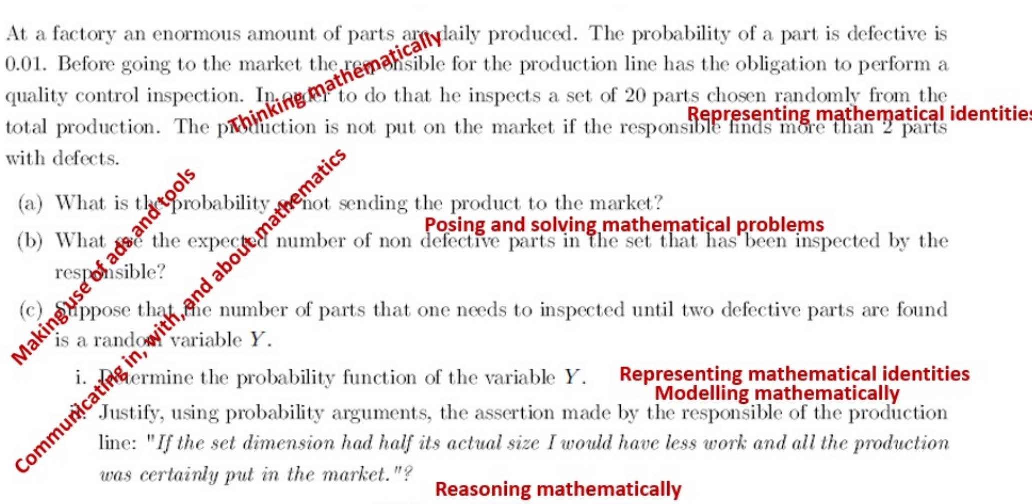


Figure 6: Matching competencies and SEFI-MWG core level 1 subjects.

Regarding Statistical Methods, students often state that the involved mathematics is different. Indeed the Mathematics that is discussed in this type of course is clearly different from Calculus or Algebra although it may use it the initial approach to the subject needs a comprehensive attitude from the subject/student and has its own literacy. A language and symbols dictionary needs to be constructed with students at the same time the contents are being explored, and the thinking and modelling mathematically are imperative competencies that this subject teachers need to take good care with and explore exhaustively.

The exercises proposed before 2017 to these students had the structure presented in Figure 7.



At a factory an enormous amount of parts are daily produced. The probability of a part is defective is 0.01. Before going to the market the responsible for the production line has the obligation to perform a quality control inspection. In order to do that he inspects a set of 20 parts chosen randomly from the total production. The production is not put on the market if the responsible finds more than 2 parts with defects.

(a) What is the probability of not sending the product to the market?

(b) What are the expected number of non defective parts in the set that has been inspected by the responsible?

(c) Suppose that the number of parts that one needs to inspected until two defective parts are found is a random variable Y.

i. Determine the probability function of the variable Y.

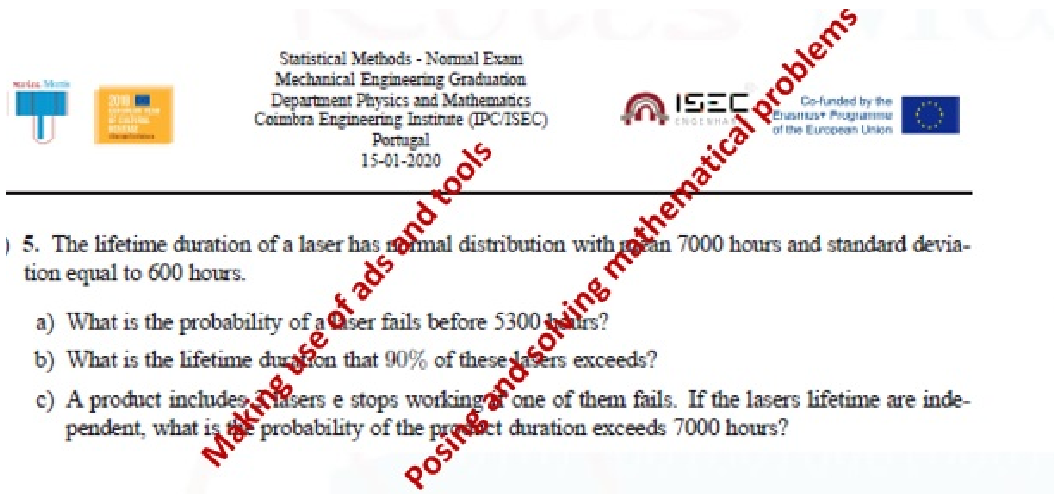
ii. Justify, using probability arguments, the assertion made by the responsible of the production line: "*If the set dimension had half its actual size I would have less work and all the production was certainly put in the market.*"?

Figure 7: Old type of questions.

As one may observe from Figure 7. some of the competencies enumerated by Niss ([2]) and later on worked by the SEFI Mathematic Working Group ([1]) were not separated and, at the same question, they were assessed simultaneously what the authors consider that might be the reason why students get confused in what reasoning should be performed in order to solve the questions correctly.

As a result of the investigation made during RULES_MATH project, a new approach to assessment was performed and the questions were, as much as possible, separated into calculus items and models identification and deduction items and inside each one of those questions'groups the competencies to be assessed were as much as possible also separated.

One example of the proposed questions is the one in Figure 8., and an example of a problem based learning used is shown in Figure 9.


Statistical Methods - Normal Exam
Mechanical Engineering Graduation
Department Physics and Mathematics
Coimbra Engineering Institute (IPC/ISEC)
Portugal
15-01-2020

5. The lifetime duration of a laser has normal distribution with mean 7000 hours and standard deviation equal to 600 hours.

a) What is the probability of a laser fails before 5300 hours?

b) What is the lifetime duration that 90% of these lasers exceeds?

c) A product includes 3 lasers e stops working if one of them fails. If the lasers lifetime are independent, what is the probability of the product duration exceeds 7000 hours?

Figure 8: New type of questions.

As the reader may observe only two competencies are present in this question and one of them (making use of aids and tools) regards the proper utilisation of a calculator which is present in almost questions where calculus have to be performed. Still in Statistical Methods the knowledge of distributions functions and procedures that are programmed in the calculator is also something that needs to be assessed and valued.

Suppose that you are playing a computer game where the objective is to destroy a section of a railway line piloting an airplane (Figure 1). You are an element of the airplane crew that receives the order to destroy and you are the one that has to do the calculations and give the instructions to your fellow colleagues.

Figure 2. Railway section to be destroyed

2. To the crew of an airplane is assigned the task of destroying a section of railway line. The crew's aim statistics reveal that the variable X, distance, in meters, from the point of impact of a bomb to the targeted line, follows a Normal distribution with standard deviation $6m$. It is considered equally probable that the bomb falls to one side (for which it is agreed to take $X > 0$) or to the other (for which it is agreed to take $X < 0$) of the line. The mission is considered fulfilled if at least one bomb hits the target, that is, if it "falls to less than $1m$ of the line".
   a. Justify that X has mean value equal to zero.
   b. What is the probability of the bomb hits the target?
   c. Characterize the distribution function of the random variable Y = "Number of bombs that hit the target, supposing that n bombs are launched ($n \in \mathbb{N}$)".
   d. Suppose that $n > 20$ and determine, using the approximate distribution of Y, the minimum number of bombs that is necessary to launch in order to have a successful mission with probability greater than 0.95.

3. Suppose the weight, W, of each bomb is a random variable with Normal distribution with mean 65 kg and standard deviation $3kg$ and that the maximum load supported by the mission plane is 3 *tones*.

Figure 9: Problem-based learning example.

The students'grades were obtained from activities and from the two tests, Figure 10., performed with them. In order to understand if the students attendance to classes were or were not important a class attendance register was also performed and data analysis between classes attendance and grades obtained was also done.

Figure 10: Students at one of the tests performed.

3 Data Analysis, Conclusions and Perspectives of Future Work

Regarding Calculus I, the authors concluded that a deeper reflection about "Student's comprehension and competencies achievement of the contents taught" related to different background when students enter higher education system is needed. Students did not acquire all the competencies and knowledge that we intended to. In general students obtained negative grades (less than 50%) due to the lack of bases in mathematics, not being motivated and having huge difficulties in mathematics. In this sense, many students need a personal attention (office hours, extra work, etc.) because their results were clearly negative. Another reflection that we also want to make is regarding "The questions itself and its usefulness in assessing competencies". We concluded that, the questions proposed in RULES_MATH guide for Analysis and Calculus covers most of the competencies that we need to evaluate. The questions difficulty is adequate to our students, in spite many of them fail to obtain positive grades. Therefore, in next year, we will utilize similar questions. We will also choose questions that allow students to test all 8 mathematical skills in a balanced way and collect throughout the year the feedback on the skills least acquired by students so that we can obtain a formative assessment. To assess communication-competence we will use other activities such as students' videos, oral presentations, etc.

With respect to Algebra, the authors concluded although the assessment cycle focuses on self-assessment, peer assessment can help learners develop a variety of skills, including collaboration, communication, conceptual understanding and problem-solving skills [21]. Likewise, providing feedback can help improve students' communication more than just analysing their work, because students practice communicating their ideas to other students. In this way, learning outcomes can be improved and the mathematical competence C8 will be more evaluated in assessment. Thus, first it is necessary to reflect on the mathematical competencies that questions must present to be useful in the evaluation of students and second to reflet on the mathematical competencies that the students acquired about the contents taught. Although the questions proposed by RULES_MATH guide covers most of the competencies that we need to evaluate and the difficulty is adequate to our students, a given assessment structure may include all mathematical competencies at different scales according to the learning outcomes that the student should acquired. Most mathematical competencies and knowledge were acquired by students, obtaining positive grades except for questions that are closely related to the C1 and C2 competencies, the competencies in which students have greater difficulty in acquiring. It was also found that some students need a personal attention in some learning outcomes and the competencies associated with them.

A next step for our research is to use the mathematical competencies to design new activities structures, and use it as a guide for future research studies to further test and validate the model. From data analysis collected, grades from different groups of questions through the three academic years from 2017 to 2020 and classes attendance, the authors raised the conclusions presented bellow.

In Statistical Methods curricular unit the students that passed were the ones that attended more than 50% of the classes as we can see in Figure 11.

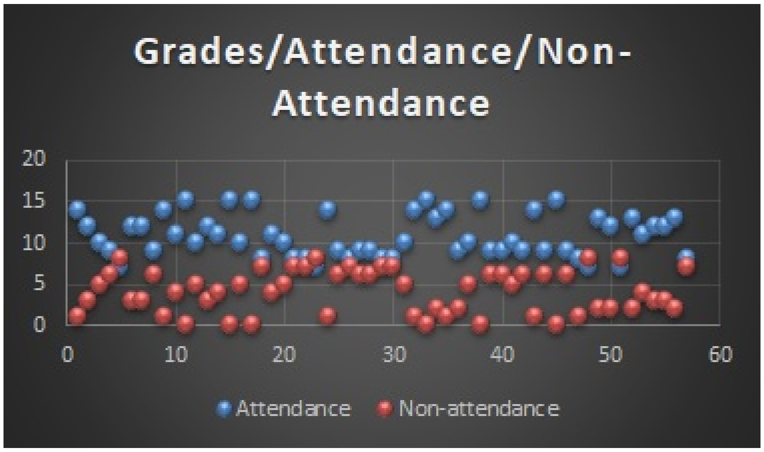


Figure 11: Grades versus classes attendance.

Considering the different questions groups (multiple choice question, development and problem based questions) explained above and afterwards dividing them into 2 major groups of questions, one regarding calculus and other regarding modelling questions we observed, as expected, that students have more difficulties on the modelling questions. The calculus questions are certainly more easily worked based on mimic and they do not involve the needed literacy referred above. Figure 12. confirms these conclusions

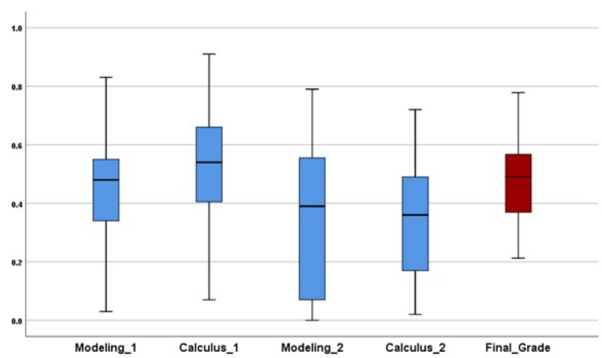


Figure 12: Calculus and Modelling questions comparison.

Notice that a very good students percentage grade, 76.3% of students present at the assessment moments, is above 10 out 20, which ensures students course approval as we can observe in Figure 13.

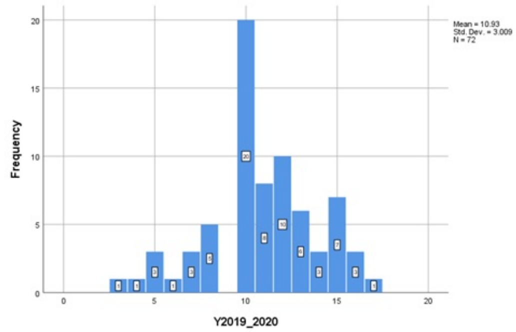


Figure 13: 2019/2020 Grades Histogram.

Comparing grades obtained through the 3 consecutive years (from 2017 to 2020), Figure 14., we may observe that the final grades globally are not that different although the maximum grade is higher than it used to be. In spite the referred just before, the authors believe that competence-methodology empowers students with a better preparation to face and deal with real-life problems, students became more critics and analysed questions and solutions in a more professional way.

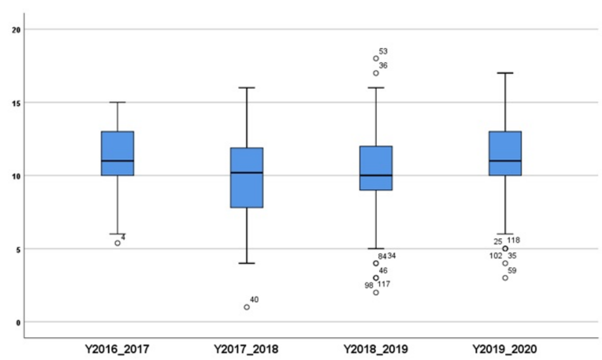


Figure 14: Grades comparison 2017/2018/2019/2020.

Another conclusion that brought great satisfaction to the authors, see Table 1. as na example of what happened in Statistical Methods course, was that students are more engaged in the exams. Throughout the past decade we observed that students were increasing their absence to assessment moments and that fact was seriously a preoccupation. With competence-methodology we observe a generous setback on this situation (the last year is not an example for this observation since we faced the Covid-19 pandemic which spoil the second test calm performance).

	Valid N	Valid percent	Missing N	Missing Percent	Total N	Total Percent
Y2016_2017	53	44.2%	67	55.8%	120	100%
Y2017_2018	57	47.5%	63	52.5%	120	100%
Y2018_2019	85	70.8%	35	29.2%	120	100%
Y2019_2020	72	60.0%	48	40.0%	120	100%

Table 1: Students presence at exams through years.

From overall investigation made on this subject in the last 3 years RULES_MATH project duration we noticed that what used to be done, previous 2017, were assessment tests and a small amount of activities. Throughout the project the amount of activities increased and are leveled with the assessment tests and the results are very promising.

In general next year as future work, we have to make a continuous assessment probably during classes in order to know which competencies are less acquired by students and immediately solve those issues. Other preoccupation that the authors are going to deal with, is the initial students level up. Students at IPC/ISEC arrive from different secondary schools and different contents programs: some of them are "regular" students and others come from professional studies where the mathematical basic concepts are not dealt with the same depth as "regular" students do. Although students arriving from professional studies are, in general, more able to discuss and criticize results, problem solving techniques and even some concepts and notation are missing. As an ideal each student should be regarded as a person with his learning style and his objectives to achieve. Nevertheless the impossibility of individual attention, the student center teaching and learning process together with competence-based methodology and a previous Index of Learning Styles procedure to the course students will certainly increase students assessment results and their in market performance.

Acknowledgement

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SOME CHARACTERIZATIONS FOR SPLIT QUATERNIONS

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ABSTRACT

At this paper, we introduce some concepts of split quaternions with quaternion coefficients. To verify some results, we benefit from 4x4 quaternion coefficients matrix representation. Moreover, the difference between split quaternions with quaternion coefficients and quaternions with complex coefficients is investigated.

***K*eywords** Quaternions · Split quaternions · Quaternions with quaternion coefficients

1 Introduction

Quaternion algebra, introduced by W. Hamilton, plays significiant role in several areas of science; such as differential geometry, theory of relativity, engineering, etc. After the discovery of quaternions, split quaternions or coquaternions were initially introduced by J. Cackle. Both quaternion and split quaternion algebras are associative and non-commutative 4 -dimensional Clifford algebras. The properties and roots of quaternions and split quaternions are given in detail, see [4], [8], [10].
Matrix representation for quaternions is generally used for calculating some algebraic properties in quaternion algebra. Thus, quaternions and matrices of quaternions were studied by many authors in literature, see [5], [9]. Especially, in [5], split quaternions and split quaternion matrices were considered. Using properties of complex matrices, split quaternion matrices were investigated and the complex adjoint matrix of split quaternion matrices was defined.
A brief introduction of the generalized quaternions is provided in [2]. Split Fibonacci quaternions, split Lucas quaternions and split generalized Fibonacci quaternions were defined in [6]. The correspondence among these quaternions were given in the same study. Similarly, split Pell and split Pell- Lucas quaternions were defined in [7]. In that study, many identities between split Pell and split Pell- Lucas quaternions were mentioned.
Our motivation for this study is quaternions with complex coefficients. Some algebraic concepts for complex quaternions and complex split quaternions were given in [1], [3]. In these studies, 4x4 quaternion coefficients matrix representation was used. Moreover, the correspondence between

complex quaternions and complex split quaternions were discussed in detail.
However, there is no research about split quaternions with quaternion coefficients in literature. Hence, we give new definition for split quaternions with quaternion coefficients. This study is organized as follows: In Section 2, a brief summary of real, complex and complex split quaternions are given. In Section 3, the properties of split quaternions with quaternion coefficients are introduced. Furthermore, the correspondence between complex split quaternions and split quaternions with quaternion coefficients are represented. In Secion 4, obtained consequences are discussed.

2 Preliminaries

In this section, firstly, brief summary of real and complex quaternions are represented. Secondly, split and complex split quaternions are given. Moreover, some properties of these quaternions are denoted.

Definition 2.1. *A real quaternion is defined as*

$$q = q_0e_0 + q_1e_1 + q_2e_2 + q_3e_3, \tag{1}$$

where q_0, q_1, q_2 and q_3 are real numbers and e_0, e_1, e_2, e_3 of q are four basic vectors of Cartesian set of coordinates which satisfy the non-commutative multiplication conditions:

$$\begin{aligned} e_1^2 &= e_2^2 = e_3^2 = e_1e_2e_3 = -1, \\ e_1e_2 &= e_3 = -e_2e_1, \;\; e_2e_3 = e_1 = -e_3e_2, \;\; e_3e_1 = e_2 = -e_1e_3. \end{aligned}$$

The set of quaternions can be represented as

$$H = \{q = q_0e_0 + q_1e_1 + q_2e_2 + q_3e_3 : q_0, q_1, q_2, q_3 \in IR\}, \tag{2}$$

where it is a 4-dimensional vector space on IR. A real quaternion is defined as a couple (S_q, V_q). Here $S_q = q_0e_0 \in IR$ is scalar part and $V_q = q_1e_1 + q_2e_2 + q_3e_3 \in IR^3$ is vector part of q, respectively.
The conjugate of q is

$$\bar{q} = S_q - V_q.$$

Also the norm of quaternion is given as

$$\|q\| = \sqrt{q\bar{q}} = \sqrt{q_0^2 + q_1^2 + q_2^2 + q_3^2}. \tag{3}$$

If $\|q\| = 1$, then q is called unit real quaternion. The inverse of the real quaternion q is

$$q^{-1} = \frac{\bar{q}}{\|q\|^2}, \;\; \|q\| \neq 0.$$

Theorem 2.1. *For $p, q \in H$, the following properties are satisfied, for more details see [1]:*

$$
\begin{array}{ll}
(i) & \bar{\bar{q}} = q, \\
(ii) & \overline{pq} = \bar{q}\bar{p}, \\
(iii) & \|qp\| = \|q\|\|p\|, \\
(iv) & \|q^{-1}\| = \dfrac{1}{\|q\|}.
\end{array}
$$

A complex quaternion q is in the form of $q = A_0e_0 + A_1e_1 + A_2e_2 + A_3e_3$, where A_0, A_1, A_2, A_3 are complex numbers and the elements of $\{e_0, e_1, e_2, e_3\}$ multiply as in real quaternions.

For any complex quaternion $q = A_0e_0 + A_1e_1 + A_2e_2 + A_3e_3$, the quaternion conjugate of q is defined as

$$\bar{q} = A_0e_0 - A_1e_1 - A_2e_2 - A_3e_3.$$

The norm is

$$q\bar{q} = A_0^2 + A_1^2 + A_2^2 + A_3^2.$$

The complex and Hermittian conjugates are given as follows, see [1]:

$$
\begin{aligned}
q^c &= \bar{A}_0e_0 + \bar{A}_1e_1 + \bar{A}_2e_2 + \bar{A}_3e_3, \\
(\bar{q})^c &= \bar{A}_0e_0 - \bar{A}_1e_1 - \bar{A}_2e_2 - \bar{A}_3e_3.
\end{aligned}
$$

Theorem 2.2. *For any complex quaternions q and p, the following properties are satisfied, see [1]:*

$$
\begin{array}{ll}
(i) & \bar{\bar{q}} = q, \\
(ii) & \overline{pq} = \bar{q}\bar{p}, \\
(iii) & \|qp\| = \|q\|\|p\|, \\
(iv) & (q^c)^c = q, \\
(v) & (qp)^c = q^c p^c.
\end{array}
$$

Definition 2.2. *A complex split quaternion p is a vector form of the $p = b_0e_0 + b_1e_1 + b_2e_2 + b_3e_3$, where b_0, b_1, b_2, b_3 are complex numbers.$\{e_0, e_1, e_2, e_3\}$ denotes complex split quaternion basis with the below properties:*

$$
\begin{aligned}
e_1^2 &= -1, \;\; e_2^2 = e_3^2 = e_1e_2e_3 = 1, \\
e_1e_2 &= e_3 = -e_2e_1, \;\; e_2e_3 = e_1 = -e_3e_2, \;\; e_3e_1 = e_2 = -e_1e_3.
\end{aligned}
$$

For details of complex split quaternions, see [3].

Definition 2.3. *A complex split quaternion matrix P is of the form*

$$P = P_0E_0 + P_1E_1 + P_2E_2 + P_3E_3, \tag{4}$$

where P_0, P_1, P_2 and P_3 are complex numbers and the split quaternion matrix basis $\{E_0, E_1, E_2, E_3\}$ of P satisfy the following equalities, [1]:

$$
\begin{aligned}
E_1^2 &= -E_0, \;\; E_2^2 = E_3^2 = E_0, \\
E_1E_2 &= E_3 = -E_2E_1, \;\; E_2E_3 = E_1 = -E_3E_2, \;\; E_3E_1 = E_2 = -E_1E_3.
\end{aligned}
$$

The conjugate of P is

$$\overline{P} = S_P - V_P. \tag{5}$$

Moreover, the norm of complex split quaternion is

$$\begin{aligned} \|P\| = \sqrt{P\bar{P}} &= P_0^2 + P_1^2 - P_2^2 - P_3^2, \\ &= (P_0^2 + P_1^2) - (P_2^2 + P_3^2), \\ &= \sqrt{||p_{11}|^2 - |p_{12}|^2|}, \end{aligned}$$

where $p_{11} = P_0 + iP_1$ and $p_{12} = P_2 + iP_3$ are taken for calculations, [1].

Theorem 2.3. *For any complex split quaternions P and Q, the following properties are satisfied:*

$$\begin{array}{ll} (i) & detP = \|P\|^2, \\ (ii) & If\ P\ is\ invertible,\ then\ \overline{(P^{-1})} = \bar{P}^{-1}, \\ (iii) & If\ P\ is\ invertible,\ then\ (\tilde{P})^{-1} = \widetilde{P^{-1}}, \\ (iv) & If\ P\ is\ invertible,\ then\ [(\bar{P})^T]^{-1} = [\overline{(P^{-1})}]^T, \\ (v) & \overline{QP} = \bar{Q}\bar{P}, \\ (vi) & If\ P\ and\ Qare\ invertible,\ then\ (PQ)^{-1} = Q^{-1}P^{-1}. \end{array}$$

3 Split quaternions with quaternion coefficients

In this section, split quaternions with quaternion coefficients are introduced and some significiant properties are obtained. Furthermore, the difference between split quaternions with quaternion coefficients and split quternions with complex coefficients are denoted.
A split quaternion with quaternion coefficients is the form

$$P = P_0E_0 + P_1E_1 + P_2E_2 + P_3E_3, \tag{6}$$

where P_0, P_1, P_2 and P_3 are quaternions and the split quaternion matrix basis $\{E_0, E_1, E_2, E_3\}$ of P satisfy the following equalities:

$$\begin{aligned} E_1^2 &= -E_0,\ \ E_2^2 = E_3^2 = E_0, \\ E_1E_2 &= E_3 = -E_2E_1,\ \ E_2E_3 = E_1 = -E_3E_2,\ \ E_3E_1 = E_2 = -E_1E_3. \end{aligned}$$

Also the quaternion with quaternion coefficients P can be written as

$$P = \sum(a_m + b_m i + c_m j + d_m k)E_m, \tag{7}$$

where a_m, b_m, c_m, d_m are real numbers for $0 \leqslant m \leqslant 3$. Here i, j, k denote the quaternion units and commutes with e_1, e_2 and e_3, respectively. Furthermore, $S_P = P_0E_0$ is scalar part and $V_P = P_1E_1 + P_2E_2 + P_3E_3$ is vector part of P. For any given two split quaternions with quaternion coefficients P and Q, the addition is

$$P + Q = S_{(P+Q)} + V_{(P+Q)}$$

and the quaternion product is

$$\begin{aligned} QP &= Q_0P_0 - (Q_1P_1 + Q_2P_2 + Q_3P_3) \\ &+ E_1(Q_0P_1 + Q_1P_0 + Q_2P_3 - Q_3P_2) \\ &+ E_2(Q_0P_2 + Q_2P_0 + Q_3P_1 - Q_1P_3) \\ &+ E_3(Q_0P_3 + Q_3P_0 + Q_1P_2 - Q_2P_1), \end{aligned}$$

where $Q = Q_0E_0 + Q_1E_1 + Q_2E_2 + Q_3E_3$ and $P = P_0E_0 + P_1E_1 + P_2E_2 + P_3E_3$ are split quaternions with quaternion coefficients. Simply, we can write

$$QP = S_Q - \langle V_Q, V_P \rangle + Q_0V_P + P_0V_Q + (V_Q \times V_P). \tag{8}$$

The scalar product of P is defined as

$$\mu P = (\mu P_0)E_0 + (\mu P_1)E_1 + (\mu P_2)E_2 + (\mu P_3)E_3. \tag{9}$$

The conjugate of P is

$$\bar{P} = S_P - V_P. \tag{10}$$

Additionally, the norm of P is given as

$$\begin{aligned} \|P\| = \sqrt{P\bar{P}} &= P_0^2 + P_1^2 - P_2^2 - P_3^2, \\ &= (P_0^2 + P_1^2) - (P_2^2 + P_3^2), \\ &= \sqrt{||p_{11}|^2 - |p_{12}|^2|}, \end{aligned}$$

where $p_{11} = P_0 + iP_1$ and $p_{12} = P_2 + iP_3$ are taken for calculations. If $\|P\| = 1$, then P is called unit split quaternion with quaternion coefficients. Basis elements of 4x4 matrices are given as follows:

$$E_0 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, E_1 = \begin{pmatrix} i & 0 & 0 & 0 \\ 0 & -i & 0 & 0 \\ 0 & 0 & i & 0 \\ 0 & 0 & 0 & -i \end{pmatrix},$$

$$E_2 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}, E_3 = \begin{pmatrix} 0 & 0 & 0 & i \\ 0 & 0 & -i & 0 \\ 0 & i & 0 & 0 \\ -i & 0 & 0 & 0 \end{pmatrix}.$$

The multiplication rules of 4x4 matrices E_0, E_1, E_2, E_3 satisfy the multiplication rules of the basis elements e_0, e_1, e_2, e_3. Therefore, the vector norm has isomorphic correspondence with the matrix form of the split quaternion with quaternion coefficients.

The algebra of split quaternion with quaternion coefficients, denoted by H_S^Q, is defined as

$$H_S^Q = \{P_0E_0 + P_1E_1 + P_2E_2 + P_3E_3 : P = \begin{pmatrix} P_0 + iP_1 & 0 & 0 & P_2 + iP_3 \\ 0 & P_0 - iP_1 & P_2 - iP_3 & 0 \\ 0 & P_2 + iP_3 & P_0 + iP_1 & 0 \\ P_2 - iP_3 & 0 & 0 & P_0 - iP_1 \end{pmatrix}\}.$$

Definition 3.1. *A determinant of P is defined as*

$$detP = P_0^2 detE_0 + P_1^2 detE_1 + P_2^2 detE_2 + P_3^2 detE_3. \tag{11}$$

Simply, we can write

$$detP = P_0^2 + P_1^2 + P_2^2 + P_3^2. \tag{12}$$

Corollary 3.1. *As it is noticed easily, the determinant of matrix representation for split quaternions with quaternion coefficients has some differences apart from the definition of determinant of matrix representation for complex split quaternions, see in [1].*

If $detP \neq 0$, the inverse of P is

$$\begin{aligned} P^{-1} &= \frac{\bar{P}}{detP}, \\ &= \frac{1}{P_0^2 + P_1^2 + P_2^2 + P_3^2}(P_0E_0 - P_1E_1 - P_2E_2 - P_3E_3). \end{aligned}$$

Example 3.1. *Let $P = \sqrt{5}E_0 + (1+i)E_1 + (k+j)E_2 + iE_3$ be quaternion with quaternion coefficients. The inverse of P is*

$$P^{-1} = \sqrt{5}E_0 - (1+i)E_1 - (k+j)E_2 - iE_3, \tag{13}$$

where P is unit split quaternion with quaternion coefficients.

The conjugate, quaternionic conjugate and the total conjugate are defined, respectively, as follows:

$$\begin{aligned} \bar{P} &= P_0E_0 - P_1E_1 - P_2E_2 - P_3E_3, \\ \tilde{P} &= \bar{P}_0E_0 + \bar{P}_1E_1 + \bar{P}_2E_2 + \bar{P}_3E_3, \\ \tilde{\bar{P}} &= \bar{P}_0E_0 - \bar{P}_1E_1 - \bar{P}_2E_2 - \bar{P}_3E_3. \end{aligned}$$

Moreover, for any $P = P_0E_0 + P_1E_1 + P_2E_2 + P_3E_3$, the transpose and the adjoint matrix of P, denoted by P^T and $AdjP$ respectively, are obtained as

$$\begin{aligned} P^T &= P_0E_0 + P_1E_1 + P_2E_2 - P_3E_3, \\ AdjP &= P_0E_0 - P_1E_1 - P_2E_2 - P_3E_3. \end{aligned}$$

From above equations, we can write

$$AdjP = \bar{P}. \tag{14}$$

4 Conclusion

At this paper, we give definition for split quaternions with quaternion coefficients and obtained some spectacular theorems and linear algebraic concepts. Moreover, we examine the correspondence among the conjugate, quaternionic conjugate and total conjugate of split quaternions with quaternion coefficients.

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A New Approach for Natural Lift Curves and Tangent Bundle of Unit 2-Sphere

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Abstract

In the current paper, some characterizations about Bertrand curve and involute-evolute curve couple for natural lift curves are considered. Obtained consequences are restricted in the subset of tangent bundle of unit 2-sphere along the study. The neccesary and sufficient conditions to be Bertrand curve and involute-evolute curve couple for natural lift curves are given.

***K*eywords** Natural lift curve · tangent bundle · Involute-evolute curve couple · Bertrand curve

1 Introduction

Several studies have been done about the theory of curves in Euclidean space(or Riemannian manifold) [1]. Determining the characterizations of a regular curve is the most significiant subject in differential geometry, kinematics, etc.
In literature, definition and properties of natural lift curve that is the curve obtained by the end points of the tangent vectors of the main curve were first encountered in J. A. Thorpe's book, see for details in [2]. That is, for any parametrized curve, $\mu : I \longrightarrow U$, define the natural lift $\bar{\mu}$ of μ to be parametrized curve $\tilde{\mu} : I \longrightarrow IR^{n+1}$ given by $\tilde{\mu}(t) = \dot{\mu}(t)$. In IR^3, properties of natural lift curve were examined by E. Ergün and M. Çalışkan in [3]. Then Frenet vector fields, the curvatures of the natural lift curve were calculated in [4].
For determining characterizations of curves, one of the method is to construct the relationship among Frenet vectors of curves. Under this approximation, one of the substantial curve couple is involute-evolute curve couple for a given curve. Evolutes and involutes were studied by C. Huygens in [8]. An involute curve of the curve was defined as a curve which all tangent vectors of the given curve are normal vectors by D. Funcs in [9]. The concept of parallel curves for evolute curves were determined by S. Şenyurt and Ş. Kılıçoğlu in [10]. Many researchers have dealt with the properties of involute-evolute curves and their characterizations in different spaces, see [11-15].
Other significiant curve couple is Bertrand curve couple for a given curve. Bertrand curve of the

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curve was defined as a curve which its principal normal coincides principal normal of the main curve by Bertrand in 1850. Characterizations of Bertrand curve couple were studied by many authors in Euclidean space, three dimensional sphere, Riemannian space and Minkowski space time, see [16-20].
A correspondence among unit dual sphere DS^2, tangent bundle of unit 2-sphere TS^2 and non-cylindirical ruled surfaces was given by B. Karakaş and H. Gündoğan in [5]. In the light of this study, the relation between TS^2 and DS^2 was given by M. Bekar, F. Hathout and Y. Yaylı in [6]. In that study, the correspondence between any curve in TS^2 and ruled surface in IR^3 was given. The striction curve and the consequences of developability condition were presented. Then the isomorphism between tangent bundle of pseudo-sphere and ruled surface in Minkowski space was given by same authors in [7]. In that article, the developability condition, involute-evolute curve couple on TS_1^2 and TH^2 were presented.
Many researchers have studied Bertrand curve and involute-evolute curve couple in literature. However, no research has been carried out on natural lift curve in the subset of tangent bundle of unit 2-sphere. In this paper, some basic definitions and theorems are given in Section 2. Section 3 is about involute-evolute curve couple for natural lift curves. In this section, main theorems for these curves are proved. Section 4 covers Bertrand curve couple for natural lift curves. Section 5 is about conlusion. In this section, obtained results are denoted.

2 Preliminaries

In this section, we give some basic definitions and theorems about tangent bundle of unit 2-sphere, natural lift curves, involute-evolute curve and Bertrand curve couple for natural lift curves.

Definition 2.1. *Assume that S^2 is the unit sphere in IR^3. The tangent bundle of S^2 is defined as*

$$TS^2 = \{(\gamma, \vartheta) \in IR^3 \times IR^3 : \|\gamma\| = 1, \langle \gamma, \vartheta \rangle = 0\}. \tag{1}$$

Definition 2.2. *Assume that S^2 is the unit sphere in IR^3. The unit tangent bundle of S^2 is defined as*

$$UTS^2 = \{(\gamma, \vartheta) \in IR^3 \times IR^3 : \|\gamma\| = \|\vartheta\| = 1, \langle \gamma, \vartheta \rangle = 0\}. \tag{2}$$

Here $\langle , \rangle$ and $\|, \|$ represent the inner product and norm in IR^3, respectively.
Let $T\bar{M}$ and $UT\bar{M}$ be the subsets of TS^2 and UTS^2, respectively. $T\bar{M}$ and $UT\bar{M}$ are defined as follows:

$$\begin{aligned} T\bar{M} &= \{(\bar{\gamma}, \bar{\vartheta}) \in IR^3 \times IR^3 : \|\bar{\gamma}\| = 1, \langle \bar{\gamma}.\bar{\vartheta} \rangle = 0\}, \\ UT\bar{M} &= \{(\bar{\gamma}, \bar{\vartheta}) \in IR^3 \times IR^3 : \|\bar{\gamma}\| = \|\bar{\vartheta}\| = 1, \langle \bar{\gamma}.\bar{\vartheta} \rangle = 0\}. \end{aligned}$$

Here $\bar{\gamma}$ and $\bar{\vartheta}$ represent the derivatives of γ and ϑ, respectively.

Definition 2.3. *Let $\Gamma : I \longrightarrow \bar{M}$ be a curve. Here $\bar{M}$ represents a hypersurface on unit 2-sphere. Γ is called an integral curve of X*

$$\frac{d(\Gamma(s))}{ds} = X(\Gamma(s)). \tag{3}$$

Here X is taken as smooth tangent vector field on $\bar{M}$.

Definition 2.4. *For the curve* Γ, $\bar{\Gamma}$ *is defined as the natural lift of* Γ *on* $T\bar{M}$, *which produces in the following equation:*

$$\bar{\Gamma}(s) = (\bar{\gamma}(s), \bar{\vartheta}(s)) = (\gamma^{'}(s)_{\gamma(s)}, \vartheta^{'}(s)_{\vartheta(s)}), \tag{4}$$

$\gamma^{'}(s)_{\gamma(s)}$ and $\vartheta^{'}(s)_{\vartheta(s)}$ are the derivatives of $\gamma(s)$ and $\vartheta(s)$, respectively. Here D refers the Levi-Civita connection in IR^3. We have

$$T\bar{M} = \cup T_p\bar{M}, \;\; p \in \bar{M}, \tag{5}$$

where $T_p\bar{M}$ is the tangent space of $\bar{M}$ at p and $\chi(\bar{M})$ is the space of vector fields on the hypersurface $\bar{M}$. Accordingly, the following equation can be written:

$$\frac{d\bar{\Gamma}(s)}{ds} = \frac{d}{ds}(\Gamma^{'}(s))_{\Gamma(s)} = D_{\Gamma^{'}(s)}\Gamma^{'}(s). \tag{6}$$

Let $\{T(s), N(s), B(s)\}$ and $\{\bar{T}(s), \bar{N}(s), \bar{B}(s)\}$ be Frenet frames of the curve $\Gamma(s)$ and the natural lift $\bar{\Gamma}(s)$, respectively. Therefore, the correspondence between Frenet frames of the main curve and its natural lift curve is given as follows:

$$\begin{aligned}
\bar{T}(s) &= N(s), \\
\bar{N}(s) &= -\cos\theta T(s) + \sin\theta B(s), \\
\bar{B}(s) &= \sin\theta T(s) + \cos\theta B(s),
\end{aligned}$$

where θ is the angle between Darboux vector field and the binormal vector field of the given curve. For curvature $\kappa(s)$ and $\tau(s)$ of the curve $\Gamma(s)$, $W(s) = \tau(s)T(s) + \kappa(s)B(s)$ is called the Darboux vector field in IR^3.

Definition 2.5. *Let* $\tilde{\Gamma} : I \longrightarrow IR^3$ *be a smooth curve for the unit speed curve* $\bar{\Gamma} : I \longrightarrow IR^3$. *If the tangent vector of* $\bar{\Gamma}$ *at the point* $\bar{\Gamma}(s)$ *passes through the tangent vector of* $\tilde{\Gamma}$ *at the point* $\tilde{\Gamma}(s)$ *for* $s \in I$ *and* $\langle \tilde{T}(s), \bar{T}(s) \rangle = 0$ *is provided,* $\tilde{\Gamma}$ *is called the involute curve of* $\bar{\Gamma}$.

Definition 2.6. *Let* $\tilde{\Gamma} : I \to IR^3$ *be a smooth curve for the unit speed curve* $\bar{\Gamma} : I \to IR^3$. *If the tangent line of* $\bar{\Gamma}$ *at the point* $\bar{\Gamma}(s)$ *coincides the tangent line of* $\tilde{\Gamma}$ *at the point* $\tilde{\Gamma}(s)$ *for* $s \in I$ *and* $\langle \tilde{T}(s), \bar{T}(s) \rangle = 0$ *is provided,* $\tilde{\Gamma}$ *is called the evolute curve of* $\bar{\Gamma}$.

Definition 2.7. *Let* $\hat{\Gamma} : I \to IR^3$ *be a smooth curve for the unit speed curve* $\bar{\Gamma} : I \to IR^3$. *If the principal normal vector of* $\bar{\Gamma}$ *at the point* $\bar{\Gamma}(s)$ *coincides the principal normal vector of* $\hat{\Gamma}$ *at the point* $\hat{\Gamma}(s)$ *for* $s \in I$, $\hat{\Gamma}$ *is called the Bertrand curve of* $\bar{\Gamma}$.

3 Involute-evolute curve couple for natural lift curves

This section deals with some basic definitions and theorems about the involute-evolute curve couple for natural lift curves. Frenet vector fields, curvature and torsion of these curves are calculated. For being unit speed curve, $\bar{\Gamma}$ is taken in $UT\bar{M}$ for calculations.

Definition 3.1. *Let* $\tilde{\Gamma} = (\tilde{\gamma}, \tilde{\vartheta}) \in T\bar{M}$ *be a smooth curve for natural lift curve* $\bar{\Gamma} = (\bar{\gamma}, \bar{\vartheta}) \in UT\bar{M}$. *If the tangent vector of* $\bar{\Gamma}$ *at the point* $\bar{\Gamma}(s)$ *passes through the tangent vector of* $\tilde{\Gamma}$ *at the point* $\tilde{\Gamma}(s)$ *for* $s \in I$ *and* $\langle \tilde{T}(s), \bar{T}(s) \rangle = 0$ *is provided,* $\tilde{\Gamma}$ *is called the involute curve of* $\bar{\Gamma}$.

Here $\bar{T}$ and $\tilde{T}$ are tangent vectors of $\bar{\Gamma}$ at the point $\bar{\Gamma}(s)$ and $\tilde{\Gamma}$ at the point $\tilde{\Gamma}(s)$, respectively.

Theorem 3.1. *Assume that* $\tilde{\Gamma} = (\tilde{\gamma}, \tilde{\vartheta}) \in T\bar{M}$ *is the involute curve of* $\bar{\Gamma} = (\bar{\gamma}, \bar{\vartheta}) \in UT\bar{M}$. *For any constant* $k \in I$, *the following equation is provided:*

$$\tilde{\Gamma}(s) = \bar{\Gamma}(s) + (-s + k)\bar{T}(s). \tag{7}$$

Let $\tilde{\Gamma} = (\tilde{\gamma}, \tilde{\vartheta}) \in T\bar{M}$ be the involute curve of $\bar{\Gamma} = (\bar{\gamma}, \bar{\vartheta}) \in UT\bar{M}$. So, for the arc-length parameter s, $\tilde{\Gamma}(s)$ can be expressed as

$$\tilde{\Gamma}(s) = \bar{\Gamma}(s) + u(s)\bar{T}(s), \tag{8}$$

where $u : I \to IR$ is function. By differentiating this equation with respect to s and using Frenet formula of the main curve, we obtain

$$\begin{aligned} \tilde{\Gamma}'(s) &= \bar{\Gamma}'(s) + u'(s)\bar{T}(s) + u(s)\bar{T}'(s), \\ &= u(s)(-\kappa(s))T(s) + (1 + u'(s))N(s) + u(s)(\tau(s))B(s). \end{aligned}$$

Since $\tilde{\Gamma}(s)$ is the involute curve of $\bar{\Gamma}(s)$, we get

$$(1 + u'(s)) = 0.$$

So, $u(s) = -s + k$ is obtained. Therefore, the expression of $\tilde{\Gamma}(s)$ turns into

$$\tilde{\Gamma}(s) = \bar{\Gamma}(s) + (-s + k)N(s). \tag{9}$$

Theorem 3.2. *Let* $\tilde{\Gamma} = (\tilde{\gamma}, \tilde{\vartheta}) \in T\bar{M}$ *be the involute curve of* $\bar{\Gamma} = (\bar{\gamma}, \bar{\vartheta}) \in UT\bar{M}$. *The Frenet frame of* $\tilde{\Gamma} = (\tilde{\gamma}, \tilde{\vartheta})$ *is given by*

$$\begin{aligned} \tilde{T}(s) &= \frac{W(s)}{\|W(s)\|}, \\ \tilde{N}(s) = \left(\frac{(\frac{\tau(s)}{\kappa(s)})'\kappa^2(s)\tau(s)}{A(s)\|W(s)\|}\right)T(s) &- \left(\frac{\|W(s)\|^3}{A(s)}\right)N(s) + \left(\frac{\kappa^3(s)(\frac{\tau(s)}{\kappa(s)})'}{A(s)\|W(s)\|}\right)B(s), \\ \tilde{B}(s) = \left(\frac{\|W(s)\|^2\tau(s)}{A(s)}\right)T(s) &+ \left(\frac{(\frac{\tau(s)}{\kappa(s)})'\kappa^2(s)}{A(s)}\right)N(s) + \left(\frac{\kappa(s)\|W(s)\|^2}{A(s)}\right)B(s), \end{aligned}$$

where $A(s) = \sqrt{\|W(s)\|^6 + \kappa^4(s)((\frac{\tau(s)}{\kappa(s)})')^2}$.

By differentiating Eq. (9), we get

$$\tilde{\Gamma}'(s) = (-s + k)(-\kappa(s)T(s) + \tau(s)B(s)). \tag{10}$$

Furthermore, we obtain

$$\begin{aligned} \tilde{T}(s) &= \frac{\tilde{\Gamma}'(s)}{\|\tilde{\Gamma}'(s)\|} \\ &= \frac{W(s)}{\|W(s)\|}. \end{aligned}$$

From the derivative of Eq. (10), we acquire the following equation:

$$\tilde{\Gamma}''(s) = (\kappa(s) - \kappa'(s)(-s+k))T(s) + ((-s+k)(-\|W(s)\|)^2)N(s) + (-\tau(s) + (-s+k)\tau'(s))B(s). \tag{11}$$

Hence, the vector product of $\tilde{\Gamma}'(s)$ and $\tilde{\Gamma}''(s)$ is

$$\tilde{\Gamma}'(s) \times \tilde{\Gamma}''(s) = (\|W(s)\|(-s+k)\tau(s))T(s) + (\frac{(-s+k)(\frac{\tau(s)}{\kappa(s)})'\kappa^2(s)}{\|W(s)\|})N(s) + (\kappa(s)\|W(s)\|(-s+k))B(s). \tag{12}$$

Since

$$\tilde{B}(s) = \frac{\tilde{\Gamma}'(s) \times \tilde{\Gamma}'(s)}{\|\tilde{\Gamma}'(s) \times \tilde{\Gamma}''(s)\|}, \tag{13}$$

the following equation is obtained:

$$\tilde{B}(s) = (\frac{\|W(s)\|^2\tau(s)}{A(s)})T(s) + (\frac{(\frac{\tau(s)}{\kappa(s)})'\kappa^2(s)}{A(s)})N(s) + (\frac{\kappa(s)\|W(s)\|^2}{A(s)})B(s). \tag{14}$$

Since

$$\tilde{N}(s) = \tilde{B}(s) \times \tilde{T}(s),$$

it can be seen as the following equation:

$$\tilde{N}(s) = (\frac{(\frac{\tau(s)}{\kappa(s)})'\kappa^2(s)\tau(s)}{A(s)\|W(s)\|})T(s) - (\frac{\|W(s)\|^3}{A(s)})N(s) + (\frac{\kappa^3(s)(\frac{\tau(s)}{\kappa(s)})'}{A(s)\|W(s)\|})B(s). \tag{15}$$

Theorem 3.3. *Let $\tilde{\Gamma} = (\tilde{\gamma}, \tilde{\vartheta}) \in T\bar{M}$ be the involute curve of $\bar{\Gamma} = (\bar{\gamma}, \bar{\vartheta}) \in UT\bar{M}$. The curvature and torsion of $\tilde{\Gamma}$ are presented as follows:*

$$\tilde{\kappa}(s) = \frac{\sqrt{\|W(s)\|^6 + \kappa^4(s)((\frac{\tau(s)}{\kappa(s)})')^2}}{\|W(s)\|^{\frac{5}{2}}\sqrt{|-s+k|}} \tag{16}$$

and

$$\begin{aligned}
\tilde{\tau}(s) &= \frac{X(s)\|W(s)\|(-s+k)\tau(s)}{(-s+k)^2(A(s))^2}\|W(s)\|^2 \\
&+ \frac{Y(s)\frac{(-s+k)}{\|W(s)\|}(\frac{\tau(s)}{\kappa(s)})'\kappa^2(s)}{(-s+k)^2(A(s))^2}\|W(s)\|^2 \\
&+ \frac{Z(s)\kappa(s)\|W(s)\|(-s+k)}{(-s+k)^2(A(s))^2}\|W(s)\|^2,
\end{aligned}$$

where

$$\begin{aligned}
X(s) &= 2\kappa'(s) - \kappa(s)\|W(s)\|^2 + (-s+k)(-\kappa''(s) + \kappa(s) - 2\|W(s)\|\|W'(s)\|, \\
Y(s) &= \|W(s)\|^2(1-\tau(s)) + (-s+k)[-(\kappa(s)\tau(s))' - 2\tau(s)\|W(s)\|\|W'(s)\|], \\
Z(s) &= 2\tau'(s) + (-s+k)\tau''(s).
\end{aligned}$$

Suppose that

$$\tilde{\kappa}(s)=\frac{\|\tilde{\Gamma}'(s)\times\tilde{\Gamma}''(s)\|}{\|\tilde{\Gamma}'(s)\|^3}. \tag{17}$$

By taking the norm of Eq. (12) and substituting into Eq. (17), we obtain

$$\tilde{\kappa}(s)=\frac{\sqrt{\|W(s)\|^6+\kappa^4(s)((\frac{\tau(s)}{\kappa(s)})')^2}}{\|W(s)\|^{\frac{5}{2}}\sqrt{|-s+k|}}. \tag{18}$$

Suppose that

$$\tilde{\tau}(s)=\frac{\langle\tilde{\Gamma}'(s)\times\tilde{\Gamma}''(s),\tilde{\Gamma}'''(s)\rangle}{\|\tilde{\Gamma}'(s)\times\tilde{\Gamma}''(s)\|^2}. \tag{19}$$

By taking the derivative of $\tilde{\Gamma}''(s)$ and taking the inner product with $\tilde{\Gamma}'(s)\times\tilde{\Gamma}''(s)$, we have

$$\begin{aligned}\tilde{\tau}(s) &= \frac{\|W(s)\|(-s+k)\tau(s)X(s)}{(-s+k)^2(A(s))^2}\|W(s)\|^2\\ &+ \frac{\frac{(-s+k)}{\|W(s)\|}(\frac{\tau(s)}{\kappa(s)})'\kappa^2(s)Y(s)}{(-s+k)^2(A(s))^2}\|W(s)\|^2\\ &+ \frac{\kappa(s)\|W(s)\|(-s+k)Z(s)}{(-s+k)^2(A(s))^2}\|W(s)\|^2.\end{aligned}$$

Definition 3.2. *Let $\tilde{\Gamma}=(\tilde{\gamma},\tilde{\vartheta})\in T\bar{M}$ be a smooth curve for natural lift curve $\bar{\Gamma}=(\bar{\gamma},\bar{\vartheta})\in UT\bar{M}$. If the tangent line of $\bar{\Gamma}$ at the point $\bar{\Gamma}(s)$ coincides the tangent line of $\tilde{\Gamma}$ at the point $\tilde{\Gamma}(s)$ for $s\in I$ and $\langle\tilde{T}(s),\bar{T}(s)\rangle=0$ is provided, $\tilde{\Gamma}$ is called the evolute curve of $\bar{\Gamma}$.*

Theorem 3.4. *Assume that $\tilde{\Gamma}=(\tilde{\gamma},\tilde{\vartheta})\in T\bar{M}$ is the evolute curve of $\bar{\Gamma}=(\bar{\gamma},\bar{\vartheta})\in UT\bar{M}$. For any constant $k\in I$, the following equation is provided:*

$$\begin{aligned}\tilde{\Gamma}(s)=\bar{\Gamma}(s) &+ \frac{-\cos\theta T(s)+\sin\theta B(s)}{\kappa(s)\cos\theta+\tau(s)\sin\theta+\tan(\phi(s)+c)\kappa(s)\sin\theta-\tau(s)\cos\theta}\\ &- \frac{\tan(\phi(s)+c)\sin\theta T(s)+\cos\theta B(s)}{\kappa(s)\cos\theta+\tau(s)\sin\theta+\tan(\phi(s)+c)\kappa(s)\sin\theta-\tau(s)\cos\theta}.\end{aligned}$$

Suppose

$$\tilde{\Gamma}(s)-\bar{\Gamma}(s)=\lambda(s)\bar{N}(s)+\mu(s)\bar{B}(s). \tag{20}$$

By differentiating Eq. (20) and using Frenet formula of the main curve, the following equation is obtained:

$$\begin{aligned}\tilde{\Gamma}'(s) &= (-\lambda'(s)\cos\theta+\mu'(s)\sin\theta+\theta'(\lambda(s)\sin\theta+\mu(s)\cos\theta))T(s)\\ &+ (\kappa(s)(-\lambda(s)\cos\theta+\mu(s)\sin\theta)+\tau(s)(-\lambda(s)\sin\theta-\mu(s)\cos\theta)+1)N(s)\\ &+ ((\lambda'(s)-\mu(s)\theta')\sin\theta+\cos\theta(\theta'\lambda(s)+\mu'(s))B(s).\end{aligned}$$

According to the definition of evolute, the vector field $\tilde{\Gamma}'$ is parallel to the vector field $\tilde{\Gamma}-\bar{\Gamma}$. Therefore, we give

$$\tilde{\Gamma}(s)-\bar{\Gamma}(s)=(-\lambda(s)\cos\theta+\mu(s)\sin\theta)T(s)+(\lambda(s)\sin\theta+\mu(s)\cos\theta)B(s). \tag{21}$$

Furthermore, we get

$$\begin{aligned} & \frac{-\lambda'(s)\cos\theta + \mu'(s)\sin\theta + \theta'(\lambda(s)\sin\theta + \mu(s)\cos\theta)}{-\lambda(s)\cos\theta + \mu(s)\sin\theta} \\ = & \frac{(\lambda'(s) - \mu(s)\theta')\sin\theta + \cos\theta(\theta'\lambda(s) + \mu'(s))}{\lambda(s)\sin\theta + \mu(s)\cos\theta}. \end{aligned}$$

After some calculations, the correspondence between $\lambda(s)$ and $\mu(s)$ is

$$\mu(s) = -\lambda(s)\tan(\phi(s) + c). \tag{22}$$

Hence, we obtain

$$\begin{aligned} \lambda(s) &= \frac{1}{\kappa(s)\cos\theta + \tau(s)\sin\theta + \tan(\phi(s)+c)(\kappa(s)\sin\theta - \tau(s)\cos\theta)}, \\ \mu(s) &= \frac{-\tan(\phi(s)+c)}{\kappa(s)\cos\theta + \tau(s)\sin\theta + \tan(\phi(s)+c)(\kappa(s)\sin\theta - \tau(s)\cos\theta)}. \end{aligned}$$

Example 3.1. *Assume that $\bar{\gamma}(s) = (\cos s, \sin s, 0)$ is the curve on $\bar{M}$ and $\bar{\vartheta}(s) = (\sin s, -\cos s, 0)$ is the vector in IR^3. Since $\|\bar{\gamma}(s)\| = \|\bar{\vartheta}(s)\| = 1$ and $\langle\bar{\gamma}, \bar{\vartheta}\rangle = 0$, the natural lift curve $\bar{\Gamma} = (\bar{\gamma}, \bar{\vartheta})$ is in $UT\bar{M}$. The involute curve $\tilde{\Gamma} = (\tilde{\gamma}, \tilde{\vartheta})$ of $\bar{\Gamma} = (\bar{\gamma}, \bar{\vartheta})$ is expressed as*

$$\tilde{\Gamma}(s) = \bar{\Gamma}(s) + (-s + k)\bar{T}(s). \tag{23}$$

Hence, we can write

$$\begin{aligned} \tilde{\gamma}(s) &= (\cos s, \sin s, 0) + (-s + k)(-\sin s, \cos s, 0), \\ \tilde{\vartheta}(s) &= (\sin s, -\cos s, 0) + (-s + k)(\cos s, \sin s, 0). \end{aligned}$$

For $k = 0$, above equations turn into

$$\begin{aligned} \tilde{\gamma}(s) &= (\cos s + s\sin s, \sin s - s\cos s, 0), \\ \tilde{\vartheta}(s) &= (\sin s - s\cos s, -\cos s - s\sin s, 0). \end{aligned}$$

As we convert $\tilde{\gamma}(s)$ as unit vector, we compute

$$\tilde{\beta}(s) = \frac{\tilde{\gamma}(s)}{\|\tilde{\gamma}(s)\|} = \frac{1}{\sqrt{1+s^2}}(\cos s + s\sin s, \sin s - s\cos s, 0).$$

Since $\|\tilde{\beta}(s)\| = 1$ and $\langle\tilde{\beta}, \tilde{\vartheta}\rangle = 0$, the involute curve couple $\{\tilde{\beta}(s), \tilde{\vartheta}(s)\}$ is in $T\bar{M}$.

4 Bertrand curve couple for natural lift curve

In this section, we proceed to study about Bertrand curve couple for natural lift curves.
There is not a Bertrand curve $\hat{\Gamma} = (\hat{\gamma}, \hat{\vartheta}) \in T\bar{M}$ of natural lift curve $\bar{\Gamma} = (\bar{\gamma}, \bar{\vartheta}) \in UT\bar{M}$. Let $\hat{\Gamma} = (\hat{\gamma}, \hat{\vartheta})$ be a Bertrand curve of $\bar{\Gamma} = (\bar{\gamma}, \bar{\vartheta})$. From the definition of Bertrand curve, we can write the following equation using the arc length parameter s:

$$\hat{\Gamma}(s) = \bar{\Gamma}(s) + \lambda\bar{N}(s) \tag{24}$$

Thus, we can obtain $\hat{\Gamma} = (\bar{\Gamma}, \bar{N})$. For being $\hat{\Gamma} \in T\bar{M}$, it must be provided that $\langle \bar{\Gamma}, \bar{N} \rangle = 0$. After some algebraic equations, we obtain $\langle \bar{\Gamma}, \bar{N} \rangle = \frac{1}{\bar{\kappa}} \neq 0$.

5 Conclusion

In this study, the difference of some special curve couple such as Bertrand curves, involute-evolute curves between IR^3 and tangent bundle of unit 2-sphere are given. The neccesary and sufficient conditios to be Bertrand and involute-evolute curve couple are denoted. Obtained results are very beneficial who specialized in kinematics, engineering and physics for modelling structure.

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Applications of Eigenvalue and Eigenvectors in Engineering

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Abstract

In engineering and science, many matrix applications benefit from eigenvalue and eigenvector. For any n-square matrix A, the special numbers which are called as eigenvalues and some special vectors which are called as eigenvectors have great importance. Here we examplified their applications in stretching of an elastic membrane, population growth, using eigenvalues and eigenvectors to study vibrations, Google's page rank.

***K*eywords** Eigenvalue · Eigenvector · Vibration

1 Introduction

Linear equations $Ax = b$ come from steady state problems. Eigenvalues have their greatest importance in dynamic problems. The solution of

$$\frac{du}{dt} = Au$$

is changing with time - growing or decaying or oscillating. We can't find it by elimination. This new part of linear algebra, is based on $Ax = \lambda x$ and all matrices are square. A good model comes from the power $A, A^2, A^3, \ldots\ldots$ of a matrix. Suppose you need the hundredth powerA^{100}. The starting matrix A becomes unrecognizable after a few steps:

$$A = \begin{bmatrix} 0.8 & 0.3 \\ 0.2 & 0.7 \end{bmatrix},$$

$$A^2 = \begin{bmatrix} 0.70 & 0.45 \\ 0.30 & 0.55 \end{bmatrix},$$

$$A^3 = \begin{bmatrix} 0.650 & 0.525 \\ 0.350 & 0.475 \end{bmatrix}$$

$$\cdots\cdots$$
$$A^{100} = \begin{bmatrix} 0.6000 & 0.6000 \\ 0.4000 & 0.4000 \end{bmatrix}$$

A^{100} was found by using the eigenvalues of A, not by multiplying 100 matrices.

To explain eigenvalues, we first explain eigenvectors. Almost all vectors change direction, when they are multiplied by A. Certain exceptional vectors $\boldsymbol{x}$ are in the same direction as Ax. Those are the “eigenvectors”. Multiply an eigenvector by A, and the vector Ax is a number λ times the original x.

The basic equation is $Ax = \lambda x$. The number λ is an eigenvalue of A.

The eigenvalue λ tells whether the special vector x is stretched or shrunk or reversed or left unchanged—when it is multiplied by A. We may find $\lambda = 2$ or $1/2$ or -1 or 1. The eigenvalue λ could be zero! Then $Ax = 0x$ means that this eigenvector is in the nullspace.

If A is the identity matrix, every vector has $Ax = x$. All vectors are eigenvectors of I. All eigenvalues “lambda” are $\lambda = 1$. This is unusual to say the least. Most 2 by 2 matrices have two eigenvector directions and two eigenvalues. We will show that $\det(A - \lambda I) = 0$.

If the matrix A has two eigenvalues $\lambda = 1$ and $\lambda = 1/2$.Look at $\det(A - \lambda I)$:

$$A = \begin{bmatrix} 0.8 & 0.3 \\ 0.2 & 0.7 \end{bmatrix}$$
$$det \begin{bmatrix} 0.8 - \lambda & 0.3 \\ 0.2 & 0.7 - \lambda \end{bmatrix} = \lambda^2 - \frac{3}{2}\lambda + \frac{1}{2} = (\lambda - 1)\left(\lambda - \frac{1}{2}\right)$$

The two eigenvalues $\lambda = 1$ and $\lambda = 1/2$. For those numbers, the matrix $A - \lambda I$ becomes singular (zero determinant). The eigenvectors x_1 and x_2 are in the nullspaces of $A - I$ and $A - \frac{1}{2}I$.

$(A - I)\, x_1 = 0$ and $Ax_1 = x_1$ and the first eigenvector is $(0.6,\ 0.4)$

$(A - \frac{1}{2}I)\, x_1 = 0$ and $Ax_1 = \frac{1}{2}x_1$ and the first eigenvector is $(1\ , -1)$.

$$x_1 = \begin{bmatrix} 0.6 \\ 0.4 \end{bmatrix} \text{ and } Ax_1 = \begin{bmatrix} 0.8 & 0.3 \\ 0.2 & 0.7 \end{bmatrix} \begin{bmatrix} 0.6 \\ 0.4 \end{bmatrix} = x_1$$

$$x_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix} \text{ and } Ax_2 = \begin{bmatrix} 0.8 & 0.3 \\ 0.2 & 0.7 \end{bmatrix} \begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 0.5 \\ -0.5 \end{bmatrix} \left(this\ is\ \tfrac{1}{2}x_2,\ so\ \lambda_2 = \tfrac{1}{2}\right)$$

If x_1 is multiplied again by A we still get x_1. Every power of A will give $A^n x_1 = x_1$. Multiplying x_2 by $\frac{1}{2}$ gave $\frac{1}{2}x_2$, and if we multiply again we get $\left(\frac{1}{2}\right)^2$times x_2. This pattern keeps going, because the eigenvectors stay in their own directions and never get mixed. The eigenvectors of A^{100} are the same $x_1$and x_2. The eigenvalues of A^{100} are $1^{100} = 1$ and $\left(\frac{1}{2}\right)^{100}$ a very small number. Other vectors do change direction. But all other vectors are combinations of the two eigenvectors. The first column of A is the combination $x_1 + 0.2x_2$:

Separate into eigenvectors

$$\begin{bmatrix} 0.8 \\ 0.2 \end{bmatrix} = x_1 + 0.2x_2 = \begin{bmatrix} 0.6 \\ 0.4 \end{bmatrix} + \begin{bmatrix} 0.2 \\ -0.2 \end{bmatrix}$$

Multiplying by A gives $(0.7,\ 0.3)$ the first column of A^2 . Do it separately for x_1 and $0.2x_2$ .Of course $Ax_1 = x_1$. And A multiplies $x_2$by its eigenvalue $\frac{1}{2}$.

Multiply each $x_i$by λ_i

$$A\begin{bmatrix} 0.8 \\ 0.2 \end{bmatrix} = \begin{bmatrix} 0.7 \\ 0.3 \end{bmatrix} \text{is } x_1 + \tfrac{1}{2}(0.2)\,x_2 = \begin{bmatrix} 0.6 \\ 0.4 \end{bmatrix} + \begin{bmatrix} 0.1 \\ -0.1 \end{bmatrix}.$$

Each eigenvector is multiplied by its eigenvalue, when we multiply by A. We didn't need these eigenvectors to find A^2 . But it is the good way to do 99 multiplications. At every step x_1 is unchanged and x_2 is multiplied by $\frac{1}{2}$ so we have $\left(\frac{1}{2}\right)^{99}$:

$$A^{99}\begin{bmatrix} 0.8 \\ 0.2 \end{bmatrix} \text{is really } x_1 + (0.2)\left(\tfrac{1}{2}\right)^{99} x_2 = \begin{bmatrix} 0.6 \\ 0.4 \end{bmatrix} + \begin{bmatrix} very \\ small \\ vector \end{bmatrix}$$

This is the first column of A^{100} . The number we originally wrote as 0.6000 was not exact.

We left out $(0.2)\left(\frac{1}{2}\right)^{99}$ which wouldn't show up for 30 decimal places. The eigenvector x_1 is a "steady state" that doesn't change (because $\lambda_1 = 1$). The eigenvector x_2 is a "decaying mode" that virtually disappears (because $\lambda_1 = 0.5\boldsymbol{5}$) The higher the power of A, the closer its columns approach the steady state. We mention that this particular A is a *Markov matrix*. Its entries are positive and every column adds to 1. Those facts guarantee that the largest eigenvalue is $\lambda = 1$ as we found). Its eigenvector $x_1 = (0.6; 0.4)$ is the steady state—which all columns of A^k will approach.

2. Some Illustrations From Real Life In engineering and science, many matrix applications benefits from eigenvalue and eigenvector. Here we are looking for some special numbers which are called as eigenvalues and some special vectors which are called as eigenvectors. Consider $n-$square matrix A and it multiplies vector x. There are most vectors that Ax is in-points in some diffrent direction. But there are certain vectors where Ax comes out parallel to x. Those are the eigenvectors. In other words, eigenvectors are any vectors that x parallel to Ax. Namely, a vector that run in only scaling without any rotation is known as the eigenvector. At this section, we present some examplifications for them.

Stretching of an Elastic Membrane

An elastic membrane in the x_1x_2-plane with boundary círcle ${x_1}^2 + {x_2}^2 = 1$ (Fig. 1) is stretched so that a point P: (x_1, x_2) goes over into the point Q: (y_1, y_2) given by:

$$y = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = Ax = \begin{bmatrix} 5 & 3 \\ 3 & 5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

in components,

$$y_1 = 5x_1 + 3x_1$$
$$y_2 = 3x_1 + 5x_1$$

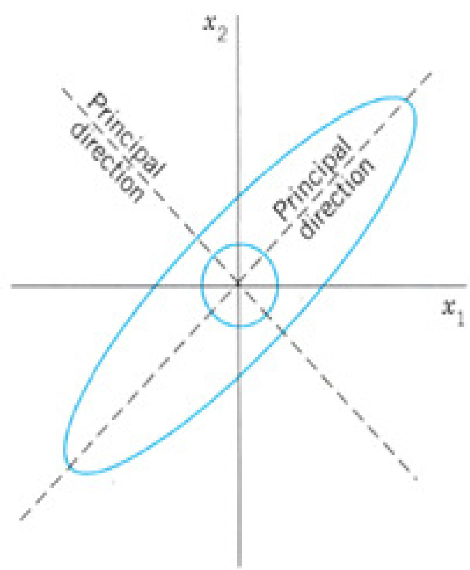

Fig. 1: Streching an elastics membrane

Find the ***principal directions***, that is, the directions of the position vector x of P for which the direction of the position vector y of Q is the same or exactly opposite. What shape does the boundary circle take under this deformation?

Solution We are looking for vectors x such that $y = \lambda x$. Since $y = Ax$, this gives $Ax = \lambda x$, the equation of an eigenvalue problem. In components, $Ax = \lambda x$ is

$$5x_1 + 3x_1 = \lambda x_1$$
$$3x_1 + 5x_1 = \lambda x_2$$

or

$$(5 - \lambda)\, x_1 + 3x_1 = 0$$
$$3x_1 + (5 - \lambda)\, x_1 = 0$$

The characteristic equation is

$$\begin{vmatrix} 5-\lambda & 3 \\ 3 & 5-\lambda \end{vmatrix} = (5-\lambda)^2 - 9 = 0$$

Its solutions are: $\lambda_1 = 8$ and $\lambda_2 = 2$. These are the eigenvalues of our problem. For $\lambda = \lambda_1 = 8$, our system becomes:

$$-3x_1 + 3x_1 = 0$$
$$3x_1 - 3x_1 = 0$$

Solution x_1, x_2, x_1 arbitrary, for instance $x_1 = x_2 = 1$

We thus obtain as eigenvectors of A**,** for instance, $\begin{bmatrix} 1 & 1 \end{bmatrix}^T$corresponding to λ_1 and $\begin{bmatrix} 1 & -1 \end{bmatrix}^T$ corresponding to λ_1 (or a nonzero scalar multiple of these). These vectors make 45^0 and 135^0 angles with the positive x_l $\_\_$ direction. They give the principal directions, the answer to our problem. The eigenvalues show that in the principal directions the membrane is stretched by factors 8 and 2, respectively; see Fig. 1.

Accordingly, if we choose the principal directions as directions of a new Cartesian u_1u_2-coordinate system, say, with the positive ,u_1-semi-axis in the first quadrant and the positive u_2-semi-axis in the second quadrant of the x_1x_2-system, and if we set

$$u_1 = rcos\phi, u_2 = rsin\phi$$

then a boundary point of the unstretched circular membrane has coordinates *cos* Φ, *sin* Φ. Hence, after the stretch we have

$$z_1 = 8cos\phi, z_2 = 2sin\phi$$

Since $cos^2\phi + sin^2\phi = 1$, this shows that the deformed boundary is an ellipse (Fig. 1).

Population Growth: Eigenvalues and Eigenvectors

Many problems look simple in a uni-variate setting but complicated in a multivariate setting. For example, consider a simple model of population.

The population growth example, let $x_t = Ax_{t-1}$ **.** If x_t is a scalar (a number, say representing the population of an European state, named E1) and so is A, this has a simple solution and $x_t = A^T x_0$ and we know immediately what x_t is for all t. However, if x_t is a vector (say two numbers, representing the populations of two European states, named $E1$ and $E2$) and A is a matrix, then we can still write$x_t = A^T x_0$, but this has matrix multiplication and it is hard to see what is going on:

$$\begin{pmatrix} x_{1,t} \\ x_{2,t} \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \cdots \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \begin{pmatrix} x_{1,0} \\ x_{2,0} \end{pmatrix}$$

For example, it is hard to see from this what is the long-term ratio of E1 population to E2 population.

Suppose there is no population movement between E1 and E2. Then the matrix **A** is diagonal:

$$A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$$

and therefore

$$A^T x_0 = \begin{pmatrix} a_{11}^t & 0 \\ 0 & a_{22}^t \end{pmatrix} \begin{pmatrix} x_{1,0} \\ x_{2,0} \end{pmatrix} = \begin{pmatrix} a_{11}^t x_{1,0} \\ a_{22}^t x_{2,0} \end{pmatrix}$$

In this case, we can see exactly what is happening. The state with the higher population growth rate $(a_{22} - 1 \; or \; a_{22} - 1)$ will have an increasing fraction of the population over time. This case of a diagonal matrix is useful but too special for most applications.

Returning to the population growth example, let

$$A = \begin{pmatrix} 1.03 & 0.005 \\ 0.02 & 1.03 \end{pmatrix}$$

and take $x_0 = (6; 13)\;^T$(millions of people).

We are taking A as given, but here is a story about where the entries might come from. Assume that the birth rate in $E1$ is 6%, the death rate 1%, and the rate at which people move to $E2$ is 2%, which is where we get $A_{11} = 1 + 6\% - 1\% - 2\% = 1.03$ and $A_{21} = 2\%$. Furthermore, we can assume the birth rate in $E2$ is 5%, the death rate 1.5%, and the rate at which people move to $E1$ is 0.5%, which is where we get $A_{12} = 0.5\%$. and $A_{11} = 1 + 5\% - 1.5\% - 0.5\% = 1.03$.

To found the eigenvalues let's solve $\det(A - \lambda I) = 0$. Then, A has eigenvalue $\lambda_1 = 1.04$ with corresponding eigenvector $x_1 = (1,\ 2)$ and eigenvalue $\lambda_1 = 1.02$ with corresponding eigenvector $x_1 = (1,\ -2)$. To write the solution in terms of the eigenvalues, we first want to write the initial vector of populations as a sum of the eigenvectors, i.e. to write $x_0 = c_1 x_1 + c_2 x_{2.}$.

Letting S be the matrix whose columns are the eigenvectors, we can use matrix notation to write this as $x_0 = Sc$ to obtain $c = S^{-1}$, which has the unique solution $c_1 = 6.25$ and $c_1 = -0.25$.

So the general solution of $x_t = Ax_{t-1}$ (where A is given above) subject to the initial condition $x_0 = (6,\ 13)^T$is

$$x_t = A^T x_0 = A^T(6.25x_1 - 0.25x_2) = 6.25A^T x_1 - 0.25A^T x_2 =$$
$$= 6.25 \cdot 1.\,04^{\,t} x_1 - 0.25 \cdot 1.\,02^{\,t} x_2 = 6.25 \cdot 1.\,04^{\,t} \begin{pmatrix} 1 \\ 2 \end{pmatrix} - 0.25 \cdot 1.\,02^{\,t} \begin{pmatrix} 1 \\ -2 \end{pmatrix}$$
$$= \begin{pmatrix} 6.25 \cdot 1.\,04^{\,t} - 0.25 \cdot 1.\,02^{\,t} \\ 12.5 \cdot 1.\,04^{\,t} + 0.5 \cdot 1.\,02^{\,t} \end{pmatrix}$$

Note that in the long run, the terms with $1.\,04^{\,t}$ dominate the terms with $1.\,02^{\,t}$. Therefore, in the long run both populations grow at a 4% rate, with $E2'$ population being about twice $E1's$.

We can see that the eigenvalue formulation makes the solution much clearer and easier to interpret than simply writing $x_t = A^T x_0$. The eigenvalue decomposition is also an efficient way to compute the value at distant times. In an example with a higher dimension (population in all 27 states or many countries), the eigenvalues and eigenvectors would have to be computed numerically, but the interpretation would be similar, since in the long term the population growth would be the largest eigenvalue less one, and the population proportions would tend to the proportions in the corresponding eigenvector.

Using Eigenvalues and Eigenvectors to Study Vibrations

Consider the system shown below with 2 masses and 3 springs (See Fig.2 and Fig.3). The masses are constrained to move only in the horizontal direction (they can't move up an down):

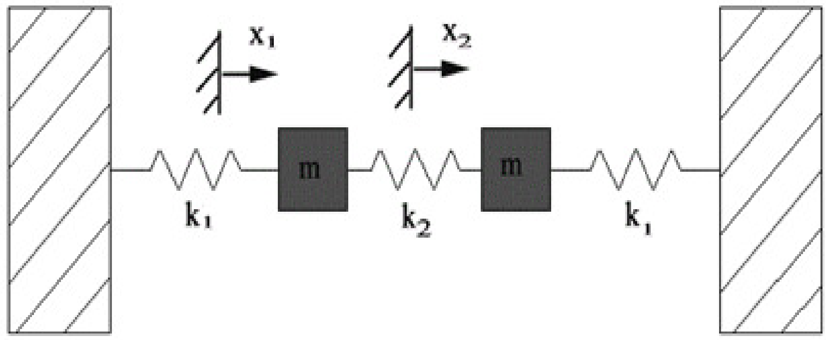

Fig. 2: The constrained masses

We can draws the free body diagram for this system:

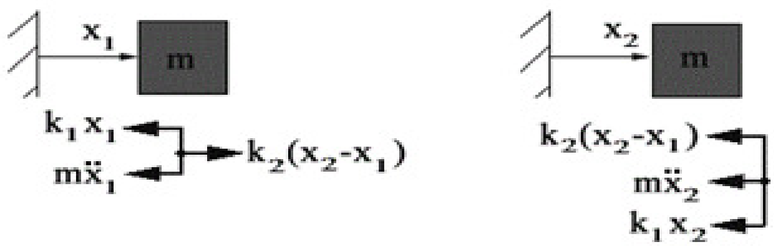

Fig. 3: The free body figure

From this, we can get the equations of motion:

$$m\ddot{x_1} + (k_1 + k_2)\, x_1 - k_2 x_2 = 0$$
$$m\ddot{x_2} + (k_1 + k_2)\, x_2 - k_2 x_1 = 0$$
$$-\frac{k_1 + k_2}{m} x_1 + \frac{k_2}{m} x_2 = \ddot{x_1}$$
$$\frac{k_2}{m} x_1 - \frac{k_1 + k_2}{m} x_2 = \ddot{x_2}$$

We can rearrange these into a matrix form (and use α and β for notational convenience).

$$\begin{bmatrix} -\frac{k_1+k_2}{m} & \frac{k_2}{m} \\ \frac{k_2}{m} & -\frac{k_1+k_2}{m} \end{bmatrix} \begin{bmatrix} x_1 \\ x_1 \end{bmatrix} = \begin{bmatrix} \ddot{x_1} \\ \ddot{x_2} \end{bmatrix}$$

$$\begin{bmatrix} -\beta & \alpha \\ \alpha & -\beta \end{bmatrix} X = \ddot{X}$$

Now we proceed by assuming the form of solution (just as with differential equations). In this case, since there is no damping, we choose a purely oscillatory solution.

$$X = Ve^{j\omega t} = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} e^{j\omega t}$$

So

$$\ddot{X} = -\omega^2 V e^{j\omega t} = -\omega^2 X$$
$$\begin{bmatrix} -\beta & \alpha \\ \alpha & -\beta \end{bmatrix} X = -\omega^2 X$$

This is obviously just an eigenvalue problem.

$Ax \;=\; \lambda x$, where $\lambda = -\omega^2$

We can solve for the eigenvalues by finding the characteristic equation (note the "+" sign in the determinant rather than the "-" sign, because of the opposite signs of λ and ω^2).

$$\left|A + \omega^2 I\right| = 0 = \begin{vmatrix} \omega^2 - \beta & \alpha \\ \alpha & \omega^2 - \beta \end{vmatrix}$$
$$\left(\omega^2 - \beta\right)^2 - \alpha^2 = \omega^4 - 2\beta\omega + \left(\beta^2 - \alpha^2\right) = 0$$

So, $\omega^2 = \frac{2\beta \pm \sqrt{4\beta^2 - 4(\beta^2 - \alpha^2)}}{2} = \beta \pm \alpha$

To make the notation easier we will now consider the specific case where $k_1 = k_2 = m = 1$, so

$$\omega_1^2 = \beta + \alpha = \frac{k_1 + 2k_2}{m} = 3$$
$$\omega_2^2 = \beta - \alpha = \frac{k_1}{m} = 1$$

Now we can also find the eigenvectors. For the first eigenvector:

$$\left(A + \omega_1^2 I\right) v_1 = 0$$

$$\left(\begin{bmatrix} -2 & 1 \\ 1 & -2 \end{bmatrix} + \begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix}\right) v_1 = 0$$

$$\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} v_{1,1} \\ v_{1,2} \end{bmatrix} = 0$$

which clearly has the solution:

$$v_{1,1} = -v_{1,2}$$

So we'll choose the first eigenvector (which can be multiplied by an arbitrary constant).

$$v_1 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

For the second eigenvector:

$$\left(A + \omega_2^2 I\right) v_2 = 0$$

$$\left(\begin{bmatrix} -2 & 1 \\ 1 & -2 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right) v_2 = 0$$

$$\begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} v_{2,1} \\ v_{2,2} \end{bmatrix} = 0$$

$$v_{2,1} = v_{2,2}$$

$$v_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

Google's PageRank

From the time it was introduced in 1998, Google's methods for delivering the most relevant result for our search queries has evolved in many ways, and PageRank is not really a factor any more in the way it was at the beginning.

Let's assume the Web contains 6 pages only. The author of Page 1 thinks pages 2, 4, 5, and 6 have good content, and links to them. The author of Page 2 only likes pages 3 and 4 so only links from her page to them. The links between these and the other pages in this simple web are summarised in Fig. 4.

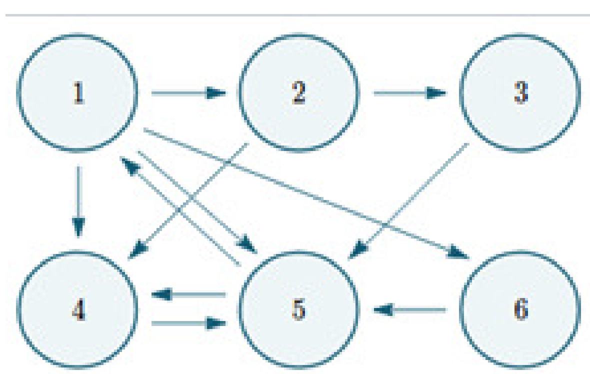

Fig. 4: Links with pages

Google engineers assumed each of these pages is related in some way to the other pages, since there is at least one link to and from each page in the web.

Their task was to find the "most important" page for a particular search query, as indicated by the writers of all 6 pages. For example, if everyone linked to Page 1, and it was the only one that had 5 incoming links, then it would be easy - Page 1 would be returned at the top of the search result.

However, we can see some pages in our web are not regarded as very important. For example, Page 3 has only one incoming link. Should its outgoing link (to Page 5) be worth the same as Page 1's outgoing link to Page 5?

The beauty of PageRank was that it regarded pages with many incoming links (especially from other popular pages) as more important than those from mediocre pages, and it gave more weighting to the outgoing links of important pages.

For the 6-page web illustrated above, we can form a "link matrix" representing the relative importance of the links in and out of each page.

Considering Page 1, it has 4 outgoing links (to pages 2, 4, 5, and 6). So in the first column of our "links matrix", we place value 1/4 in each of rows 2, 4, 5 and 6, since each link is worth 1/4 of all the outgoing links. The rest of the rows in column 1 have value 0, since Page 1 doesn't link to any of them.

Meanwhile, Page 2 has only two outgoing links, to pages 3 and 4. So in the second column we place value $1/2$ in rows 3 and 4, and 0 in the rest. We continue the same process for the rest of the 6 pages.

$$A = \begin{bmatrix} 0 & 0 & 0 & 0 & \frac{1}{2} & 0 \\ \frac{1}{4} & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{2} & 0 & 0 & 0 & 0 \\ \frac{1}{4} & \frac{1}{2} & 0 & 0 & \frac{1}{2} & 0 \\ \frac{1}{4} & 0 & 1 & 1 & 0 & 1 \\ \frac{1}{4} & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

We have

$$|A - \lambda I| = \begin{vmatrix} -\lambda & 0 & 0 & 0 & \frac{1}{2} & 0 \\ \frac{1}{4} & -\lambda & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{2} & -\lambda & 0 & 0 & 0 \\ \frac{1}{4} & \frac{1}{2} & 0 & -\lambda & \frac{1}{2} & 0 \\ \frac{1}{4} & 0 & 1 & 1 & -\lambda & 1 \\ \frac{1}{4} & 0 & 0 & 0 & 0 & -\lambda \end{vmatrix}$$

$$= \lambda^6 - \frac{5\lambda^4}{8} - \frac{\lambda^3}{4} - \frac{\lambda^2}{8}$$

This expression is zero for $\lambda = -0.72031,\ -0.13985 \pm 0.39240i,\ 0,\ 1$. We can only use non-negative, real values of λ (since they are the only ones that will make sense in this context), so we conclude $\lambda = 1$. (In fact, for such PageRank problems we always take $\lambda = 1$.)

We could set up the six equations for this situation, substitute and choose a "convenient" starting value, but for vectors of this size, it's more logical to use a computer algebra system. We find the corresponding eigenvector is:

$$v_1 = [4\ 1\ 0.5\ 5.5\ 8\ 1]^T$$

As Page 5 has the highest PageRank (of 8 in the above vector), we conclude it is the most "important", and it will appear at the top of the search results.

We often normalize this vector so the sum of its elements is 1. (We just add up the amounts and divide each amount by that total, in this case 20.) This is OK because we can choose any "convenient" starting value and we want the relative weights to add to 1. I've called this normalized vector P for "PageRank".

$$P = [0.2\ 0.05\ 0.025\ 0.275\ 0.4\ \ 0.05]^T$$

3. Conclusion

At this paper, we present some applications of eigenvalues and eigenvectors from real life, showing the importance of mathematics in everyday life. In this way, we can understand the applicability and usefulness of mathematics.

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THE WAY OF ASSESSMENT OF MATHEMATICAL COMPETENCIES IN RULES_MATH PROJECT

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ABSTRACT

The main goal of the project RULES_MATH was to design the standards for assessment of mathematical competence at technical bachelor studies via contents models. FME STU as one of the partners of the project prepared testing models in the topics Analysis and Calculus and Probability and Statistics. The models were designed with respect to main pedagogical aim, to develop and to assess the mathematical competency through mutually overlapping eight competencies elaborated by SEFI MWG in the Framework and previously introduced by Danish KOM project. We will deal with two guidelines covering the learning outcomes in each model. The mathematical competencies were determined for each learning outcome in tables: *Learning outcomes with degree of coverage of competencies involved in this assessment activity*. With respect to scope and curriculla, we prepared tests, by which the problems in guidelines were verified at FME STU: in the topic Analysis and Calculus at bachelor study programmes in subject Mathematics I for the first year of the study and in the topic Probability and Statistics in compulsorily elective subject Basics of Statistical Analysis for the second year of the study. Levels of acquirement of mathematical competencies were determined by means of given assessment tests. Since not every competence was present in the test with the same importance, it was necessary to determine all competencies covered by the test. In this article, we introduce the process of testing the acquirement of mathematical competencies covered by the test and its results. The main idea for assessment of acquirement of the mathematical competencies is based on division of the points amount achieved by a student in the test into problems/subproblems/subtopics covering the corresponding competencies. After that, these points are distributed on particular competencies in given learning outcome following the "table of competencies" built by teacher for each test. It was needed to norm the resulting values of acquirement of competence in order to analyse it correctly. It means we were supposed to compare these values with maximal possible achievable value for each competence. Data we obtained were quantitatively processed by program developed in Wolfram Mathematica software. The two input tables and one vector were transformed into one resulting table. The first input table contained the points acquired by each student in each test problem.

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The second input table contained the importance of each competence in each test problem. Entries of the input vectors were the maximal point ammount, that one could earn in each test problem. Entries in the output table give the acquirement of each competence for each student. Consequently, these results (this output table) were statistically processed by software Statgraphics, in order to obtain quanlitative analysis of acquired mathematical competencies.

***K*eywords** Engineering · Mathematics · Mathematical competence · Assessment · A competencies-based methodology · Rules$_{M}athproject$

1 New approaches to student assessment

Today, the goals of mathematics education suitable for technical tertiary schools can be found in the form of recommended learning outputs in the document "A Framework for Mathematics Curricula in Engineering Education" elaborated by SEFI Mathematics Working Group (MWG). This document has come to be the basis for Erasmus Plus Project: 2017-1-ES01-KA203-038491, New Rules for assessing Mathematical Competencies [4] coordinated by University of Salamanca, with 8 partners. STU Slovak University of Technology in Bratislava, Slovakia, is also one of the partners.

2 Assessment of mathematical competencies – experiment implementation

At Institute of Mathematics and Physics, FME STU we decided to carry out our part through a pedagogical experiment. The main goal of the experiment was to design test models focused on assessment of mathematical competencies of students in selected learning units through eight mathematical competencies elaborated by SEFI MWG [1] and introduced by Danish KOM project [2]. The experiment has been carried out in three phases: 1. Design phase; 2. Testing phase; and 3. Evaluation phase.

2.1 Design phase

The first phase of experiment was conducted during the academic year 2018/2019. With respect to scope and curriculla, we prepared tests, by which the problems in guidelines were verified at FME STU: in the topic Analysis and Calculus at bachelor study programmes in subject Mathematics I for the first year of the study and in the topic Probability and Statistics in compulsorily elective subject Basics of Statistical Analysis for the second year of the study.

Overal we have prepared guidelines related to four subheadings:

- two guidelines covering the learning units Complex Numbers and Differentiation in the topic Analysis and Calculus,
- two guidelines covering the learning units Data Handling and Statistical Inference in the topic Probability and Statistics learnt in Basics of Statistical Analyses.

The list of learning outcomes together with suggested level of particular competencies importance, which were examined in proposed assessing models, following the goals of mathematical education in specific courses are displayed in tables Learning outcomes with degree of coverage of competencies involved in this assessment activity in [3].

Since the partial competencies do not act separately, their particular importance in one of three levels was asserted for each of the proposed learning outcomes. The level of importance was rated

V = very important   M = medium important   L = less important

by experienced university teachers of mathematics they formated the working team of Rules_Math STU partner.

2.2 Testing phase

The second phase of experiment was conducted in the academic years 2018/2019 and 2019/2020 .

2.2.1 The course MATHEMATICS I

The test was conducted during the practical classes in the winter semester for the first year engineering students at the bachelor study programmes. Participating in the evaluated test version were 24 students.

The version of test created in the Learning outcomes AC5 Differentiation differs from version used in this phase of Project. Students got different test. The reason is such that not all Learning Outcomes declared in Project schedule were covered by curricula of subject Mathematics I.

Students were allowed to use calculator in order to deal with computational obstacles. No other help nor devices nor software were provided. The test was theoretical one.

2.2.2 The course BASICS OF STATISTICAL ANALYSIS

The test was conducted during the practical classes of the basic Basics of statistical analysis (BSA) course in the summer semester for the second year engineering students at the bachelor study programmes. Participating in the evaluated test version were 25 students.

One version of this test was created in the learning outcomes SP1: Data Handling and SP7: Statistical inference for this working group. It had two parts:

- part “Test of Data Handling and Statistical inference” – the students answered the question in a written form that was collected. Students were allowed to use statistical tables a statistical formulas in order to deal with computational obstacles.
- part “On-line test of Statistical inference ” – it was realized in computer classes. This test was multiple-choice questions.

In this course the experiment was completed with a questionnaire. The questionnaire was aimed to discover attitudes and opinions of students on the motivation and students' competencies in using mathematics to solve real problems and was obtained free interviews with students.

2.3 Evaluation phase

Mathematical competencies can be evaluated by different ways with respect to different learning outcomes. Each author can apply different approach. One way how to evaluate is to evaluate one specific competence in all Learning Outcomes. Another way is to evaluate all present/stated-by-teacher competencies in one learning outcome. Or next way is to evaluate all present competencies in all learning outcomes. Or next way could be by author determined way of evaluation which runs not on learning outcomes, but on test problems and subproblems instead.

We decided to evaluate all present learning outcomes and all present competencies there. The main idea for assessment of acquirement of the mathematical competencies is based on division of the points amount achieved by a student in the test into problems/subproblems/subtopics covering the corresponding competencies. After that, these points are distributed on particular competencies in given learning outcome following the "table of competencies" built by teacher for each test.

For example The rows like this were created for each problem/subproblem in the test.

Problem X	score	SPx	Learning outcomes: xxx	Competencies							
				C1	C2	C3	C4	C5	C6	C7	C8
X.x.	3	SPxx	xxx	V	V	M	V		M		L
Competencies score				$\frac{3}{14}\cdot 3$	$\frac{3}{14}\cdot 3$	$\frac{2}{14}\cdot 3$	$\frac{3}{14}\cdot 3$	$0\cdot 3$	$\frac{2}{14}\cdot 3$	$0\cdot 3$	$\frac{1}{14}\cdot 3$

Table 1: Competence's weight: V = weight 3, M = weight 2, L = weight 1

Consequently were the values in columns summed up.

It was needed to norm the resulting values of acquirement of competence in order to analyse it correctly. It means that these values were compared with maximal possible achievable value for each competence. Data we obtained were quantitatively processed by program developed in Wolfram Mathematica software. Consequently, these results were statistically processed by software Statgraphics, in order to obtain qualitative analysis of acquired mathematical competencies.

2.3.1 The course MATHEMATICS I

Problems/subproblems of the test of Differentiation and learning outcomes with degree of coverage of competencies involved in this test are listed in the following table.

Course: MATHEMATICS I											
AC: ANALYSIS AND CALCULUS											
Problem /subproblems	**score**	**AC5**	**Learning outcomes:** Differentiation	**Competencies**							
				C1	C2	C3	C4	C5	C6	C7	C8
Definition of local minimum.	1	AC55	Locate any points of inflection of a function *	V	V				L		
Formal derivative of expression	1	AC5	Differentiation *	L	L	L			V		
Derivative of inverse function	2	AC52	Differentiate inverse functions	M	M	V			L		
Domain of definition	0.5	AC56	Find greatest and least values of physical quantities *	V	V	L					
Derivative	1	AC56	Find greatest and least values of physical quantities *	L	L				V		
Monotonicity and local extremes	1.5	AC56	Find greatest and least values of physical quantities *	V	V	V			L		
Graph	1	AC56	Find greatest and least values of physical quantities *	V	V		V				
Domain of definition	0.5	AC55	Locate any points of inflection of a function	V	V	L					
Derivatives	1	AC55	Locate any points of inflection of a function	L	L				V		
Intervals of convexity and concavity	1.5	AC55	Locate any points of inflection of a function	V	V	V			L		
Points of inflection	1	AC56	Locate any points of inflection of a function	V	V	V			L		
Evaluation of derivative of physical quantity in the time	1	AC56	Find greatest and least values of physical quantities	V	V	V			M		
Find global maximum of physical quantity on interval	3	AC56	Find greatest and least values of physical quantities	V	V	V			M		
Estimation of derivative of physical quantity	1	AC56	Find greatest and least values of physical quantities	V	V	V	M	M	L		
Total	**17**										

Table 2: Differentiation learning outcomes with degree of coverage of competencies involved in test (sub)problems.

*In the table *Tab.2.* of assessment of activities AC5 in [3], there are some activities, which are not included in curriculum of FME of STU. And some activities are similar to activities included in this Table 2.

We would like to present three figures.

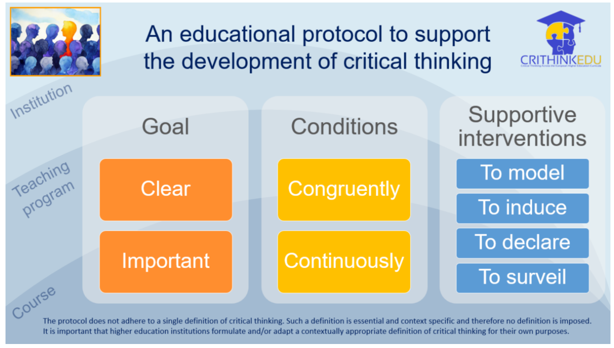

Figure 1: Averages of acquiring the competencies.

The Figure 1 illustrates the level of acquiring the competencies. It is clear that C6: Handling mathematical symbols and formalism was the best of tested competencies. On the other hand the competence C3 was the worst one in the set of tested students. (Competences C7 and C8 were not tested, competences C4 and C5 were tested only by few problems, which exclude them from significant group of competencies.)

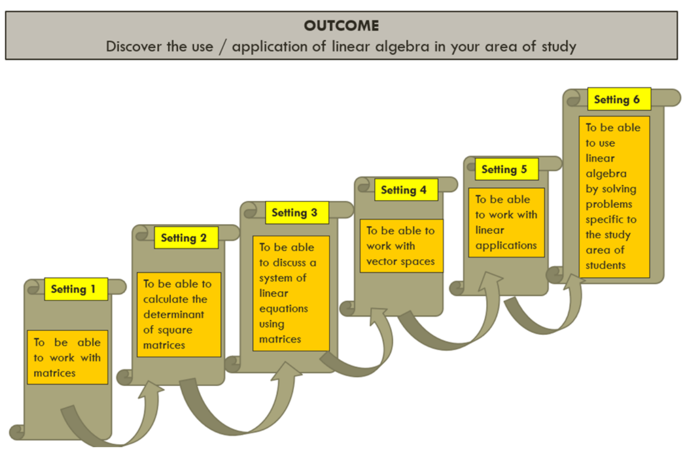

Figure 2: The comparing the competencies using confidence intervals.

According to the Figure 2 it is possible to find mean of acquiring of competence by all students in the course, not only by tested ones. For example the competence C6 (c.f. Table 3.): the average is 0.7625 and mean of acquiring of competence by all students is in interval $\langle 0.66; 0.87 \rangle$. In any case, this is good result.

	Count	Average	Median	Standard deviation	Minimum	Maximum	Range	Lower quartile	Upper quartile	Interquartile range
C1	24	0.674167	0.690	0.205933	0.19	0.89	0.7	0.575	0.845	0.270
C2	24	0.674167	0.690	0.205933	0.19	0.89	0.7	0.575	0.845	0.270
C3	24	0.603333	0.630	0.309232	0.10	1.00	0.9	0.380	0.900	0.520
C4	24	0.550000	0.700	0.342451	0.00	1.00	1.0	0.350	0.700	0.350
C5	24	0.083333	0.000	0.288675	0.00	1.00	1.0	0.000	0.000	0.000
C6	24	0.762500	0.770	0.166303	0.35	0.95	0.6	0.715	0.900	0.185
Total	144	0.557917	0.695	0.336732	0.00	1.00	1.0	0.265	0.815	0.550

Table 3: Differentiation – Summary Statistics

What about weak or clever students? Are there any (with respect to any competence)? This question can be answer by dint of the Figure 3.

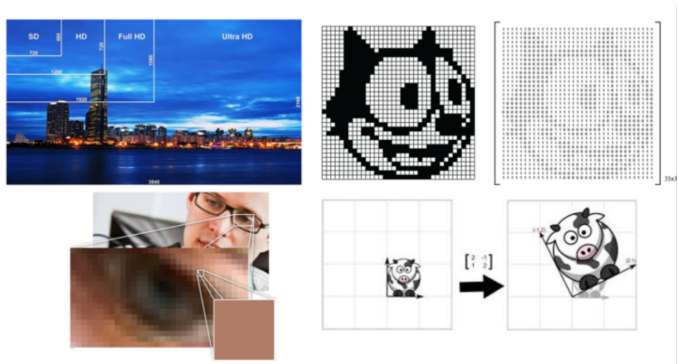

Figure 3: The comparing the competencies using Box-Plot.

Disregarding C4 and C5, we can see that in competence C6 we have a weak student. The dot below the box illustrates this situation.

2.3.2 The course BASICS OF STATISTICAL ANALYSIS

There were used next tests.

a) Test of Data Handling and Statistical inference

The purpose of the test of Data Handling and Statistical inference is to provide a more detailed analysis of the mathematical competencies and know which of these competencies were acquired by the students in highest and lowest quality.

Problem 1.

All employees of certain computer company travel every day to work by public transport. The daily travel times (min) home-to-work are shown in the Table 4 bellow. (We assume that travel time to work is a random variable with a normal distribution).

Daily time	Number of employees
6 – 15	8
16 – 25	14
26 – 35	12
36 – 45	9
46 – 55	7

Table 4: The daily travel times home-to-work

1.1. Calculate measures of central tendency of travel times to work.

1.2. Calculate measures of variance / standard deviation of travel times to work.

1.3. Sketch a box-and-whisker-plot with corresponding numerical characteristics.

1.4. Sketch a graph of normal data distribution based on the calculated measures of central tendency.

The competencies, which were measured in Problem 1 are listed in the Table 5.

Course: BASICS OF STATISTICAL ANALYSIS											
SP: STATISTICS AND PROBABILITY											
Problem 1	score	SP1	Learning outcomes: Data Handling	Competencies							
				C1	C2	C3	C4	C5	C6	C7	C8
1.1.+1.2.	2	SP15	Modification: Frequency Tabulation	V	V	M			L		
1.1.	3	SP16	Calculate measures of average median, mode for a grouped set of data	V	V	M	V		M		L
1.2.	1	SP16	Calculate measures of variance and standard deviation a grouped set of data	V	V	M	V		M		L
1.3.	3	SP11	Calculate the range, inter-quartile range for a set of data items	V	V	L			M		L
	2	SP14	Modification: Construct a suitable BOX–PLOT from a data set	V	M	L	V		L		
1.4.	1	SP14	Modification: Construct a graph of normal data distribution from a data set	V	M	L	V		L		
Total	12										

Table 5: Data Handling learning outcomes with degree of coverage of competencies in Problem 1

Note. In the table Tab.3. of assessment of activities SP1 in [3] , there are some activities, which are not included in Problem 1.

Acquiring the competencies by students was compared by means of unified score in each competence.

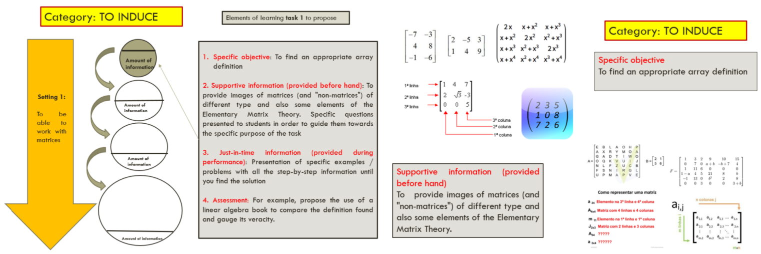

Figure 4: The comparing the competencies using Box-Plot.

It is evident from Figure 4., that there are some outliers in competence C8 . The possible interpretations are that these two student are weak at all or have unsufficient experience with Making use of aids and tools only. Personal conversation with them confirmed the second case.

	Count	Average	Median	Standard deviation	Minimum	Maximum	Range	Lower quartile	Upper quartile	Interquartile range
C1	25	0.9476	1.0	0.0700167	0.80	1.0	0.20	0.87	1.0	0.13
C2	25	0.9548	1.0	0.0602854	0.82	1.0	0.18	0.89	1.0	0.11
C3	25	0.9532	1.0	0.0641431	0.80	1.0	0.20	0.89	1.0	0.11
C4	25	0.9192	1.0	0.1085100	0.68	1.0	0.32	0.80	1.0	0.20
C6	25	0.9540	1.0	0.0621825	0.81	1.0	0.19	0.89	1.0	0.11
C8	25	0.9688	1.0	0.0523068	0.82	1.0	0.18	0.94	1.0	0.06
Total	150	0.9496	1.0	0.0722865	0.68	1.0	0.32	0.89	1.0	0.11

Table 6: Data Handling –Summary Statistics

We can see in Table 6 that the difference between averages are very small and medians are the same throughout all competencies. But there are differences between standard deviations. The greatest variance is in competence C4 and the least one in competence C8. Variances in C1, C2, C3 and C6 can be considered the same. These fact is confirmed by aj 95.0% confidence intervals (CI) for mean of CI_i, $i = 1, 2, \ldots, 8$ in Figure 5.

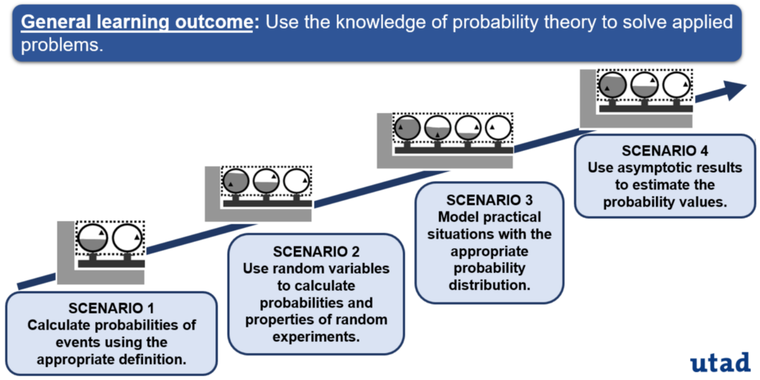

Figure 5: The comparing the competencies using confidence intervals.

We can see in Figure 5 the widths of intervals C1, C2, C3 a C6 are almost the same. It means that medians, averages and standard deviations do not signal great differences in between students' performance in competencies C1, C2, C3 and C6 – i.e. the level of acquirement of competencies C1, C2, C3 and C6 is approximately the same.

Significant difference can be seen in competencies C4 and C8. The widest confidence interval is the C4-confidence interval with the least mean and on the contrary the narrowest one is C8-confidence interval with the greatest mean. It means that the students' performance is mostly concentrated around the mean in competence C8, in which the student achieved the greatest average level 96%. Students' performance is the most heterogeneous in C4, in which the least average level is achieved. Only 91%.

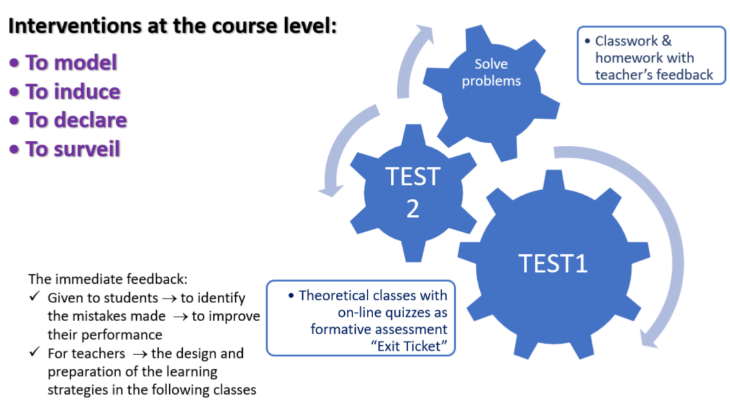

Figure 6: Averages of acquiring the competencies.

According to the result of the test, the greatest value (96%) of acquiring of competence was achieved at C8: Making use of aids and tools in spite of two outliers. The least value (91%) was achieved at C4: Modeling mathematicaly. In the future it is important to focus on C4 and to engage students with problems targeting on development of C4. We appreciate very positively the level of acquiring of each competence which is about 95%.

Problem 2.

Two production lines fill bottles with beer. 25 bottles are taken out randomly from the first production line and it was found out that the value of sample mean of the beer volume in these bottles was 2.04 liters. Subsequently, 20 bottles were taken out from the second production line and we found out that value of sample mean of the beer volume in this bottle was 2.07 liters. It is supposed that two random samples come from a normal distribution and we known that both production lines fill bottles of beer with different precision.

Find out at the level of significance $\alpha = 0.05$ whether both production lines fill bottles of beer with the same accuracy.

2.1. Test the hypothesis using the test statistic.

2.2. Test the hypothesis using the confidence interval.

2.3. Test the hypothesis using the *P–value* graphically only.

The competencies, which were measured in Problem 2, are listed in Table 7.

Course: BASICS OF STATISTICAL ANALYSIS											
SP: STATISTICS AND PROBABILITY											
Problem 1	**score**	**SP7**	**Learning outcomes: Statistical Inference**	**Competencies**							
				C1	C2	C3	C4	C5	C6	C7	C8
	2	SP71	Modification: Point estimate of parameters – understand the difference between the characteristic values of a population and of a sample	M	V	L	L	M	V	L	M
	1	SP79	Recognise whether an alternative hypothesis leads to a one-tail or a two-tail test	M	M	M	M	V	V	L	M
2.1.	6	SP72	Follow the main steps in a test of hypothesis.	V	V	V	M	V	M	M	V
	1	SP74	Modification: Understand the level of a test (error of the first kind)	V	V	M	L	M	L	V	M
	2	SP78	Test claims about the population mean using results from sampling	V	V	V	V	L	L	V	V
	2	SP77	Place confidence intervals around the sample estimate of a population mean	V	V	M	M	M	L	M	V
2.2.	2	SP80	Compare the approaches of using confidence intervals and hypothesis tests	V	V	M	M	M	L	M	V
2.3	2	SP73	Modification: Compare graphically the approaches of using a test of hypothesis and a significance test (*P*–value)	M	V	L	V	M	M	M	V
Total	**18**										

Table 7: Statistical Inference learning outcomes with degree of coverage of competencies in Problem 2

Note. In the table Tab.4. of assessment of activities SP7 in [3], there are some activities, which are not included in Problem 2.

Acquiring the competencies by students was compared by means of unified score in each competence.

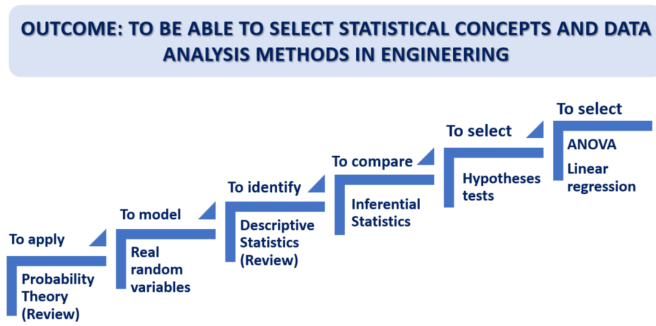

Figure 7: The comparing the competencies using Box-Plot.

It is evident in Figure 7 that in each competence there is no outliers. It signals that there are no very weak student (with respect to acquiring any competence) in the group of tested students. (It is good.) It signals that there are no very clever students (with respect to acquiring any competence). (What a pity.)

	Count	*Average*	*Median*	*Standard deviation*	*Minimum*	*Maximum*	*Range*	*Lower quartile*	*Upper quartile*	*Interquartile range*
C1	25	0.6372	0.65	0.124248	0.43	0.82	0.39	0.53	0.75	0.22
C2	25	0.6568	0.67	0.125091	0.44	0.84	0.40	0.55	0.77	0.22
C3	25	0.6312	0.62	0.126699	0.43	0.82	0.39	0.51	0.75	0.24
C4	25	0.6532	0.66	0.116859	0.46	0.84	0.38	0.55	0.76	0.21
C5	25	0.6728	0.68	0.129567	0.45	0.88	0.43	0.57	0.78	0.21
C6	25	0.7228	0.74	0.129245	0.48	0.91	0.43	0.60	0.85	0.25
C7	25	0.6384	0.65	0.122566	0.42	0.83	0.41	0.54	0.75	0.21
C8	25	0.6448	0.66	0.122818	0.45	0.83	0.38	0.54	0.76	0.22
Total	200	0.5715	0.66	0.125596	0.42	0.91	0.49	0.55	0.76	0.21

Table 8: Statistical Inference –Summary Statistics

We can see in Table 8. that the difference between averages, medians and standard deviations are very small. It would be interesting to find out the presence of statistical significant difference in between some of values of averages and standard deviations.

Then we investigated whether there is a statistical significant difference in between any mean. A t-test was used to test a specific hypothesis about the difference between the means of the populations from which the two samples come. By this test we have found out that a statistically significant difference between the means of the populations is between competencies C1-C6, C3-C6, C6-C7 and C6-C8.

95.0% confidence intervals (CI) for mean of CI_i, $i = 1, 2, \ldots, 8$ are graphicaly presented in Figure 8. We can see the narrowness of these intervals. It means that mean of competences are estimated with greater accuracy and grade of average acquiring of any competernce is more accurate.

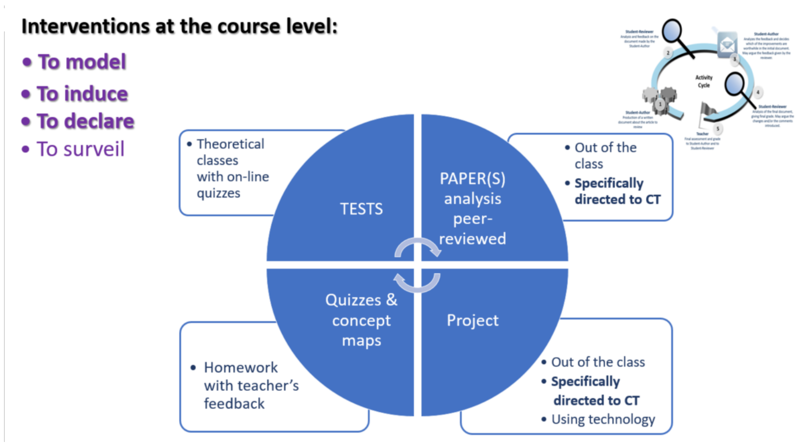

Figure 8: The comparing the competencies using confidence intervals.

Average acquiring of the competences is displayed in Figure 9

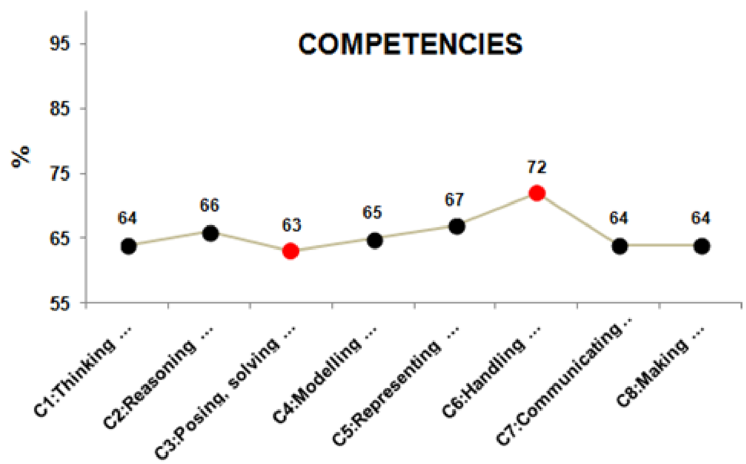

Figure 9: Averages of acquiring the competencies.

According to the result of the test, the greatest value (72%) of acquiring of competence was achieved at C6: Handling mathematical symbols and formalism. The least value (63%) was achieved at C3: Posing and solving mathematical problems. In future it is important to focus on C3. To engage student with problems targeting on development of C3.

Level of acquiring of each competence is over 60% . This is positive. In the group of tested students, there is no one with very low acquiring level for any competence. We would appreciate the occurence of students with higher acquiring level (as high as possible) for some competence.

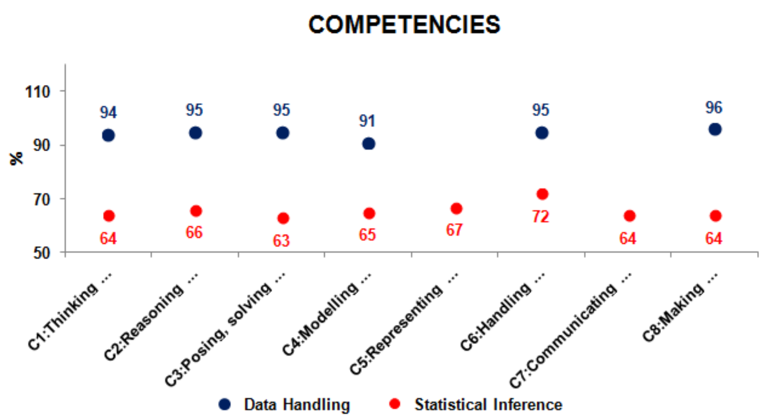

Figure 10: Comparing of averages of acquiring the competencies.

We can see in Figure 10 the level of acquiring of each competence in Data Handling is about 95% and the level of acquiring of each competence in Statistical Inference is over 60%. „Routine“ solving of mathematical problems prevails in Data Handling. Students adopt it more easily. Problems of Statistical Inference are more demanding for students because one is supposed to write hypothesis test by words and mathematically, to understand the difference between the characteristic values of a population and of a sample, to define the level of a test (error of the first kind), to understand the difference between a test of hypothesis and a significance test (*P–value*) and to find a critical value in tables.

Regarding the test as a tool to assess competencies, we may conclude that it covers most of the competencies that we need to evaluate. Difficulty of test is adequate for our students.

b) Online-test of Statistical inference

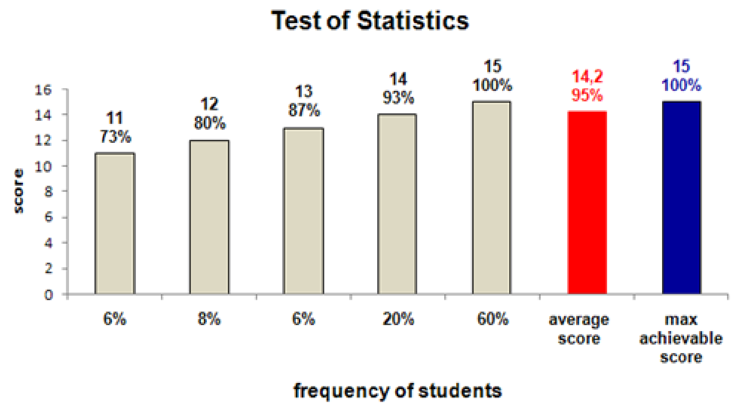

Figure 11: Test results.

Each student obtained at least 73% in the test. Only 1/5 (20%) of students obtained less than 93% score. 80% of student obtained in this test over 93% and 2/3 of all student obtained 100%. With respect to excellent results there is no need to analyse competencies.

3 Conclusion

Our approach to assessment of competencies started fom the initial idea which was about analysing of only one competence throughout the test. We tried to upgrade the assessment of competencies on all possible/present competencies in the test. Thus at the end after quantitative and qualitative analysis, it is possible to made an comparison between levels of acquiring the different competencies. What would be a nice idea? To compare different methodologies (the ones created by project partners) about assessment of competencies on the same set of students.

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Understanding the Failure in Differential and Integral Calculus in the Degrees of Engineering at a Higher Education School in Portugal

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Abstract

Major difficulties are often found among students of engineering degrees in the Curricular Units (CU) of Mathematical Science area, particularly those related to Differential and Integral Calculus (DIC), which lead to a high failure rates. A research work has been developed with the objective of finding the reasons that lead the students to fail in the CU of DIC (CU-DIC), taught in the 1st year of undergraduate degrees of engineering, in the Coimbra Engineering Institute (ISEC), in Portugal. Applying a case study methodology, this study will present a current diagnosis of CU-DIC in the ISEC, with the objective of defining a set of actions that will guide a research on the causes of failure in mathematics in engineering degrees at ISEC. We will intend to establish relationships between teaching methods and how students learn, and besides, build learning environments that lead them to higher success with the co-responsibility of all actors in the educational process. The collected data analysis allow us to conclude that the CU-DIC in the ISEC maintain an identical distribution in the hourly load in several engineering degrees, contents are adjusted to each context taking into account the CU of each degree. The data analysis found better results in the process that includes two examination periods with no relation between class attendance, dropout and pass rates. We propose some different teaching/learning strategies in CU-DIC and new learning environments that enhance freshmen students engagement and participation in their own learning processes.

Keywords Differential and Integral Calculus · Mathematical Knowledge in Engineering · Teaching and Learning

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1 Framework

In Portugal, there is growing democratization but decreasing investment in higher education. Portuguese students access to higher education through very different programs (Scientific – Humanistic graduates, Professional undergraduates Courses, Technological undergraduates courses, Technological Specialization undergraduate courses, over 23 years old, etc), which increased the heterogeneity of personal and motivational features of students. On the particular case of students from engineering degrees, they reveal heterogeneous mathematical skills, asymmetries in the essential mathematics knowledge and difficulties on integration into higher education. Curricular Units of Differential and Integral Calculus (CU-DIC) are responsible for the theoretical basis necessary for professional's future in engineering, and for this, they are present in most of the undergraduate degrees taught in various higher education institutions (HEI).

However, it has been noticed over the years that this basic science is the cause of high failure rates in engineering undergraduate degrees, resulting in several problems, such as absenteeism and, consequently, dropping out higher education studies. The failure and drop out rates in CU-DIC are one of the most important issues in the investigation, frequently appear in national and international discussions being addressed in several published works [26, 11, 1, 12, 13, 9]. So, it is necessary to question what methodologies and teaching approaches are applied, which learning environments are developed that best allow students to be co-responsible in their educational process, and which assessment practices are related to their school success and lead to significant learning [3, 25, 10, 20]. The acquisition of basic and elementary knowledge, essential to the full integration of students in UC-DIC, has been one of the main reasons for failure in higher education. This discussion has led to the definition of multiple strategies aimed to overcome the difficulties detected and the consequent analysis of the impact of the implementation of the measures [18, 14, 15, 23, 16, 22, 6, 17].

The insufficient preparation on mathematics, that students reveal when they arrive at higher education, is not new neither exclusive of the Portuguese education [2, 24]. These difficulties, experienced by students, are a big concern expressed by teachers and led to many adaptations of curricular programs and the definition of different strategies to implement, that allowed those students to follow a positive learning process [21, 19].

On the other hand, Bologna reforms centred the study on the student and reduced contact times between teacher and student, which limits the ability of teachers to intervene directly.

2 Differential and Integral Calculus in ISEC

Teachers in ISEC have proven, over the years of teaching, what the literature presents, noting the great difficulties of the students, in particular those referring to the Basic Infinitesimal Calculus. Indeed, these performances lead to a high failure rate and therefore to a demotivation of all those involved in the educational process, placing teachers in constant self-reflection. Engineering teachers should develop appropriate strategies to adapt, as best as possible, to the ever-increasing heterogeneity found in the knowledge and skills acquired by students in secondary education [5, 8, 7].

Aware of the difficulties in accessing to higher education, ISEC's Department of Physics and Mathematics (DFM) has developed several activities that allow students the opportunity to bridge gaps in mathematics. One of the measures found by the DFM was to offer CU-DIC in a sliding regime. This extraordinary regime arises from the attempt to overcome the failure detected over the years, in the CU of Mathematics, taught in the 1st year/1st semester of engineering studies. In fact, in the academic year 2002/2003, the Scientific Committee (CC) of the Scientific Area of Mathematics (ACM) of ISEC implemented the pedagogical experience "sliding disciplines" which, after analysis and corrections introduced to optimize resources and improvement of results, began to integrate the distribution of teaching service. These CUs work in alternative semesters, complementing the curricular program of the degree. This operation also allows students over 23 years to access ISEC, prior preparation during the first semester to acquire the basic essential knowledge to the integration in the curricular units of DIC.

Trying to reverse these trends in students' performance it was decided to carry out an exploratory study, which would allow a diagnosis of the situation. So, a research work was developed to find the reasons that lead the students to fail in the CU of DIC taught in the 1st year of the undergraduate degrees of engineering at ISEC in general, and in particular in the CU, whose head is one of the authors. Applying a case study methodology, the authors tried to establish relationships between teaching evaluating and learning processes to build environments that lead to higher success. The analysis of these data may subsequently lead to the need for an investigation that seeks to design, develop and evaluate an intervention at the level of teaching and assessment practices of the CU-DIC that leads to better learning and increased success rates.

3 The Study

Coimbra Engineering Institute is an organic unit of the Polytechnic Institute of Coimbra (IPC) that offers degrees in engineering, such as: Civil (EC), Industrial Management (EGI), Chemical (EQ), Electromechanics (EEM), Electrotechnology (normal-EE and post-work regime-EE (PL)) and Informatics (normal-EI, post-work regime-EI (PL)) and European course-EI (CE)), Mechanics (EM), Bioengineering (BioE) and Biomedical (EBiom). In order to meet the weak demand for EC, in the 2018/19 school year, ISEC created a new degree in Sustainable Cities Management (EGC) which was presented with a curricular plan focused on economic, environmental and social sustainability of learning methodologies that allow students to develop professional skills.

3.1 Methodology

This exploratory study follows a methodology of quantitative investigation, considering the observation and analysis of collected data. Taking into account that the analysis may allow us to understand and explain the factors that influence students' failure in CU-DIC integrated into the curricular plan of engineering degrees, the approach of this case study will be done according to an interpretative paradigm. Therefore, it is intended, without any kind of control over the situation, to obtain explanations that allow the establishment of relations between the operation of the various CU-DICs taught at ISEC and their respective pass rates. The conclusions reached may lead to the implementation of teaching, learning and assessment strategies that contribute to the promotion of success in those CUs. The data treatment had a descriptive statistical approach.

3.2 Instruments

From the academic year 2010/11, CC-ACM requested CU managers from several Disciplinary Groups (groups of teachers who teach CU in the same mathematical area: analysis, algebra, statistics and applied mathematics) the reports of information systematization (RCU) that include class attendance, dropout and pass rates. These reports are completed semi-annually by the professors responsible for each CU related to DIC and constitute the data collection to be evaluated in order to carry out a continuous analysis of the results obtained in the CU-CDI.

These instruments refer to the period from 2011/12 to 2017/18, corresponding to 7 academic years, associated with 11 CU-DIC, and integrated into the Disciplinary Group of Analysis.

Taking into account the context and the objective of the study that was proposed, we collected the following relevant information:

- Dropout rate in the examination moments given by the ratio between the number of evaluated students (A) and the number of present students at the assessment (P), i.e. A/P.
- Pass rate in examination moments are given by the ratio between the number of students approved (Ap) and the number of evaluated students (A), i.e. Ap/A.

The analysis of these assessment rates corresponds to the exam (Ex) and distributed evaluation (AD), which includes 2 or more mid-term examination moments. The study was conducted in different forms of assessment in order to understand which strategy has the best final results.

However, since the RCUs integrate final information elaborated in a systematized form by the teachers responsible for each CU-DIC, they do not allow to draw more specific conclusions on the relation of the pass rates with the alternative assessment methods. Consequently, a complementary study was carried out in the degrees of EBiom and EI. For this purpose, the assessment guidelines were analysed in these 7 years under review, from 2011/12 to 2017/18, and which integrate the data of the pass rate in AD, Ex and the final pass rate (T). It should be noted that similar to the RCU, these pass rates are calculated in relation to the number of students evaluated, thus excluding dropouts.

3.3 Sample

The results presented here come from the analysis over a period of 7 years, between the academic year 2011/2012 and 2017/2018. They were analysed UC-DICs evaluation reports (RCU) of 10 engineering degrees (EC, EGI, EQ, EEM, EE, EI, EM, BioE, EBiom and GSC), for the 1st semester and 4 of these UC-DICF (EM, EEM, EI and EE) taught in the 2nd semester, under the CU sliding regime. In total, 75 RCU were analysed, 51 of the first semester and 24 of the second semester referring to the CU sliding regime.

4 Data analysis

4.1 General Scope

About the dropout rate, it is observed that only 9 CU have percentages below 75% and 50% of the RCU refer to values above 90%. Also, in this approach, the lowest values are recorded in the CUs that operate in a sliding regime.

The pass rates in the 7 years under analysis is summarized in Figure 1. The data in the RCU for the first semester concludes that EBiom presents the best results with a mean of 73.10% (between 62.5% and 88.2%), with EE showing lower results than the other undergraduates (between 27.9% and 52.1%), presenting an average of 42.24%. It is also verified that the overall average of the pass rate in the first half of the year is 58.40%, with a standard deviation of 14.34% and an average deviation of 12.4%.

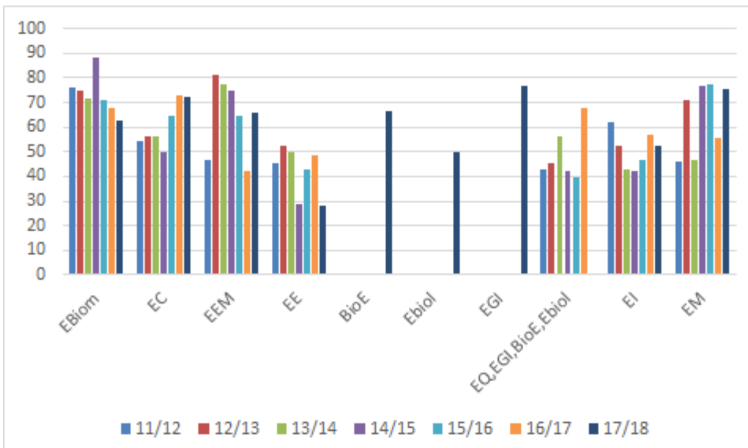

Figure 1: Pass rate distribution (1st semester).

Regarding the second semester, the results evidenced in the RCU were summarized in Figure 2. From the collected data it can be evidenced that EM is the degree that has the highest pass rates, with an average of 66.32%, followed by EE with 52.26% and EI with 50.07%. It is also verified that the overall average of the pass rate in the 2nd semester of the year is 57.36%, with a standard deviation of 17.39% and an average deviation of 12.96%.

4.2 Private Sector - Biomedical and Informatics

In the analysis of the assessment guidelines of CU-DIC, we can infer from the EBiom results that the pass rate, the AD is the students' preferred modality, although in 2016/17 there was a reversal of the situation (Figure 3). The AD pass rates are between 83.33% and 100%, while Ex varies between 25% and 66.67%. Total pass rates are between 71.79% and 88.24%, with an average of 78.82%.

With regards to Informatics, the CU-DIC works in the first semester and in a sliding regime (2nd semester). Regarding the pass rates in the first half, the Ex ratio is between 13.30% and 38.60%,

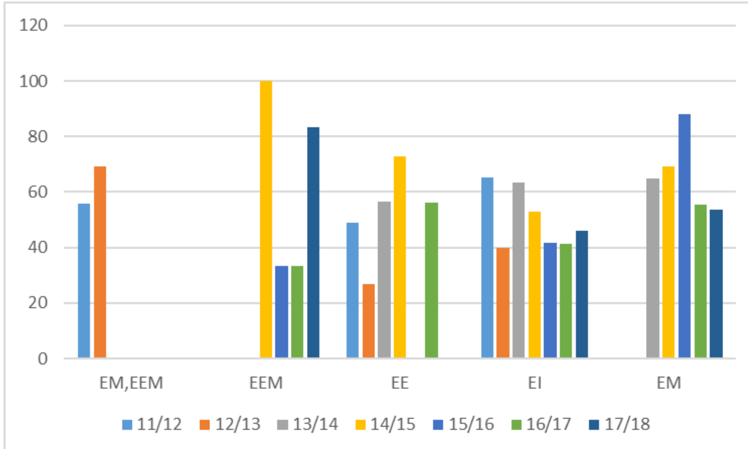

Figure 2: Pass rates distribution (2nd semester).

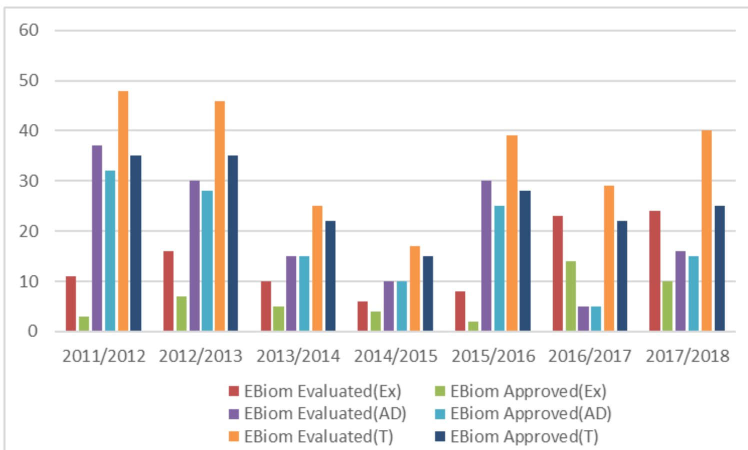

Figure 3: Pass rates distribution (AD, Ex, T): Biomedical.

while in the AD there are pass rates between 67.27% and 81.25% (Figure 4). Total pass rates range from 41.15% to 57%, with an average of 48.02%.

About to the second semester, regarding the pass rates in the second half of the year, the Ex ratio is between 20% and 32.84%, while in the AD there are pass rates between 47.50% and 86.11% (Figure 5). It should be noted that the worst percentages in AD were in the years 2011, 2012 and 2016, which integrated a component distributed by 4 moments of mid-term examinations. Total pass rates range from 40.23% to 51.09%, with an average of 44.69%.

In summary, we can conclude that the pass rates obtained by AD induce to think this to be a strategy leading to the success in CU-DIC taught in EBiom and EI degrees (Table 4). It should also be noted that the difference in performance between the students of both degrees can be explained by the ease of integration of EBiom students in the CU-DIC. This corroborates the study carried out which concludes that in terms of the median, EBiom students perform better on the diagnostic test, applied to the entrance of higher education in ISEC, while EI students present the worst classifications [4].

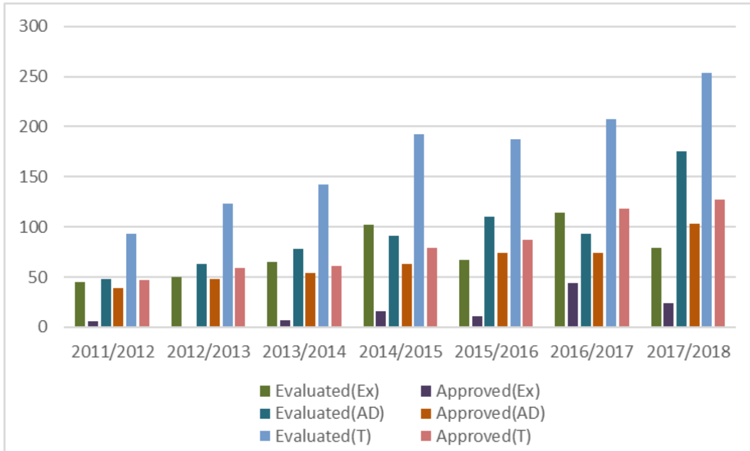

Figure 4: Pass rates distribution (AD, Ex, T): Informatics – 1st semester.

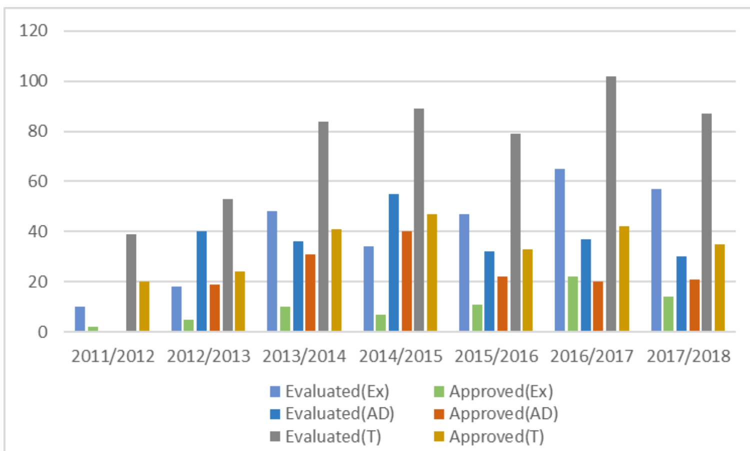

Figure 5: Pass rates distribution (AD, Ex, T): Informatics – 2nd semester.

5 Conclusions and Future Work

In the study carried out in the CU-DIC at ISEC to make a diagnosis of the situation regarding the teaching/learning of those CUs in engineering degrees, we found the following conclusions.

The pass rates of the CUs that run on a sliding regime do not differ between the two semesters. Therefore, it is possible to infer the need to review the functioning of these CUs. It will be important to apply a concerted strategy of investment in the learning carried out by the student, that it can't be considered only one more opportunity for success. In the particular context of EBiom and EI it is concluded, after analyzing the data that students obtain better results in the processes that include distributed evaluation, preferably the one that integrates two mid-term examination moments. For the 7 school years analyzed, EBiom has an average pass rate of 78.82%. EI shows an average pass rate of 48.02% in the 1st semester and 44.69% in the 2nd semester. Future work will need to be taken into account to understand what is the student profile in higher education that leads to better pass rates.

	Ex		AD	
	Mean	Std. deviation	Mean	Std. deviation
Biomedical	45.03%	12.13%	93.84%	5.2814%
Informatics (1stS)	18.48%	9.15%	71.66%	6.30%
Informatics (2ndS)	24.29%	3.52%	64.37%	11.46%

Table 1: Pass rates measures: Ex and AD

As already mentioned, the teaching of CU-DIC has been evidenced in many studies, namely about difficulties demonstrated by students in basic and elementary contents, essential to their full integration in that subject. This inevitably leads to an adaptation of the curricular organization and the definition of actions that allow modifying the situation. Since 2015, ISEC is implementing a Mathematics Support Center that aims to help students overcome gaps in essential math concepts. Another solution may be the introduction of teaching strategies that allow students to adapt their learning styles to the desired learning outcomes.

The low participation of the students in the curricular assessment process together with the pass rate obtained in the distributed evaluation can lead us to enunciate a set of questions that are related to:

1. Student's profile attending and participating in different examination models proposed by teachers.
2. Teaching/learning strategies to be applied, aimed at reaching students who do not carry out the examination and understand the consequent reasons that led them to dropout.
3. The set of basic and elementary level knowledge that students need to master upon entering higher education.
4. Mistakes made in basic and elementary knowledge that allow the definition of a structured intervention in overcoming gaps.
5. Educational environments that lead to meaningful learning and involve all actors (teachers and students) in the educational process.

The answer to these and other questions that may be related shall form the basis for future work.

Acknowledgement

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IMPROVING ENGINEERING THERMODYNAMICS LEARNING WITH MATHEMATICA

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ABSTRACT

Sophomore students from the Chemical Engineering Degree at the University of Salamanca, are involved in a Mathematics course during the first semester and in an Engineering Thermodynamics course during the second one. When they participate in the latter they are already familiar with mathematical software to solve numerical methods problems, including non-linear equations, interpolation or differential equations. On one hand, we present in this paper some of the materials elaborated in both courses with the Wolfram Mathematica package, and on the other, the didactic organization of the Engineering Thermodynamics course. The objective of the experience is to increase the interrelationship between different subjects, to promote transversal skills, and to make the subject closer to real working procedures the students will find in their future careers. The satisfactory results of the experience are exposed in this work.

***K*eywords** Engineering Education · Engineering Thermodynamics · Numerical and computational Methods

1 Introduction

Chemical Engineers must be professionals capable of developing their work through innovation and continuous improvement of processes and products with analytical, creative and critical thinking, entrepreneurial spirit and the ability to lead highly productive teams. In their professional performance they will have to plan, analyze and interpret, design, implement, evaluate, investigate and put into practice possible solutions to needs that arise in society in their area of work or company [1]. All these skills are also essential for other types of engineering degrees. This work is focused on chemical engineering degree but the objectives, procedures and methodology could be easily applied to other specialities.

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Traditionally, there has been a mismatch between the way in which universities have evaluated the results of their educational processes in Mathematics and Engineering Thermodynamics and the way in which society, in general, and companies, in particular, do so. We could say that until a few years ago, teachers at the university wanted to find out what knowledge the student had, but nowadays this has changed and now we focus on what skills and abilities do they have. Competency-based assessment seeks to change the educational process to guide it in this direction [2].

Competency-based learning is one remarkable change that the Bologna Process have brought. The European Higher Education Area implies a way of teaching and learning, where competencies represent the central axis of the new system. This means that it is no longer enough for a student to learn technical or specific knowledge. In this framework, students must also acquire a series of competencies that guarantee that they are capable of effectively and adequately exercise the work for which they are prepared. In any case, the contents are still necessary and essential, although they acquire a practical and applicable nature [3].

The definition of competencies arises from the need to understand an increasingly diverse and interconnected environment. Individuals need to master technologies and manage huge amounts of information. In these contexts, the competencies that individuals need to satisfy to achieve their goals have become more complex, requiring greater mastery of certain skills [4]. Based on the need for university students to acquire competencies, teachers must adapt the teaching-learning processes [5].

This is the reason why teachers at the University of Salamanca, who teach Mathematics III and Engineering Thermodynamics of second year of the Chemical Engineering Degree, in successive semesters, participate in this study. So, the methodology regarding the acquisition of competencies carried out in Mathematics, has explicit continuation in Engineering Thermodynamics, enriching the teaching-learning process of both subjects. Of course, coordination among all the teachers who tutor the same grade is essential, but also it is basic the coordination in subjects such as Mathematics and Physics. It is not necessary to justify this collaboration, since it is evident the need for mathematics as a language to express relationships in Physics. However, there are studies that analyze the elements that influence thermodynamic learning at university level, and they show the advantages of establishing collaborations between teachers of mathematics, physics and engineering, that is, collaborations between different disciplines [6, 7].

Furthermore, there are serious difficulties in teaching mathematics to engineering students. These students often face with difficulties in learning mathematical contents, in acquiring mathematical competencies and ultimately, in being proficient in mathematics. When we teach our engineering students subjects like Calculus, Linear Algebra, Numerical Methods, etc., one of our main concerns is usually how we can motivate them to learn mathematics. Engineering students often do not see the relationship between mathematics and other subjects, such as electricity, mechanics, mechanisms, automation, electronics or thermodynamics [2]. When students from engineering courses find the connection between topics from different subjects, they are more motivated [8]. With this proposal we try to make the students aware that the tools that are worked in Mathematics are useful to solve the problems of Engineering Thermodynamics.

In general, the use of the computer is of special relevance in Science and Engineering courses. Today, it is difficult to understand laboratory data processing without computational support. In the same way, programs with mathematical tools allow for solving Mathematics and Engineering Thermodynamics problems much more efficiently and similar to the real professional world.

Our objective is that students know how to use mathematical tools to formulate and solve problems that arise in other subjects, more specifically in Engineering Thermodynamics. It is generally accepted that computer-based (or computer-enhanced) problem solving is a very important application of the computer in engineering education and practice [9]. The *Wolfram Mathematica* package, without going any further, is a very suitable tool for solving mathematical problems. The use of symbolic software packages such as *Mathematica* has been steadily rising in academic instruction and specifically in Mathematics, and this trend is likely to gain strength in the upcoming years. There are numerous publications that reveal various applications of *Mathematica* for the training of engineers in general [8]. Moreover, for instance, interesting works were developed in the field of of chemical engineering [10, 11].

With all the arguments explained above, during the last two courses, we have organized a project in which students of Mathematics III (Numerical Methods) and Thermodynamic Engineering solve problems of these subjects with *Mathematica* program. In the first semester, in Mathematics III, as part of the training of the whole group, they carry out practices of numerical algorithms described in the theoretical classes using that software. The activity takes place in small groups in a computer room. In the second semester, the realization of some Engineering Thermodynamics problems with *Mathematica* software has been proposed to a test group of volunteer students. In five sessions, one per topic, work is done, together with the teacher, to solve problems with the help of the computer. This initiative is included in the university activities as an innovation and teaching improvement project "*Assessment mathematical competencies in science and engineering degrees ID2018/027*" [12]. The project has been carried out in coordination with all teachers (from physics and mathematics courses). On the other hand, it is part of the European project RULES_MATH [13], where we work on competencies-based learning.

This paper is organized as follows. Firstly, in Section 2 the context of the project carried out with students of the Chemical Engineering Degree will be exposed. This project is encompassed within the global methodology of the Thermodynamic Engineering course. Methodology is summarized in Sec. 3. Some highly relevant examples of the work done are introduced in Sec. 4. Finally, the evaluation methodology and the most significant conclusions are exposed (Secs. 5 and 6 respectively).

2 Context

The number of new students entering in the Chemical Engineering Degree at the University of Salamanca (USAL) each academic year is in the interval $60-70$. This Degree enables to exercise the regulated profession of Industrial Technical Engineer. The curriculum lasts 4 years (60 credits per year to complete 240). Adequate training in this scientific field involves the acquisition of the basic knowledge and skills that guarantee getting to know and being able to develop the design of processes and products characteristic of the chemical industry and the multiple sectors related to it (pharmaceutical, biotechnology, energy, food, environmental, etc.).

In this project we have focused on the following competencies established in the curriculum of the Chemical Engineering Degree, from the University of Salamanca [14], related with mathematics and engineering thermodynamics:

Transversal skills (the nomenclature corresponds to the official university program):

1. Computer knowledge in the field of study (TI5).
2. Problem resolution (TI8).
3. Critical thinking (TP8).
4. Ability to apply knowledge in practice (TS1).

Specific skills:

1. Ability to solve mathematical problems that may arise in Chemical Engineering, applying knowledge of algebra, geometry, calculus, numerical methods, statistics and optimization (DB1).
2. Basic knowledge on the use of computers, programming, operating systems, databases and programs with engineering applications (DB3).
3. Knowledge of the basic principles of thermodynamics and heat transmission and their application to the resolution of engineering problems (DR1).

2.1 Mathematics III

Mathematics III complements the basic mathematical training of the future Chemical Engineer, with elementary knowledge of Numerical Analysis, essential to translate an engineering problem into a mathematical problem. Moreover, to promote the capacity to solve the stated problems and to interpret the possible solutions are also objectives of the subject. Mathematics III is a course of 7.5 credits, developed in the first semester of the second year.

2.2 Engineering Thermodynamics

The main objective of Engineering Thermodynamics is the thermodynamic analysis of projected systems to carry out conversions among different energy sources. Among these, special attention is paid to cyclically operating devices (thermodynamic machines) designed for power generation and refrigeration. This subject covers 4.5 credits in the second semester of the second year.

2.3 *Wolfram Mathematica* Package

Information and Communications Technologies (ICTs) help to achieve higher levels of quality in teaching and, furthermore, allow to mimic the skills acquired by the students to those they will be required in their professional careers. The teaching of Mathematics and Engineering Thermodynamics cannot be an exception and should not be left out of the use of these methods. ICT provides students with the possibility of simulating experiences and posing very different situations, and comparing them. Sometimes, to do this manually can be difficult or at least tedious. For example, it allows the student to understand the true scope of a problem or the effectiveness of an algorithm by analyzing the results obtained by varying hypotheses, initial conditions, etc.

It is in these specialties where the use of specific software is really useful, such as symbolic calculation packages like Wolfram Mathematica package, among others. These programs, called CAS (*Computer Algebra System*) have an easy syntax to learn, since the syntax and commands resemble the mathematical operations they execute and, therefore, their learning is quick and intuitive. In addition, the help they offer is very complete and it is illustrated with numerous examples.

The University of Salamanca has a "*campus*" license for Mathematica, which gives legal coverage and allows its installation in the computer rooms and personal computers, so its utilization by the entire university community. Particularly, it is installed in all the computers in the computer rooms in the Chemical Sciences Faculty. It is widely used in various undergraduate and master degrees. All these points makes this program an excellent software to carry out computer practices in different subjects.

3 Methodology

During a typical Thermodynamic Engineering course different methodological resources are used: master class, seminars, individual works, laboratory practices, tutoring and a final exam.

- Part of the training is given in the form of theory lectures. Videos and applets are used to enrich these sessions, which help to clarify the concepts and allow viewing experiences that would otherwise be difficult to carry out.
- An essential complement to theoretical classes are the problem resolutions in seminars. Facing problems and trying to solve them students can apply the knowledge acquired in theoretical classes and improve their skills.
- A third component are individual works of the students: not all the proposed problems are solved in class, therefore, students are asked to make two deliveries, throughout the course, with problems in which they have worked individually. These tasks contribute to the ongoing assesssment of the student.
- Because of bureaucratic issues in the developing of the official university program of the subject, no practical laboratory hours were established for Thermodynamic Engineering. However, we consider that it would be a really enriching complement for the in-depth understanding of the thermodynamic cycles that are addressed. For this reason, some seminars are dedicated to take students to the Thermodynamics laboratory, for example to observe the operation of a Stirling engine, for didactic purposes, as well as to experimentally measure cycle performance, maximum and minimum temperatures, and to analyze the pressure vs. volume diagram, among other aspects.

- Personal attention to students through face-to-face tutoring is essential to solve questions and doubts. In this way, we facilitate students to deepen their knowledge, while reinforcing direct and personal contact with teacher.
- To carry out the evaluation by competencies. This includes written tests with theoretical and numerical problems to solve. Students are allowed to use the class notes.

The use of the web-based *Studium* platform of the University of Salamanca (*Moodle*-based digital platform) is proposed for the subjects both to make available to students notes, presentations and figures used in class, proposed problems, and to enrich the teaching-learning process through forums, experience videos, etc.

An extra activity was planned the last two academic years: the realization of problems of the subject with the *Mathematica* tool. Briefly, the main objective was to allow the student a direct interaction with the topics developed in class. The display of the results "*in real time*" and using all possible graphic resources, is very effective in capturing the interest of the students. Moreover, computer practice familiarizes the students with a working method that, without doubt, will be essential in the development of their professional activity.

4 Using Mathematica software in Mathematics and Thermodynamic Engineering

Students from Chemical Engineering Degree take a Computer Science course in the first semester of the first year, so they are familiar with the Matlab programming environment. Thus they are already familiar with symbolics calculation packages.

Sophomore students must attend to Mathematics III course during the fall semester, this is a mandatory course and it includes several computer classes where the aim is to solve problems with Mathematica. These sessions are carried out in small groups (computer rooms usually have capacity for 15-20 students). In addition to the numerical methods described during theoretical lectures, several problems that connect students to their reality are also stated. They reveal the real usefulness of mathematics courses. As an example, we present the problem of the "Angry birds" (AB) game: to launch birds with parabolic trajectories. For the Thermodynamic Engineering course (during spring semester) we propose the students the problem of fugacity (F) as a bridge between both courses. Next, both problems are detailed.

"Angry birds" problem (AB)

The "Angry birds" game, developed by Rovio Entertainment Corporation, consists in destroying structures of different materials in order to elliminate the pigs inside or around them (green images in Fig. 2). With the help of a slingshot, the player launches an angry bird (in red) with the angle and strength necessary to achieve the proposed objective. To avoid a tower collision and not to reach its destination (Figure 2), the bird must pass as close to the first two towers as possible, but without touching them.

Figure 1: Image of the situation that raises the 'Angry birds" problem.

In a particular example, it is assumed, for instance, that the slingshot always shoots from a height of 1 cm. Tower 1, which is 1 cm from the slingshot measures 4.5 cm, Tower 2, at 2 cm, measures 6.5 cm and the desired point of impact, in the Tower 3, which is 4 cm from the slingshot, is exactly 5 cm. The question is set as: Draw the curve that the Angry Bird describes in the shot, knowing that we do not take much risk and go 0.5 cm above the towers.

Figure 3 shows the solution of the problem using Mathematica. In this way, a simple interpolation problem acquires meaning and a motivating application when connected to a popular videogame. On the other hand, problem solving skills (TI8), computer use (TI5 and DB3) are being acquired,

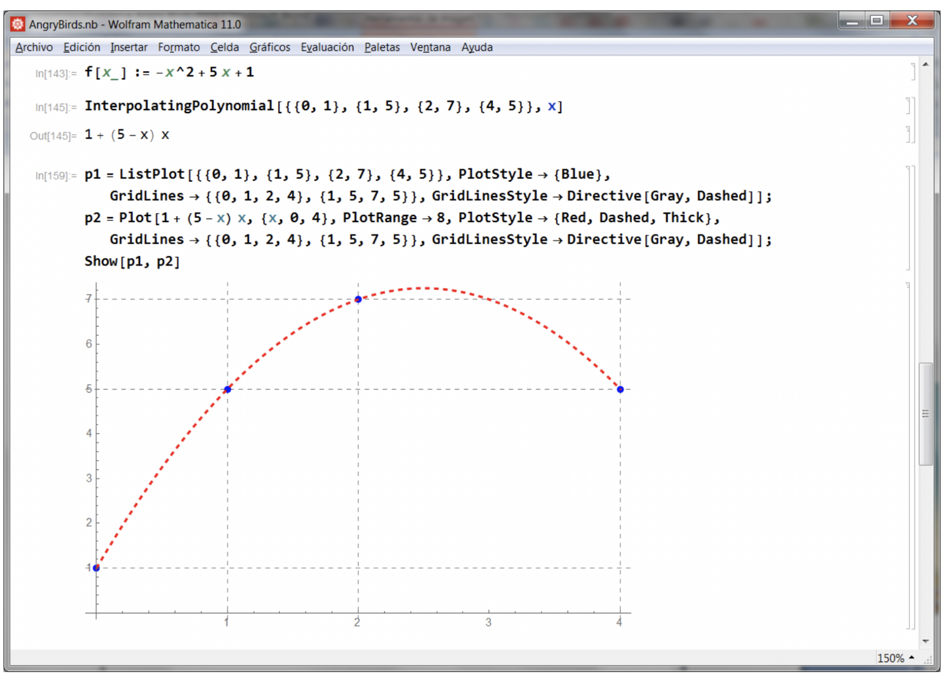

Figure 2: "Angry birds" problem solution.

as well as the ability to put knowledge into practice (TS1). Figure 4 shows a diagram where each proposed problem is related to the skills being worked on.

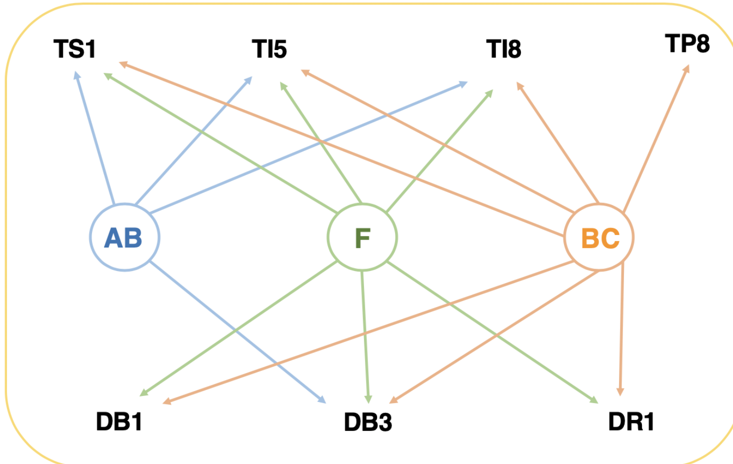

Figure 3: Relationship between the skills to be achieved (in black color) and the problems addressed (in different color).

Fugacity problem (F)

Use the following data to calculate N_2 fugacity at $0°$C and 400 atm.

p (atm)	50	100	200	400
pV/RT	0.9846	0.9863	1.0365	1.2557

To help students to solve the problem using *Mathematica*, it is broken down into the following steps (the solution of this problem is shown in Fig. 5).

The fugacity f, can be calculated as: $\ln f = \ln p + \int_0^p \left(\frac{V}{RT} - \frac{1}{p}\right) dp$ or $\ln f = \ln p + \int_0^p \left(\frac{z-1}{p}\right) dp$ where $z = \frac{pV}{RT}$.

a) Determine the compressibility factor values $\frac{z-1}{p}$.

b) Make (using blue dots) the graphic representation of $\left(p, \frac{z-1}{p}\right)$.

c) Find the interpolation polynomial that best fits these value pairs $\left(p, \frac{z-1}{p}\right)$. It shall be denoted as $f(p)$.

d) Do represent $f(p)$ (in red color), along with the pairs of data (blue color).

e) Determine the area under the curve between $p = 0$ and $p = 400$ of the function $f(p)$. As it is known, the definite integral between $p = 0$ and $p = 400$ of the interpolation function, $f(p)$, corresponds to the area under the curve.

f) Clear the fugacity, f, of the expression: $\ln f = \ln p + \int_0^p \left(\frac{z-1}{p}\right) dp$ where $z = \frac{pV}{RT}$.

As it can be seen, students use the knowledge acquired in Mathematics to solve a problem of Thermodynamics, relying on different commands and functions of the *Mathematica* software.

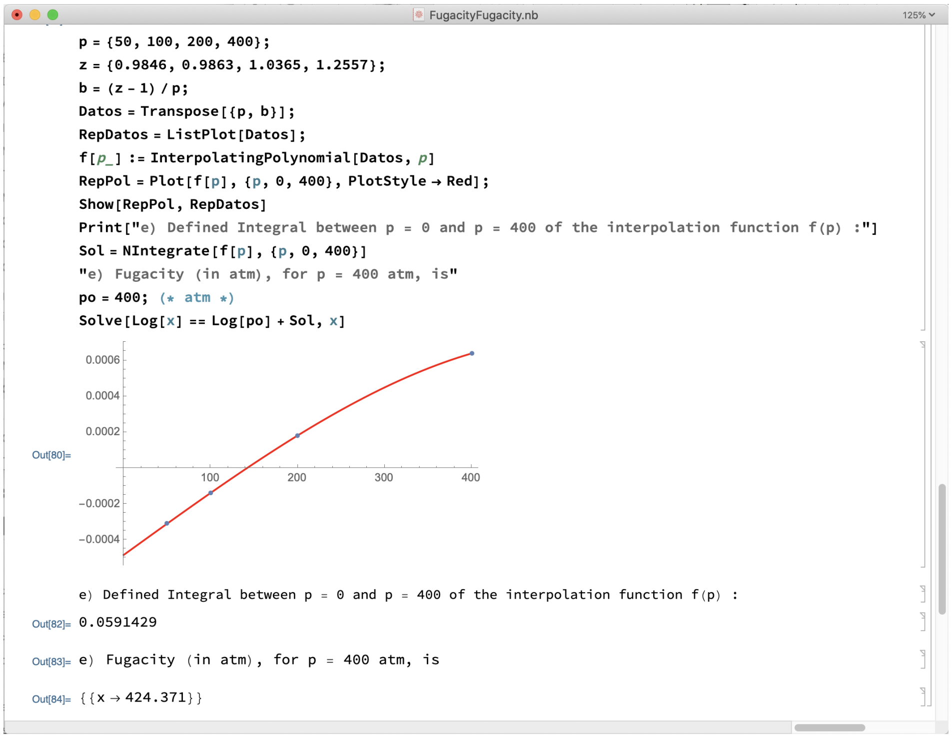

Figure 4: Fugacity problem solution.

It is clear which competencies are developed: ability to solve problems (TI8), using Maths and Thermodynamic knowledge (DB1 and DR1), use of the computer (TI5 and DB3), and applying their knowledge practically (TS1) (see Fig. 4).

When students start the Thermodynamic Engineering course, they are already familiar with the Mathematica program and the necessary commands to solve the applied thermodynamics problems. One of the difficulties that this subject presents is that problems are long, and many operations are required. Usually, problems are solved in class, using a calculator. This is conditioned by the nature of the subject, since it is not about numerical methods, and because the exams will be done in a classroom without access to computers. However, when students tackle problems with computer, time is saved for resolution, allowing more time to be spent understanding the problems in depth. Since it is possible to vary different magnitudes and see how they affect the final solution. This is the reason that led us to offer students the opportunity to work on the same class problems, but with the Mathematica tool. It has been a voluntary activity, in 5 days outside the formally established hours we explain students how to solve some problems using a mathematical software. In that session, with the teacher, a problem was solved that had already been worked on in the problems class, but this time with the mathematical tool. This way, the emphasis was placed on the programming difficulties, not on those of Thermodynamics that were already explained in a previous case. Students were then asked to solve another new problem.

In the year 2018-19, 5 students from 34 participated in this activity and in the 2019-20 academic year, 30 of 48. This shows the inertness that this project has awakened.

Usually problems in Thermodynamic Engineering are long, with chained operations, and it is very useful to understand them to be able to visualize graphs of the situation. A typical example is presented below: a typical Brayton cycle problem (BC).

Brayton cicle problem (BC)

Consider a reversible air Brayton cycle (considered as ideal gas), with a mass flow of 1 kg/s and minimum and maximum temperatures of $290\,\mathrm{K}$ and $1430\,\mathrm{K}$, respectively.

a) Calculate the different values of pressure and temperature at each vertex of the cycle and representation of the $p - T$ diagram for different compression ratios (rp): 5, 10 and 15.

b) Calculation for the three cases of the compression work, the one carried out by the turbine, the absorbed heat and the thermodynamic efficiency.

c) Repeat the exercise assuming an ideal regenerative Brayton cycle.

d) Compare and comment on the results of sections b) and c). In which cases is the use of the regenerator interesting?

As mentioned previously, one of the potentialities of Mathematica are graphs, as shown in the solution to the BC problem (Fig. 6). With a single image, three different situations can be identified, observing the consequences of modifying a variable in the problem, such as the particular case of the pressure relation (rp).

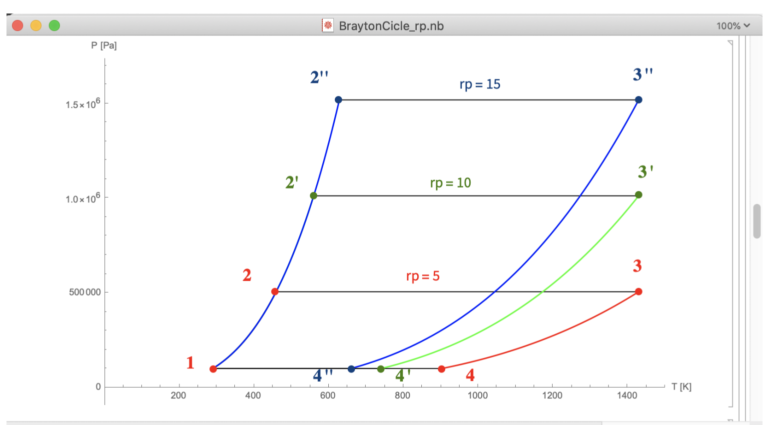

Figure 5: P-T diagram of an ideal Brayton cycle for different values of pressure ratio ($rp = 5$ $rp = 10$ and $rp = 15$). Results from section a) of Brayton cycle problem.

In the solutions of sections b) and c) of problem BC (Fig. 7 and Fig. 8), you can see how, through some orders, the problem is solved for 8 different cases (4 values of rp and 2 cycles: ideal and regenerative Brayton). Presenting the results in a visual and didactic way, through tables. This helps students to compare and deeply understand the root of the problem, leading them to achieve a better understanding of the physical phenomena involved [11].

This is one of the examples where all the skills initially planned, including the critical capacity, are worked on, since they can tackle the same problem with multiple variants (Fig. 4).

BraytonCicle_rp.nb

b) Calculation for the three cases of the compression work, the one carried out by the turbine, the absorbed heat and the thermodynamic efficiency.

- Work carried out by the turbine: $W_t = \dot{m}c_p\,(T_3\text{-}T_4)$
- Compression work: $W_C = \dot{m}c_p\,(T_2\text{-}T_1)$
- Absorbed heat: $Q_H = \dot{m}c_p\,(T_3\text{-}T_2)$
- Thermodynamic efficiency: $\eta = \frac{(T_3-T_4)-(T_2-T_1)}{(T_3-T_2)} = 1\text{-}\,r_p^{\frac{(1-\gamma)}{\gamma}}$

In[53]:=

$$\texttt{rp = Column[Table[1*rp, \{rp, 5, 20, 5\}]];}$$

$$\texttt{wt = Column}\left[\texttt{Table}\left[(\texttt{fm / Pm})*\texttt{cp}*\left(\texttt{T3} - \left(\texttt{T3}*\left(\frac{1}{\texttt{rp}}\right)^{\frac{\gamma-1}{\gamma}}\right)\right), \{\texttt{rp}, 5, 20, 5\}\right]\right];$$

$$\texttt{wc = Column}\left[\texttt{Table}\left[(\texttt{fm / Pm})*\texttt{cp}*\left(\left(\texttt{T1}*(\texttt{rp})^{\frac{\gamma-1}{\gamma}}\right) - \texttt{T1}\right), \{\texttt{rp}, 5, 20, 5\}\right]\right];$$

$$\texttt{q = Column}\left[\texttt{Table}\left[(\texttt{fm / Pm})*\texttt{cp}*\left(\texttt{T3} - \left(\texttt{T1}*(\texttt{rp})^{\frac{\gamma-1}{\gamma}}\right)\right), \{\texttt{rp}, 5, 20, 5\}\right]\right];$$

$$\eta\ \texttt{= Column}\left[\texttt{Table}\left[\frac{\left(\texttt{T3} - \left(\texttt{T3}*\left(\frac{1}{\texttt{rp}}\right)^{\frac{\gamma-1}{\gamma}}\right)\right) - \left(\left(\texttt{T1}*(\texttt{rp})^{\frac{\gamma-1}{\gamma}}\right) - \texttt{T1}\right)}{\texttt{T3} - \left(\texttt{T1}*(\texttt{rp})^{\frac{\gamma-1}{\gamma}}\right)}, \{\texttt{rp}, 5, 20, 5\}\right]\right];$$

```
res = {{"rp ", "Wc (J)", "Wt (J)", "Qabs (J)", "η"}, {rp, wc, wt, q, η}};
cptabla = Grid[res, Frame → All, {Background → {None, {RGBColor[1, 0, 1, .3], White}, Dividers → {Black, {2 → Black}}},
   Frame → True, Spacings → {2, {1, {1.1}, 1}}}]
```

Out[59]=

r_p	W_c (J)	W_t (J)	Q_{abs} (J)	η
5	170 131.	529 681.	975 410.	0.368615
10	271 214.	692 685.	874 326.	0.482053
15	340 318.	774 100.	805 223.	0.53871
20	394 436.	826 403.	751 104.	0.575109

Figure 6: Results from section b) of Brayton cycle problem.

BraytonCicle_rp.nb

c) Calculation for the three cases of the compression work, the one carried out by the turbine, the absorbed heat and the thermodynamic efficiency assuming an **ideal regenerative Brayton cycle**.

- Work carried out by the turbine: $W_t = \dot{m}c_p\,(T_3\text{-}T_4)$
- Compression work: $W_C = \dot{m}c_p(T_2\text{-}T_1)$
- Absorbed heat: $Q_H = \dot{m}c_p(T_3\text{-}T_4) = W_t$
- Thermodynamic efficiency: $\eta = \frac{(T_3-T_4)-(T_2-T_1)}{(T_3-T_2)} = 1\text{-}\,r_p^{\frac{(1-\gamma)}{\gamma}}$

In[72]:= $\tau = \frac{\texttt{T3}}{\texttt{T1}};$

$$\eta\eta\ \texttt{= Column}\left[\texttt{Table}\left[1 - \left(\frac{\texttt{rp}^{(\gamma-1)/\gamma}}{\tau}\right), \{\texttt{rp}, 5, 20, 5\}\right]\right];$$

$$\texttt{qq = Column}\left[\texttt{Table}\left[(\texttt{fm / Pm})*\texttt{cp}*\left(\texttt{T3} - \left(\texttt{T3}*\left(\frac{1}{\texttt{rp}}\right)^{\frac{\gamma-1}{\gamma}}\right)\right), \{\texttt{rp}, 5, 20, 5\}\right]\right];$$

```
resreg = {{"rp ", "Wc (J)", "Wt (J)", "Qabs (J)", "η"}, {rp, wc, wt, qq, ηη}};
cptabla =
 Grid[resreg, Frame → All, {Background → {None, {RGBColor[0, 0, 1, .3], White}, Dividers → {Black, {2 → Black}}},
   Frame → True, Spacings → {2, {1, {1.1}, 1}}}]
```

Out[76]=

r_p	W_c (J)	W_t (J)	Q_{abs} (J)	η
5	170 131.	529 681.	529 681.	0.678806
10	271 214.	692 685.	692 685.	0.60846
15	340 318.	774 100.	774 100.	0.560369
20	394 436.	826 403.	826 403.	0.522707

Figure 7: Results from section c) of Brayton cycle problem.

Another reason for encourages our students to learn Mathematica software is that the professors responsible for Mathematics and Thermodynamic Engineering offer end-of-degree projects in which it is necessary to use such software.

5 Evaluation

With this methodology, the results in the last three academic years, $2017-2018$, $2018-2019$ and $2019-2020$, are shown in Table 1. Although the results of the 2019-2020 are really good, we consider that they should not be taken into account for this study, since half of the teaching and the entire evaluation (continuous and final) has been carried out on-line because of coronavirus pandemic. Both the success rate (percentage of students who passed the subject with respect to those presented) and the performance rate (percentage of students who pass the subject with respect to the total of enrolled students) are above 65%, which can be consider good results within an Engineering Degree.

Table 1: Results obtained in Thermodynamic Engineering in the last three academic years: $2017-2018$, $2018-2019$ and $2019-2020$.

Academic year	$2017-18$	$2018-19$	$2019-20$
Students involved in the project	–	5	30
Rate	–	14.7%	62.5%
Students that pass the course	–	100%	100%
Total number of students	41	34	48
Performance rate	65.8%	82.3%	95.8%
Success rate	75%	96.6%	100%

It is observed that the results have improved since 2018-2019, which is when we started with the project presented here. Although there are many variables to consider, the difference between 2017-2018 and 2018-2019 is the fact of starting to work through Mathematica problems in Thermodynamic Engineering. It is true that there are more hours of work both between teacher and student, as well as personal work of the student and closer student-teacher relationship.

Besides, the interest that the initiative has aroused has been verified: in academic year 2018-2019 it was carried out with 5 student volunteers, out of 34 enrolled (15%) in Thermodynamic Engineering, and in 2019-2020, 30 volunteer students participated out of 48 enrolled (63%), which is a considerable increase. It is remarkable the fact that all the students who have participated in the project have passed the course. It is true, that being a voluntary activity, those students who are most interested in learning generally sign up. We will think about the possibility of expanding it to the entire group of students enrolled in the subject, for future courses.

6 Conclusions

Our final goal is that the students of the Chemical Engineering Degree know how to use mathematical tools to formulate and solve problems that arise in Thermodynamic Engineering. So that the students understand that knowledge acquired in different subjects are interdependent. For reaching it, teachers at the University of Salamanca, who teach Mathematics III and Engineering Thermodynamics of second year of the Chemical Engineering Degree, in successive semesters, have worked together.

First of all, in Mathematics III computer practices, it have been solved problems with Mathematica. Covering both classic problems and others that connect students with their close reality and motivate them, such as the case of the "Angy bird". In addition to Thermodynamic exercises where they discover the application of the knowledge acquired in Mathematics, as “fugacity case”. In second semester, the realisation of some Engineering Thermodynamics problems through Mathematica software has been proposed to a group of volunteers students. In five sessions, one per topic, work is done, together with the teacher, to solve problems raised with the help of this tool.

This work aims to achieve competency-based teaching, such as: problem resolution (Mathematics III and Engineering Thermodynamics); and basic knowledge of the use of computers, and operating systems, with engineering applications.

And, above all, to introduce in the teaching process, methods that are closer to those of the graduate’s later work and that involve an approach to the reality of engineering, such as solving problems with the program.

The interest that the initiative has aroused has been verified: in academic year 2018-2019 it was carried out with 5 student volunteers, out of 34 enrolled (15%) in Thermodynamic Engineering, and in academic year 2019-20, 30 volunteer students participated out of 48 enrolled (63%). Which is a considerable increase.

It is also notable that the degree of satisfaction of the teachers involved is high. On the one hand, perceiving that such a voluntary activity which students carry out outside the established schedule, arouses their interest. And on the other hand, we think that the fact that students perceive that teachers care about what they learn (not just their grade), establishes another type of perception, positive, towards the subject. This has been verified with the attendance to the usual classes, which has been very high, compared with that of other subjects of the same course.

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Analytical investigation of a two-mass system connected with linear and nonlinear stiffnesses

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Abstract

An iteration procedure has been developed based on the Mickens iteration method. This procedure also offers the angular frequencies and corresponding periodic solutions to the nonlinear vibration of a two-mass system connected with linear and nonlinear stiffnesses. A real-world case of this system is analysed and introduced. In this paper, the truncated terms of the Fourier series have been used and utilized in every step of iterations. The approximated results are compared with existing and corresponding numerical (considered to be exact) results. The obtained results are valid for whole ranges of vibration amplitude of the oscillations. The error analysis has carried out and shown acceptable results for the proposed iteration procedure. Effectiveness of the proposed iteration procedure found from comparison with other existing methods. The method is demonstrated by an example.

Keywords Nonlinear stiffnesses · Mickens iterative method · Two-mass system · Two-degree-of-freedom oscillation systems · Duffing equation

1 Introduction

The motion of multidegree of freedom oscillating system was widely considered by researchers in previous century. An approximate method was proposed by Moochhala and Raynor [1] for the motions of unequal masses connected by $(n + 1)$ nonlinear springs and anchored to rigid end walls. Huang [2] studied the harmonic oscillations of nonlinear two-degree-of-freedom (TDOF) systems. The free oscillations of conservative quasilinear systems with TDOF has been analysed by Gilchrist [3]. A research was done by Efstathiades [4] on the existence and characteristic behaviour of combination tones in nonlinear systems with TDOF. Alexander and Richard [5] considered the resonant dynamics of a TDOF system. The system is composed of a linear oscillator weakly coupled to a strongly nonlinear associated with an essential (nonlinearizable) cubic stiffness nonlinearity. A generalized Galerkin's method was applied by Chen [6] to the nonlinear oscillations of TDOF systems. Ladygina and Manevich [7] used the multiscale method to study the free oscillations of a conservative system with TDOF having cubic nonlinearities (of symmetric nature) and close natural frequencies. A combination of Jacobi elliptic and trigonometric functions

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has been used by Cveticanin [8, 9]. An analytical solution has been obtained for the motion of a two-mass system with TDOF in which the masses are connected by three springs.

Currently, TDOF systems have a huge importance in the fields of physics and engineering discipline. Many practical engineering vibration systems, such as the elastic beams supported by two springs and the vibration of a milling machine [10], can be studied by considering them as TDOF systems. The TDOF oscillation systems consist of two second-order differential equations with cubic nonlinearities. Solving the equations of motion for a mechanical system which is associated with linear and nonlinear springs was obtained through the transformation into a set of differential algebraic equations using intermediate variables; here the equations of motion for a TDOF system transformed into the well-known Duffing equation [11].

In general, finding an exact solution for nonlinear equations is extremely difficult and this perception has led to intensive research over many decades. Many analytical and numerical approaches are currently being investigated. The most widely used analytical methods for approximating nonlinear equations are traditional perturbation methods. These are not effective for strongly nonlinear equations and have many limitations. In the recent past, many new analytical techniques have been investigated to overcome these limitations. Among of them, the Newton-harmonic balance method [12], He's variational approach [13], the energy balance method [14], the max-min approach [15] and He's improved amplitude-frequency formulation method [16], the harmonic balance method [17] have been used to derive approximate angular frequencies and corresponding periodic solutions to the TDOF system. Very recently, researchers have been applied several analytical methods include as the He's amplitude-frequency formulation [18], the max-min approach [19], the energy balance method [20] to the well-known Duffing equation. In fact, to the best of our knowledge, in most of these methods, only the first-order approximation has been considered which does not lead to sufficient accuracy.

The iteration method is a well-known general technique for calculating approximate angular frequencies and corresponding periodic solutions to strongly nonlinear oscillators for both small as well as large vibration amplitudes of the oscillation. The methodology of iteration method was the first introduced by Mickens [23]. Further, a generalized iteration procedure has been applied by Mickens [24] to the truly nonlinear oscillators. Afterwards, Mickens [25] used the iterative technique to calculate a higher order approximation to the periodic solutions to a conservative oscillator for which the elastic force term is proportional to $x^{1/3}$. Hu [26, 27] applied the modified iteration technique of Mickens [24] to find approximate solutions to nonlinear oscillators with fractional powers and quadratic nonlinear oscillator consecutively. Zheng et al. [28] has applied Micken's extended iteration method and Micken's direct iteration method to determine approximate periodic solutions to a class of nonlinear jerk equations. They have shown that the Micken's direct iteration procedure is more efficient for nonlinear jerk equation than the Micken's extended iteration procedure. Recently, Haque et al. [29] utilize the iteration method of calculating the approximate angular frequencies and corresponding periodic solutions to the oscillator with cubic nonlinearity.

In this study, an iteration procedure has been applied to determine the approximate angular frequencies and corresponding periodic solution to the nonlinear vibration of a two-mass system connected with linear and nonlinear stiffnesses. In the proposed iteration procedure, only linear non-homogeneous differential equations are required to be solved at every step of the iteration. These is an important issue for obtaining higher order iteration of the solutions. The higher order iterations (up to a third order iteration) have been obtained to the nonlinear vibration of a two-mass system connected with linear and nonlinear stiffnesses. The approximated results are compared with existing results those obtained by using already published methods including the variational approach [13], the energy balance method [14], the max-min approach [15, 16] and He's improved amplitude-formulation [16].

The rest of this paper is organized as follows: In Section 2, we give the outline of the proposed iteration procedure based on the Mickens iteration method. In Section 3, we give a detailed description of a two-mass system connected with linear and nonlinear stiffnesses both geometrically and mathematically. In Section 4, we apply the proposed iteration procedure to the nonlinear vibration of a two-mass system connected with linear and nonlinear stiffnesses. In Section 5, results and discussions are discussed in detail. We analyse the consistency and convergence of the proposed iteration procedure in Section 6. Finally, in Section 7, concluding remarks are given.

2 The Methodology

Consider a second order nonlinear differential equation is as the following

$$\ddot{y} + f(y, \ddot{y}) = 0, \quad y(0) = A_0, \quad \dot{y}(0) = 0, \tag{1}$$

where over dots denote differentiation with respect to time t. Now consider the angular frequency ω of this system. Then adding $\omega^2 y$ to both sides of Eq. (1), it can be obtained

$$\ddot{y} + \omega^2 y = \omega^2 y - f(y, \ddot{y}) \equiv G(y, \ddot{y}), \tag{2}$$

where $\omega^2 y$, the constant is currently unknown.
Now, we formulate the iteration scheme of Eq. (2) in the following way

$$\ddot{y}_{q+1} + \omega_q^2 y_{q+1} = G(y_q, \ddot{y}_q); \quad q = 0, 1, 2, 3, \cdots, \tag{3}$$

together with

$$y_0(t) = A_0 cos(\omega_0 t). \tag{4}$$

Herein y_{q+1} satisfies the initial conditions

$$y_{q+1}(0) = A_0, \quad \dot{y}_{q+1}(0) = 0. \tag{5}$$

At each step of the iteration, ω_q is determined by the requirement taking that the secular terms [21] should not occur in the full solution of $y_{q+1}(t)$. This procedure gives the sequence of solutions: $y_0(t), y_1(t), y_2(t), \cdots$. The method can be proceeded to any order of approximation; but due to growing algebraic complexity the solution is confined to a reasonable order usually the third. Although, Eq. (1) is of odd parity, the solution will only contain odd multiples of the angular frequency [22].

3 Formulation and mathematical modelling of the problem

The schematic view of two-mass system connected with linear and nonlinear stiffnesses [9] is considered as shown in Figure 1.

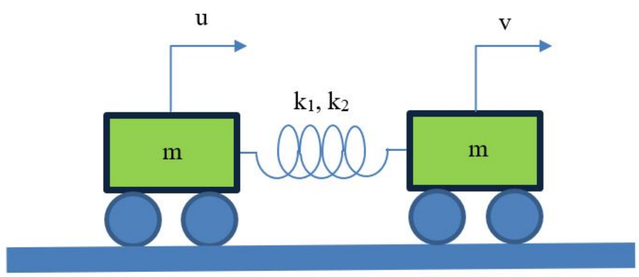

Figure 1: Schematic view of two-mass system connected by linear and nonlinear stiffnesses.

The equations of motion [9] are defined as

$$m\ddot{u} + k_1(u - v) + k_2(u - v)^3 = 0, \tag{6}$$

$$m\ddot{v} + k_1(v - u) + k_2(v - u)^3 = 0, \tag{7}$$

with the initial conditions

$$u(0) = u_0, \quad \dot{u}(0) = 0, \tag{8}$$

$$v(0) = v_0, \quad \dot{v}(0) = 0, \tag{9}$$

where the double dots in Eqs. (6)-(7) represent double differentiation with respect to time t, k_1 and k_2 are linear and nonlinear coefficients of the spring stiffness respectively. Dividing Eqs. (6)-(7) by mass m it can be written as

$$\ddot{u} + \frac{k_1(u-v)}{m} + \frac{k_2(u-v)^3}{m} = 0, \tag{10}$$

$$\ddot{v} + \frac{k_1(v-u)}{m} + \frac{k_2(v-u)^3}{m} = 0. \tag{11}$$

Introducing the intermediate variables x and y [11] as follows

$$u := x, \tag{12}$$

$$v - u := y, \tag{13}$$

and transforming Eqs. (10)-(11), it becomes

$$\ddot{x} - \beta y - \varepsilon y^3 = 0, \tag{14}$$

$$\ddot{y} + \ddot{x} + \beta y + \varepsilon y^3 = 0, \tag{15}$$

where $\beta = \frac{k_1}{m}$ and $\varepsilon = \frac{k_2}{m}$. Rearranging Eq. (14) as follows

$$\ddot{x} = \beta y + \varepsilon y^3 \tag{16}$$

Substituting Eq. (16) into Eq. (15), it becomes

$$\ddot{y} + 2\beta y + 2\varepsilon y^3 = 0, \tag{17}$$

with initial conditions

$$y(0) = v(0) - u(0) = v_0 - u_0 = A_0, \quad \dot{y}(0) = 0. \tag{18}$$

Therefore, Eq. (17) is obviously similar to the well-known Duffing equation $\ddot{y} + \delta y + \sigma y^3 = 0$ with $\delta = 2\beta$ and $\sigma = 2\varepsilon$. For solving Eq. (17) using the proposed iteration procedure, the approximate solutions of $y(t)$ can be substituted back into Eq. (16) as

$$\ddot{x} = \beta y + \varepsilon y^3,$$

with the initial conditions

$$x(0) = u(0) = u_0, \quad \dot{x}(0) = 0,$$

and to obtain the intermediate variable $x(t)$ by double integration.

4 Solution procedure

Adding $\omega^2 y$ to both sides of Eq. (17), it can be transformed into

$$\ddot{y} + \omega^2 y = \omega^2 y - 2\beta y - 2\varepsilon y^3. \tag{19}$$

Now, we formulate the iteration scheme according to Eq. (3) as in the following

$$\ddot{y}_{q+1} + \omega_q^2 y_{q+1} = \omega_q^2 y_q - 2\beta y_q - 2\varepsilon y_q^3, \tag{20}$$

together with the initial approximation

$$y_0(t) = A_0 cos(\omega_0 t). \tag{21}$$

First, we take $q = 0$ for Eq. (20), it can be supposed as

$$\ddot{y}_1 + \omega_0^2 y_1 = \omega_0^2 y_0 - 2\beta y_0 - 2\varepsilon y_0^3. \tag{22}$$

Substituting Eq. (21) into Eq. (22), it can be reduced as

$$\ddot{y}_1 + \omega_0^2 y_1 = \omega_0^2 A_0 cos(\omega_0 t) - 2\beta A_0 cos(\omega_0 t) - 2\varepsilon A_0^3 cos^3(\omega_0 t). \tag{23}$$

Expanding $cos^3(\omega_0 t)$ into a Fourier cosine series, Eq. (23) can be transformed into

$$\ddot{y}_1 + \omega_0^2 y_1 = (A_0\omega_0^2 - 2\beta A_0 - \frac{3\varepsilon A_0^3}{2}) cos(\omega_0 t) - \frac{1}{2}\varepsilon A_0^3 cos(3\omega_0 t). \tag{24}$$

No secular term in the solution for $y_1(t)$ requires that the coefficient of $cos(\omega_0 t)$ term equal to zero from the right-hand side of Eq. (24). Thus, we have

$$\omega_0 = \sqrt{2\beta + \frac{3\varepsilon A_0^2}{2}}. \tag{25}$$

Therefore, the solution of Eq. (24) satisfying the initial condition $y_1(0) = A_0$, can be obtained as

$$y_1(t) = c_1 cos(\omega_0 t) + c_3 cos(3\omega_0 t), \tag{26}$$

where

$$c_1 = (A_0 - \tfrac{\varepsilon A_0^3}{32\beta + 24\varepsilon A_0^2})$$

$$c_3 = (\tfrac{\varepsilon A_0^3}{32\beta + 24\varepsilon A_0^2})$$

This is the first approximate solution of Eq. (17) where ω_0 is given by Eq. (25) and the related ω_1 is to be determined further.

Taking $q = 1$ for Eq. (20), it can be assumed as

$$\ddot{y}_2 + \omega_1^2 y_2 = \omega_1^2 y_1 - 2\beta y_1 - 2\varepsilon y_1^3. \tag{27}$$

The angular frequency of ω_1 will be determined from Eq. (27). Substituting y_1 from Eq. (26) into the right-hand side of Eq. (27), it can be obtained

$$\begin{aligned} \ddot{y}_2 + \omega_1^2 y_2 = {} & (\omega_1^2 c_1 - 2\beta c_1 - \frac{3\varepsilon c_1^3}{2} - \frac{3}{2}\varepsilon c_1^2 c_3 - 3\varepsilon c_1 c_3^2) cos(\omega_1 t) \\ & + (\omega_1^2 c_3 - 2\beta c_3 - \frac{\varepsilon c_1^3}{2} - 3\varepsilon c_1^2 c_3 - \frac{3\varepsilon c_3^3}{2}) cos(3\omega_1 t) \\ & - (\frac{3}{2}\varepsilon c_1^2 c_3 + \frac{3}{2}\varepsilon c_1 c_3^2) cos(5\omega_1 t) - \frac{3}{2}\varepsilon c_1 c_3^2 cos(7\omega_1 t) - \frac{\varepsilon c_3^3}{2} cos(9\omega_1 t). \end{aligned} \tag{28}$$

Avoiding secular terms in the solution of Eq. (28), it can be determined as

$$\omega_1 = \sqrt{2\beta + \frac{3\varepsilon c_1^2}{2} + \frac{3\varepsilon c_1 c_3}{2} + 3\varepsilon c_3^2}. \tag{29}$$

Hence, the solution of Eq. (28) and satisfying the initial condition $y_2(0) = A_0$, can be obtained as

$$y_2(t) = d_1 cos(\omega_1 t) + d_3 cos(3\omega_1 t) + d_5 cos(5\omega_1 t) + d_7 cos(7\omega_1 t) + d_9 cos(9\omega_1 t), \quad (30)$$

where

$$d_1 = A_0 - \frac{40\beta c_3 + (10c_1^3 + 70c_1^2 c_3 + 15c_1 c_3^2 + 31c_3^3)\varepsilon - 20c_3\omega_1^2}{160\omega_1^2}$$

$$d_3 = \frac{4\beta c_3 + \varepsilon c_1^3 + 6\varepsilon c_1^2 c_3 + 3\varepsilon c_3^3 - 2c_3\omega_1^2}{16\omega_1^2}$$

$$d_5 = \frac{3\varepsilon c_1^2 c_3 + 3\varepsilon c_1 c_3^2}{48\omega_1^2}$$

$$d_7 = \frac{3\varepsilon c_1 c_3^2}{96\omega_1^2}$$

$$d_9 = \frac{\varepsilon c_3^3}{160\omega_1^2}$$

Again taking $q = 2$ for Eq. (20), it can be written as

$$\ddot{y}_3 + \omega_2^2 y_3 = \omega_2^2 y_2 - 2\beta y_2 - 2\varepsilon y_2^3. \quad (31)$$

The angular frequency of ω_2 will be determined from Eq. (31). Substituting y_2 from Eq. (30) into the right-hand side of Eq. (31), it can be obtained

$$\begin{aligned}\ddot{y}_3 + \omega_2^2 y_3 = {} & (d_1\omega_2^2 - 2\beta d_1 - (\frac{3d_1^3}{2} + \frac{3d_1^2 d_3}{2})\varepsilon + \cdots)cos(\omega_2 t) \\ & +(d_3\omega_2^2 - 2\beta d_3 - (\frac{d_1^3}{2} + \frac{3d_3^3}{2})\varepsilon + \cdots)cos(3\omega_2 t) \\ & +(d_5\omega_2^2 - 2\beta d_5 - (\frac{3d_1^2 d_3}{2} + \frac{3d_1 d_3^2}{2})\varepsilon + \cdots)cos(5\omega_2 t) \\ & +(d_7\omega_2^2 - 2\beta d_7 - (\frac{3d_1 d_3^2}{2} + \frac{3d_1^2 d_5}{2})\varepsilon + \cdots)cos(7\omega_2 t) \\ & +(d_9\omega_2^2 - 2\beta d_9 - (\frac{d_3^3}{2} - 3d_1 d_3 d_5)\varepsilon + \cdots)cos(9\omega_2 t) + \cdots .\end{aligned} \quad (32)$$

Avoiding secular terms in the solution of Eq. (32), it can be determined as

$$\omega_2 = \sqrt{2\beta + (\frac{3d_1^2}{2} + \frac{3d_1 d_3}{2} + 3d_3^2 + 3d_3 d_5 + 3d_5^2 + 3d_9^2)\varepsilon + \cdots}. \quad (33)$$

Hence, the solution of Eq. (32) and satisfying the initial condition $y_3(0) = A_0$, can be obtained as

$$y_3(t) = e_1 cos(\omega_2 t) + e_3 cos(3\omega_2 t) + e_5 cos(5\omega_2 t) + e_7 cos(7\omega_2 t) + e_9 cos(9\omega_2 t), \tag{34}$$

where

$$e_1 = A_0 - (\tfrac{d_3}{8} + \tfrac{d_5}{24} + \tfrac{d_7}{48} + \tfrac{d_9}{80}) + (\tfrac{d_3}{4} + \tfrac{d_5}{12} + \tfrac{d_7}{24})\tfrac{\beta}{\omega_2^2} + \cdots$$

$$e_3 = \frac{(2\beta d_3 + (\frac{d_1^3}{2} + 3d_1^2 d_3 + \frac{3d_3^3}{2} + \frac{3d_1^2 d_5}{2} + 3d_1 d_3 d_5)\varepsilon + \cdots)}{8\omega_2^2}$$

$$e_5 = \frac{(2\beta d_5 + (\frac{3d_1^2 d_3}{2} + \frac{3d_1 d_3^2}{2} + 3d_1^2 d_5 + 3d_3^2 d_5)\varepsilon + \cdots)}{24\omega_2^2}$$

$$e_7 = \frac{(2\beta d_7 + (\frac{3d_1 d_3^2}{2} + \frac{3d_1^2 d_5}{2} + 3d_1 d_3 d_5 + \frac{3d_3 d_5^2}{2})\varepsilon + \cdots)}{48\omega_2^2}$$

$$\cdots .$$

In the similar way, the fourth order approximate angular frequency is obtained as

$$\omega_3 = \sqrt{2\beta + (\frac{3e_1^2}{2} + \frac{3e_1 e_3}{2} + 3e_3^2 + 3e_3 e_5 + \frac{3e_3^2 e_5}{2e_1} + 3e_5^2 + \frac{3e_3^2 e_7}{2e_1})\varepsilon + \cdots}. \tag{35}$$

5 Results and discussions

In this section, we demonstrate and verify the accuracy of the proposed iteration procedure. Comparison the obtained results with the existing results [13, 14, 15, 16] and exact ones are presented in Table 1 to Table 2 for different values of parameters m, k_1, k_2 and initial conditions u_0 and v_0. For the values of parameters $m = 10$, $k_1 = 5$, $k_2 = 5$, $u_0 = 10$ and $v_0 = 20$, the approximated maximum relative error is 0.0021% which is much lower than the errors found using the existing methods including the variational approach [13], the energy balance method [14], the max-min approach [15, 16] and He's improved amplitude-formulation [16]. Hence, it is particularly noted that an excellent agreement with the exact solutions for the nonlinear Duffing equation is provided. The proposed iteration procedure is convergent and the obtained results are consistent and can also easily be generalized to the two-degree-of-freedom oscillation systems by combining the transformation technique.

In Table 1 to Table 2, m, k_1, k_2, u_0 and v_0 respectively denote the various parameters of the systems which have already defined in Section 3. $\omega_{[MMA=IAFF]}^{[16]}$ and $\omega_{[VA=EBM]}^{[13,\,14]}$ represent the existing

Table 1: Comparison of the approximated angular frequencies with existing results and exact frequencies to the various parameters of the system:

m	k_1	k_2	u_0	v_0	$\omega^{[16]}_{[MMA=IAFF]}$ Er(%)	$\omega^{[16]}_{[exact]}$	$\omega^{[this\ study]}_{[0]}$ Er(%)	$\omega^{[this\ study]}_{[1]}$ Er(%)	$\omega^{[this\ study]}_{[2]}$ Er(%)	$\omega^{[this\ study]}_{[3]}$ Er(%)
1	0.5	0.5	1	5	3.605551 (1.8735)	3.539243	3.605551 (1.8735)	3.545978 (0.1902)	3.539847 (0.0170)	3.539294 (0.0014)
1	1	1	5	1	5.099020 (1.8735)	5.005246	5.099019 (1.8734)	5.014771 (0.1903)	5.006100 (0.0170)	5.005318 (0.0014)
10	5	5	10	20	8.717798 (2.1586)	8.533586	8.717797 (2.1586)	8.553930 (0.2383)	8.535586 (0.0234)	8.533771 (0.0021)
20	40	50	20	10	19.46792 (2.1708)	19.05429	19.467922 (2.1708)	19.100121 (0.2405)	19.058811 (0.0237)	19.054709 (0.0021)
50	100	50	-10	20	36.79674 (2.2065)	36.00234	36.796738 (2.2065)	36.091218 (0.2468)	36.011204 (0.0246)	36.003174 (0.0023)
100	400	100	50	-50	122.5071 (2.2179)	119.8489	122.50714264 (2.2179)	120.147247 (0.2489)	119.878785 (0.0249)	119.851754 (0.0023)

Table 2: Comparison of the approximated angular frequencies with existing results and exact frequencies to the various parameters of the system:

m	k_1	k_2	u_0	v_0	$\omega^{[13,\ 14]}_{[VA=EBM]}$ Er(%)	$\omega^{[13]}_{[exact]}$	$\omega^{[this\ study]}_{[0]}$ Er(%)	$\omega^{[this\ study]}_{[1]}$ Er(%)	$\omega^{[this\ study]}_{[2]}$ Er(%)	$\omega^{[this\ study]}_{[3]}$ Er(%)
1	5	5	5	1	11.4018 (1.8736)	11.1921	11.401754 (1.8732)	11.213369 (0.1900)	11.193981 (0.0168)	11.192231 (0.0011)
1	1	1	10	-5	18.4255 (2.1924)	18.0302	18.425525 (2.1925)	18.074248 (0.2443)	18.034558 (0.0241)	18.030591 (0.0021)
5	10	10	20	30	17.4356 (2.1585)	17.0672	17.435595 (2.1585)	17.107861 (0.2382)	17.071173 (0.0232)	17.067543 (0.0020)
10	50	-0.01	-20	40	2.1448 (3.1401)	2.0795	2.144761 (3.1383)	2.076061 (0.1653)	2.079404 (0.0046)	2.079493 (0.0003)
1	10	5	20	25	14.4049 (1.7913)	14.1514	14.404860 (1.7910)	14.176462 (0.1770)	14.153575 (0.0153)	14.151568 (0.0011)
100	200	300	400	200	424.2688 (2.2203)	415.053	424.268782 (2.2203)	416.087898 (0.2493)	415.156636 (0.0249)	415.062802 (0.0023)

results which have been obtained using the max-min approach [16], He's improved amplitude-frequency formulation method [16], the variational approach [13] and the energy balance method [14] respectively. $\omega^{[13]}_{[exact]}$ and $\omega^{[16]}_{[exact]}$ signify the exact solutions which are obtained in [13, 16]. $\omega^{[thisstudy]}_{[0]}$, $\omega^{[thisstudy]}_{[1]}$, $\omega^{[thisstudy]}_{[2]}$, $\omega^{[thisstudy]}_{[3]}$ denote the first, second, third and fourth order approximate angular frequencies respectively obtained by using the proposed iteration procedure. $E(\%)$ represents the percentage error which is obtained from the relation $\mid \frac{\omega_{[q]}-\omega_{[exact]}}{\omega_{[exact]}} \mid 100$; $q = 0, 1, 2, 3, \cdots$.

To further illustrate and verify the correctness of the approximated solutions, a comparison of the time history oscillatory displacement response for the two masses for different values of initial conditions and stiffnesses with fourth-order Runge-Kutta (consider to be exact) solutions are plotted in Figure 2 to Figure 4. The proposed iteration procedure is simple, quite easy and highly efficient and is valid for a wide range of vibration amplitudes of the oscillation. As can be seen in Figure 2 to Figure 4, it is found that the proposed iteration procedure is in excellent agreement with the numerical solution. The proposed technique is swiftly convergent and can also easily be generalized to the two-degree-of-freedom oscillation systems by combining the transformation technique.

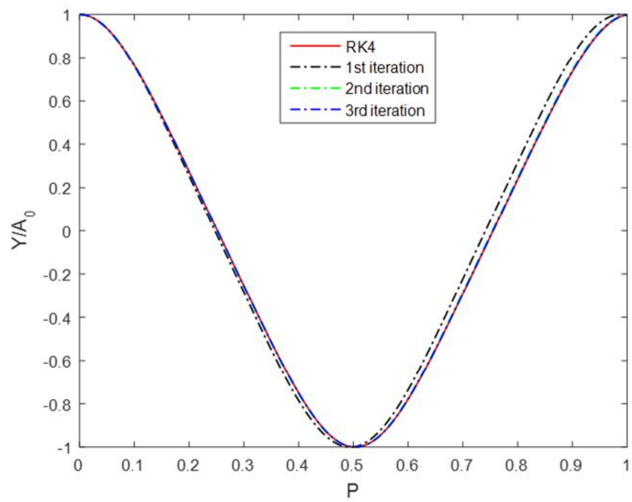

Figure 2: Comparison of the approximated solution in Eq. (17) with corresponding numerical one for various parameters $m = 1$, $k_1 = 0.5$, $k_2 = 0.5$, $u_0 = 1$, $v_0 = 5$.

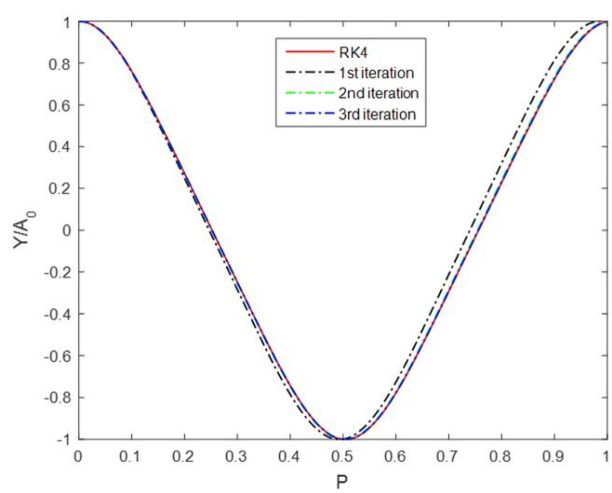

Figure 3: Comparison of the approximated solution in Eq. (17) with corresponding numerical one for various parameters $m = 100$, $k_1 = 400$, $k_2 = 100$, $u_0 = 50$, $v_0 = -50$.

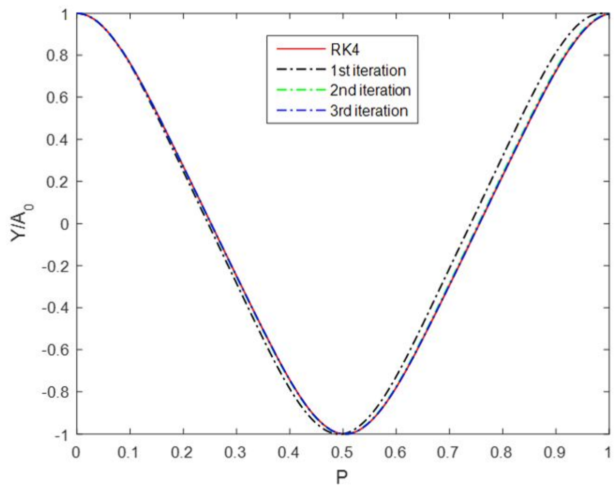

Figure 4: Comparison of the approximated solution in Eq. (17) with corresponding numerical one for various parameters $m = 100$, $k_1 = 200$, $k_2 = 300$, $u_0 = 400$, $v_0 = 200$.

6 Convergence and consistency analysis

Every iteration method is to obtain a sequence of approximate solutions y_q (as well as angular frequencies ω_q). It is the basic idea of iteration methods and has a convergence property

$$y_{ex} = \lim_{q\to\infty} y_q \quad or \quad \omega_{ex} = \lim_{q\to\infty} \omega_q, \tag{36}$$

where y_{ex} and ω_{ex} represent the exact solution and angular frequency respectively. The proposed iteration procedure has been revealed that the obtained solution gives better accuracy in every iteration step than previous one and finally $\mid \omega_3 - \omega_{ex} \mid < \varepsilon$, where ε is the small positive number and the initial oscillation amplitude A_0 is considered to be unity. Hence, it is shown that the proposed iteration procedure is convergent.

An iteration method is said to be consistent if it satisfies the followings

$$\lim_{q\to\infty} \mid y_q - y_{ex} \mid = 0 \quad or \quad \lim_{q\to\infty} \mid \omega_q - \omega_{ex} \mid = 0. \tag{37}$$

The proposed iteration procedure has already been defined in Eq. (3). The initial approximation as well as the initial condition is defined in Eq. (4)-(5). In the consistence analysis of the present study, it is observed that

$$\lim_{q\to\infty} \mid y_q - y_{ex} \mid = 0 \quad as \quad \mid y_3 - y_{ex} \mid \approx 0. \tag{38}$$

or

$$\lim_{q\to\infty} \mid \omega_q - \omega_{ex} \mid = 0 \quad or \quad \mid \omega_3 - \omega_{ex} \mid \approx 0. \tag{39}$$

Hence, the consistency of the proposed iteration procedure is attained.

7 Conclusion

An iteration procedure has been developed based on the Mickens iteration method and investigated to strongly nonlinear differential equations. The iteration procedure is powerful and highly efficient mathematical tool for solving strongly nonlinear differential equations. The proposed iteration procedure is also convergent and the approximated solutions are consistent. The iteration procedure is mostly essential when the solutions with higher accuracy are required. We obtained the approximate angular frequencies as well as periodic solutions to the nonlinear vibration of a two-mass system connected with linear and nonlinear stiffnesses. As compared with existing results and exact solutions, excellent agreement has been found. The exactness of the results shows that the proposed iteration procedure can be easily and efficiently investigated for the analysis of strongly nonlinear vibration problems with high accuracy. Hence, it is concluded that the proposed iteration procedure can be considered as a great potential and an efficient alternative to the previously existing methods for solving strongly nonlinear oscillatory systems.

8 Acknowledgement

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Fuzzy Modelling on Control of Heat Exchangers

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Abstract

Heat exchangers are an indispensable part of many chemical processes. These are the systems that allow the transfer of heat between two or more fluids. The dynamics of heat exchangers are highly nonlinear; thus, many methods have been used to control these systems. Feedback, feedforward and cascade control loops can be presented as traditional methods. The type of controller used in these systems is also crucial, and controllers such as P, PI, PID are widely used in industry. Because of the complexity of the control, intelligent controllers like fuzzy logic controllers (FLCs) are also considered to provide an alternative to existing controllers, since FLCs provide ability to implement human knowledge and experience. In this study, control of a heat exchanger with process gain Kp = 50 and process time constant τ = 30 was investigated by using a fuzzy controller with feedback control loop. By comparing the conventional membership function for the fuzzy variables with proposed membership function, the response of the heat exchanger system to a step change of 10 units applied to the setpoint was interpreted.

Keywords Control · Heat Exchanger · Fuzzy Control

1 Introduction

A heat exchanger is a system that is used to transfer heat between two or more fluids. It is used in almost all production lines, since temperature control is critical in most chemical or physical processes. Heat exchangers can be designed to tolerate wide range of pressure and temperature; thus, they are widely used in industry [1]. Applications range from power production, petroleum refining and chemicals, paper and pharmaceutical production, to aviation and transportation industries. A large percentage of world market for heat exchangers is served by the industry workhorse, the shell-and-tube heat exchanger. Recent developments in other exchanger geometries, such as plate-type and jacketed heat exchangers, have penetrated in various industry applications; however, the shell-and-tube exchanger by far remains the industry choice where maintainability and reliability are vital. [2].

Control of these heat exchangers are essential in the chemical processes since they are widely used and slight shifts in the temperature may affect many critical parameters, such as viscosity, pressure, reaction rates in reactors, concentration, vapor pressure etc. Thus, the control of these systems has been studied rigorously by researches all around the world. Several methods have been developed to achieve this goal, most important ones being feedback, feedforward and cascade control systems [3]. The advantages and disadvantages of these systems over each other are thoroughly analyzed in literature and beyond the scope of this study. The main objectives of these control systems are to track reference trajectories when a disturbance occur, and to maintain the controlled variable close to the setpoints [4].

In this study, an approach for fuzzy feedback control of heat exchanger is proposed by using Mamdani interface system built in MATLAB. Then, the definations of the fuzzy variables for the control are variated in order to see in what manner the response of the system will change.

2 Preliminaries

2.1 Feedback Control System

A feedback control system is a system whose output is controlled using its measurement as a feedback signal. This feedback signal is compared with a reference signal to generate an error signal which is filtered by a controller to produce the system's control input. Feedback control is needed to counteract disturbance signals affecting the output, to improve the system performance in the presence of model uncertinty and to stabilize an unstable output [4]. The control scheme of a feedback control is given in Figure 1.

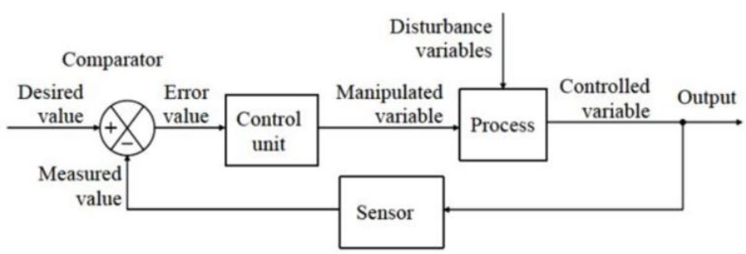

Figure 1: General control scheme of a feedback control system.

2.2 Obtainment of Process Transfer Function

In order to control a process, the process should be accurately modeled first. Thus, to model the heat exchanger, a simple energy balance is conducted for the fluid whose temperature is to be controled, as shown in the following equations (1) and (2).

$$EnergyAccumilation = EnergyIn - EnergyOut \tag{1}$$

$$V\rho c_p dT/dt = F\rho c_p T_0 + M\Delta H_b - F\rho c_p T \tag{2}$$

where V is the volume of heat exchanger, ρ is the density, c_p is the heat capacity, T is the temperature and F is the flow rate of the main process fluid. M is the flow rate and ΔH_b is the enthalpy change of the fluid which is heating or cooling the main process fluid. The fluids are assumed incompressible and the changes in the physical properties of the fluids with changing temperature are neglected for simplicity, which is a common application in modelling of the heat exchangers.

In order to control this system, we need to obtain the transfer function of the process. Thus, by taking the Laplace transform of equation (2), equation (3) is obtained. When this equation is reorganized as in equation (4), we obtain the process and disturbance transfer functions as the coefficients of M(s) and T_0(s); respectively.

$$V\rho c_p sT(s) = F\rho c_p T_0(s) + M(s)\Delta H_b - F\rho c_p T(s) \tag{3}$$

$$T(s) = 1/(\tau s + 1)T_0(s) + (\Delta H_b/(F\rho c_p))/(\tau s + 1)M(s) \tag{4}$$

where τ is the residance time, which is defined as τ = V/F.

In this study, we are not investigating the changes in the disturbance, namely T_0, thus, T_0(s)=0. This results the equation (5), where the coefficient of M(s) is known as process transfer function, denoted by G_p(s).

$$T(s) = (\Delta H_b/(F\rho c_p))/(\tau s + 1)M(s) \tag{5}$$

$$G_p(s) = K_p/(\tau s + 1) \tag{6}$$

where K_p is the process gain, defined as K_p=ΔH_b/(F ρ c_p).

For both simulations conducted in this study, K_p=50 and τ=30 will be taken since they are variables that depend on the system, and the investigated system will be assumed the same.

3 Proposed Aproach for Fuzzy Feedback Control of Heat Exchanger

Fuzzy control has several advantages over conventional controllers such as simplicity, flexibility, ability to handle problems with imprecise data etc. In order to construct a fuzzy feedback control, a fuzzy controller is needed. In this study, Mamdani interface system (built in MATLAB) was used for construction of the fuzzy controller, and the error between the set point and the controlled variable (e), derivative of the error (de) and the output (Δu) variables are defined with the fuzzy numbers NB (Negative Big), NS (Negative Small), ZE (Zero), PS (Positive Small) and PB (Positive Big) whose membership functions are illustrated in Figures 2 and 3.

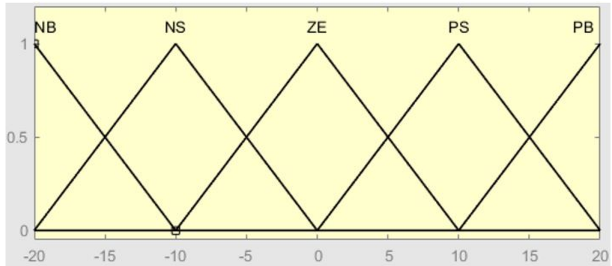

Figure 2: Membership function of input variables error (e) and difference in error (de).

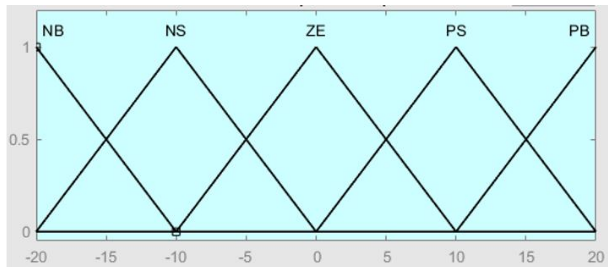

Figure 3: Membership function of fuzzy output variable Δu.

The rule table for the control was designed as presented in table 1. The non-linear control surface generated by these rules is illustrated in Figure 4. Then, feedback control system with the fuzzy controller was implemented, resulting the flow diagram given in Figure 5, which is similar to other works in the literature [5, 6].

Table 1: Rule table for the fuzzy controller.

Processes	**$\dot{m}$(kg/s)**	**T_i (°C)**	**T_o (°C)**	**ρ (kg/m³)**	**c_p (kj/kg K)**	**μ (Pa s)**	**k (W/m K)**	**R_f (m² K/W)**
Process 1								
Shellside: Methanol	27.8	95	40	750	2.84	0.00034	0.19	0.00033
Tubeside: Sea Water	68.9	25	40	995	4.2	0.00080	0.59	0.00020
Process 2								
Shellside: Kerosene	5.52	199	93.3	850	2.47	0.00040	0.13	0.00061
Tubeside: Crude Oil	18.8	37.8	76.7	995	2.05	0.00358	0.13	0.00061
Process 3								
Shellside: Distilled Water	22.07	33.9	29.4	995	4.18	0.00080	0.62	0.00017
Tubeside: Raw Water	35.31	23.9	26.7	999	4.18	0.00092	0.62	0.00017

10 units of step change is applied to the set point at t=1 in order to observe the reaction of the system. The response of the heat exchanger can be seen in the Figure 6. Note that there occurs an off-set, which is uncharacteristic for a PID-like controller, which tries to minimize the absolute value of error whenever possible. However, the occurrence of such off-set is actually resulted from nature of the fuzzy numbers, since when the temperature value adjusted 9.5 units, the controller was able to deduce that there is no big difference between 9.5 and 10, thus stopped resulting any output. In order to decrease this off-set, the definition of fuzzy variables may be altered accordingly. As an example, if the fuzzy variables e, de and Δu are defined as in Figure 7, with the same rule table, the offset may be decreased, resulting a response of the system as seen in Figure 8.

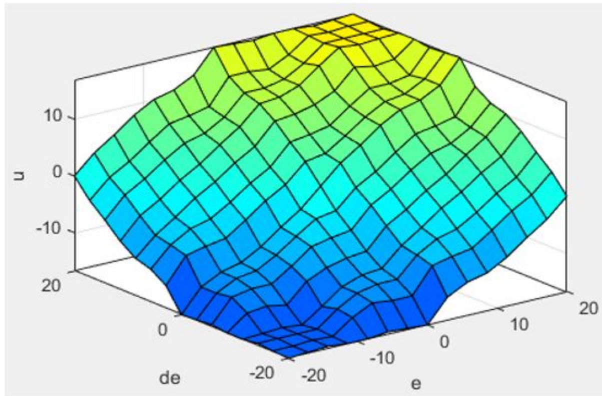

Figure 4: Control surface for non-linear control system.

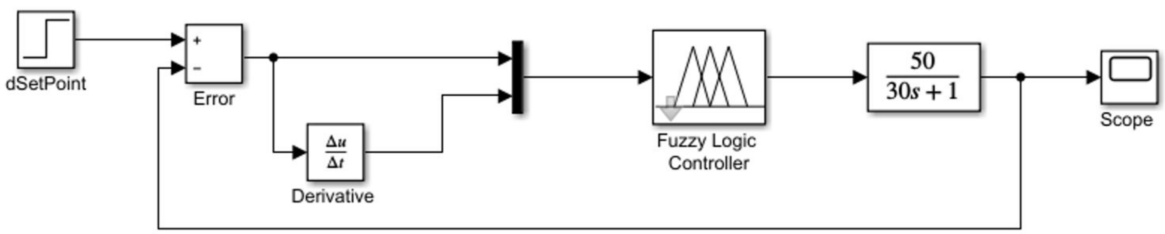

Figure 5: Flow diagram of feedback control with fuzzy controller.

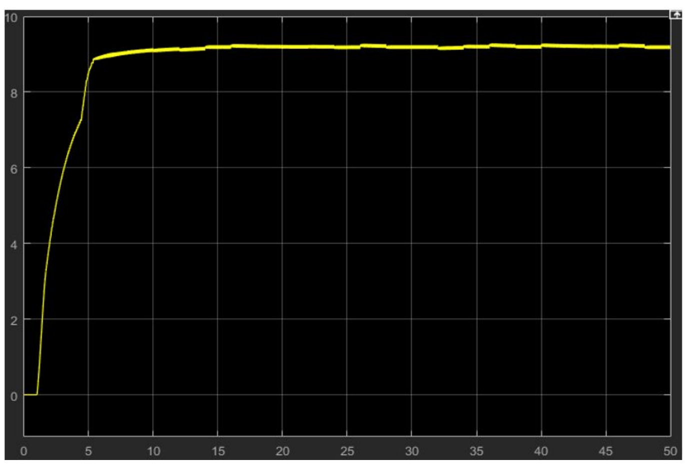

Figure 6: System response after a 10 units of step change in the setpoint with the controller variables defined as in Figures 2 and 3.

4 Conclusion

In this study, we have presented an aproach for the fuzzy control of a simple heat exchanger system. The response of the system was quick to react, but an off-set was observed which is quite contrary to the nature of PID controllers. This contradiction stems from the nature of the fuzzy numbers. Because the "stop condition" was defined with the fuzzy number ZE, the controller was able to deduce that 0.5 is within the bounds of ZE and it does not increase the temperature more than necessary, which is a significant thing to be understood by a controller. We have also shown

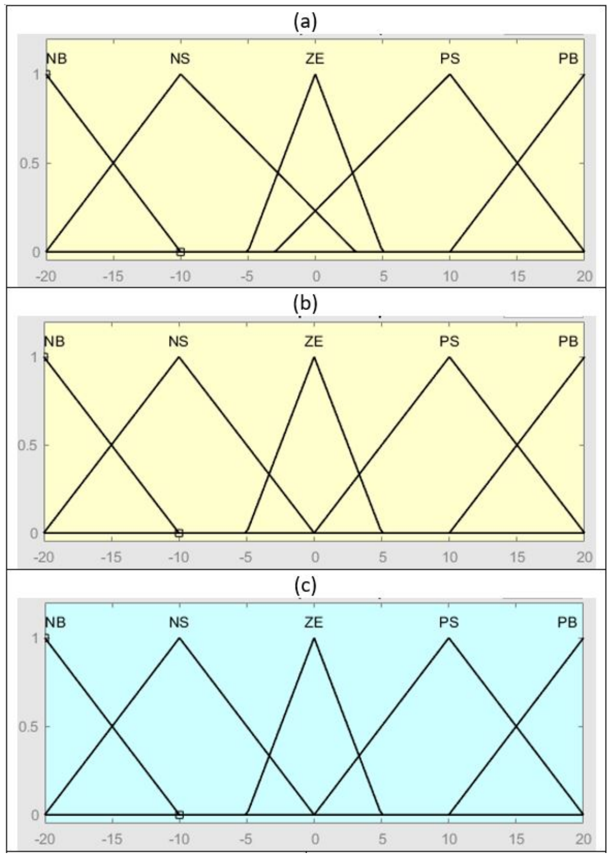

Figure 7: New definitions for variables e (a), de (b) and Δu (c).

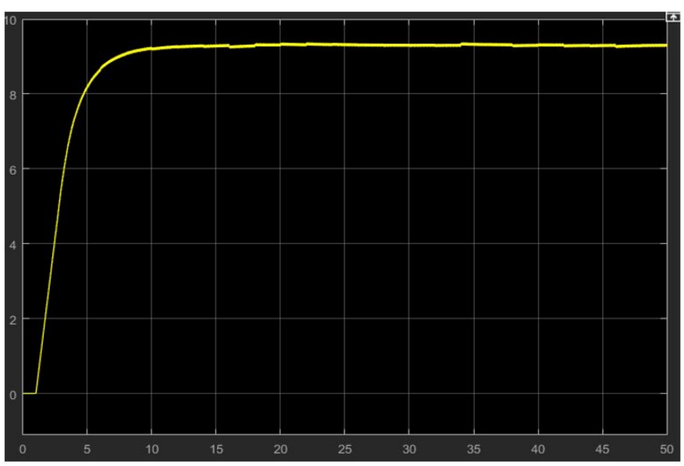

Figure 8: System response after a 10 units of step change in the setpoint with the controller variables defined as in Figure 7.

that when the definations of the fuzzy variables are variated, different responses of the system are obtained. As a future direction, this topic should be investigated more rigorously to obtain a relation between the response and the definitions of the fuzzy variables.

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COMPARISON OF NONTRADITIONAL OPTIMIZATION TECHNIQUES IN OPTIMIZATION OF SHELL AND TUBE HEAT EXCHANGER

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ABSTRACT

Heat exchangers, which are the systems that allow heat transfer between two or more uids, are widely used in industry, and they are an indispensable part of chemical processes. Thus, a considerable amount of research has been conducted in the optimization of these systems, especially regarding cost optimization. There are two main parameters affecting the cost of a heat exchanger. One is the surface area of the exchanger, which mainly affects the capital investment cost. The other one is the pressure drop, which mainly affects the operating cost. These parameters are interlinked; however, since the change in the surface area may result in a change in the pressure drop and vice versa due to the physical laws. In the case which is analyzed in this study, there are four main variables for the heat exchanger, which will be adjusted in order to obtain the optimum values for surface area and pressure drop: the number of tube side passages, shell inside diameter, bafes spacing, and tube outside diameter. In this study, we compare the success of the nontraditional optimization techniques such as generic algorithms, particle swarm optimization, articial bee colony, and biography-based optimization for a shell and tube heat exchanger. In the considered three processes in which these optimization methods are utilized, we see that biography-based optimization has given the minimum cost for each process.

***K*eywords** Optimization · Heat Exchanger · Cost Minimization

1 Introduction

A heat exchanger is a system that is used to transfer heat between two or more fluids. They are utilized in almost all production lines, since temperature control is critical in most chemical or physical processes. Heat exchangers can be designed to tolerate wide range of pressure and temperature; thus, they are widely used in industry [1]. There are several kinds of heat exchangers,

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and one of the most used ones is shell and tube heat exchanger [2]. The schematic representation of shell and tube heat exchangers is given in Figure 1.

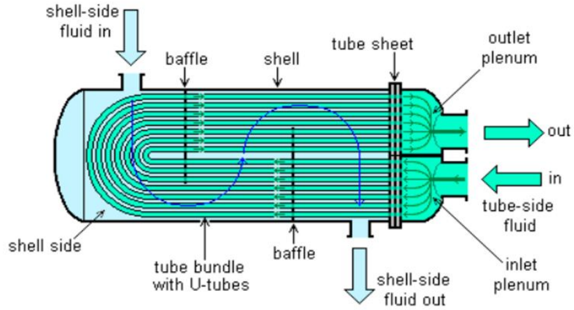

Figure 1: Schematic representation of shell and tube heat exchanger [3].

These exchangers allow indirect contact between two fluids, and thus heat transfer between the fluids takes place. In shell and tube heat exchangers, baffles are used in order to increase the turbulence and contact time to increase the heat transfer as much as possible.

Optimization of shell and tube heat exchangers is of great importance since slight changes in the temperature of process fluids may substantively affect other chemical and physical properties of the materials that are produced in a process. Different options for the optimization of shell and tube heat exchangers exist where gradient descent etc. are utilized. The traditional design methods for shell and tube heat exchangers includes evaluation of a large number of different heat exchanger geometries to identify those that meet a particular heat duty and a range of geometric and operational constraints. Since the optimization problem highly interconnected and nonlinear, obtainment of the optimum parameters has proven to be challenging.

In this study we will be focusing on cost optimization shell and tube heat exchangers. The optimum heat exchanger parameters which are obtained by different optimization algorithms will be examined for three different scenarios. In terms of helping engineers and producers in determining which optimization method should be used to yield the minimum cost for the production of a shell and tube heat exchanger, the presented study is remarkable.

2 Nontraditional Optimization Algorithms

There have been several heat exchanger designs proposed as a result of different kind of optimization methods. These methods include generic algorithm (GA), particle swarm optimization (PSO), artificial bee colony algorithm (ABC) and biography-based optimization (BBO). The first design for three different cases was conducted by Sinnot R.K. [10], in 2005, which will be referred as “Original Study” from now on. Then, in 2008, Caputo A.C. et. al. [11] used GA and obtained lower cost than before. Later, Patel V.K. et. al. [12] used PSO in 2010 and Şahin A.Ş. et. al. [4] used ABC algorithm in 2011 to further improve the design parameters and obtained better results. And then BBO is used in 2013 by Hadidi A. [9] and the cost is slightly more minimized by this algorithm.

2.1 Generic Algorithm

GA is an evolution inspired optimization procedure in a binary search space, where a set of solutions called population are evaluated rather than a single solution like in traditional hill climbers [14]. The most interesting advantages of GA when compared to the conventional optimization algorithms are parallelism, simple programmability, robustness regarding the input data and its large and wide solution space search ability [15].

2.2 Particle Swarm Optimization

PSO was developed in 1995 by Kennedy and Eberhart, inspired by behavior of social organisms such as ants and fish in a group [15]. This procedure emulates the interaction of each individual in a social group to share information. The optimal solution is obtained by evaluating these individual solutions, referred as particles, whose trajectories are adjusted accordingly. In each iteration, the particles adjust their velocity according to the "best particle" in the group [16].

2.3 Artificial Bee Colony Algorithm

ABC is a type of swarm intelligence method where, the behavior of honeybees is emulated. It combines local search methods, carried out by employed and onlooker bees, with global search methods, managed by onlookers and scouts, attempting to balance exploration and exploitation process. [4, 17]

2.4 Biography Based Optimization

BBO is based on biography theory, where the geographical distribution of biological species is investigated. It is developed through simulation of emigrations, immigrations, evolutions and extinctions of species between habitats which are in the multi-dimensional solution space [18].

3 Calculation of the Surface Area

The main variable determining the capital cost of heat exchanger is the surface area of the heat exchanger. This variable can be obtained in the following manner. The subindices in the equations s and t represent belonging to the shell and tube side fluid respectively.

The necessary energy rate to change the temperature of a flowing fluid is represented as in Equation 1.

$$q = \dot{m} c_p \Delta T \tag{1}$$

In order to provide this energy, a fluid can be contacted with another fluid in a heat exchanger. The surface area of this connection is calculated as in Equation 2.

$$S = q/(FU\Delta T_{LM}) \tag{2}$$

where,

$$U = 1/(1/h_s + R_{fs} + d_o/d_i(R_{fs} + 1/h_t)) \tag{3}$$

To evaluate U, the overall heat transfer coefficient, heat transfer coefficients should be evaluated by experimental correlations of Reynolds and Prandtl numbers. One such correlation for tube side is given in Equation 4 [4, 5, 6].

$$Nu = (h_t d_i)/k = 0.023 Re_t^{0.8} Pr_t^{0.4} \tag{4}$$

Thus, the tube side heat transfer coefficient can be written as in Equation 5 [4, 5, 6].

$$h_t = (kNu)/d_i \tag{5}$$

A similar correlation for the shell side heat transfer coefficient is given as in Equation 6 [4, 5, 6].

$$Nu = (\mu_t/\mu_w)^{0.14} 36 Re_s^{0.55} Pr_s^{0.4} \tag{6}$$

And, the shell side heat transfer coefficient can be determined as in Equation 7.

$$h_s = (kNu)/D_e \tag{7}$$

here, the equivalent diameter represents the diameter of the crossectional area which the fluid flows. For square tube pitch with two tube side passages, equivalent shell diameter can be obtained by the Equation 8. For triangular tube pitch with four tube side passages, equivalent shell diameter can be obtained by the Equation 9.

$$D_e = 4(Pt^2 - \pi d_o/4)/(\pi d_o) \tag{8}$$

$$D_e = 8((0.43Pt^2 - \pi d_o/2))/(\pi d_o) \tag{9}$$

To be used in the calculation of Reynolds number, which is defined as in Equation 10, and to be used in the calculations of heat transfer coefficients above, the velocities of shell and tube side fluids are given as in Equations 11 and 12, respectively.

$$Re = (vd)/\nu \tag{10}$$

$$v_t = n/N_t \dot{(m)}_t/((\pi d_o)/4\rho_t) \tag{11}$$

$$v_s = \dot{m}_s/(a_s\rho_s) \tag{12}$$

where the shell side crossectional area clearance is given in Equation 13.

$$a_s = (D_s B(Pt - d_o))/Pt \tag{13}$$

Also, the Prandtl number is defined from the properties of the fluids as in Equation 14.

$$Pr = (c_p\mu)/k \tag{14}$$

Thus, by using Equations 1 - 14, overall heat transfer coefficient (U) can be computed to be used in the determination of heat transfer surface area. The surface area is one of the two main parameters which will be considered in the cost optimization. From the equations, the number of tubes and the necessary tube length can also be obtained by the Equations 15 and 16, respectively. The length of the tube, among with velocities of shell and tube side fluids interling the surface area and pressure drop, resulting the highly nonlinear system of equations for the cost optimization.

$$N_t = K_1(D_s/d_o)^{n_1} \tag{15}$$

$$L = S/(\pi d_o N_t) \tag{16}$$

4 Calculation of the Pressure Drop

While the capital investment is a function of the surface area, operating cost of the heat exchanger is a function of pressure drop. The surface area and the pressure drop are interlinked, however, since the parameters which affect the surface area also affects the pressure drop of the fluids.

The pressure drop for the tube side can be calculated as in Equation 17 [4, 5, 6].

$$\Delta P_t = \Delta P_{tube_elbow} + \Delta P_{tube_length} = (1/2)\rho_t v_t^2 (L/d_i f_t + p)n \tag{17}$$

where, for the tube side, the friction factor is as in Equation 18.

$$f_t = (1.82log_{10}(Re_t) - 1.64)^2 \tag{18}$$

The pressure drop for the shell side can be calculated as in Equation 19 [4, 5, 6]..

$$\Delta P_s = (1/2)\rho_t v_s^2 f_s L D_s/(D_e B) \tag{19}$$

where, for the shell side, the friction factor is as in Equation 20 [9].

$$f_s = 2b_0 Re_s^{-0.15} \tag{20}$$

here, b_0=0.72 is valid for Re<40000 [9].

5 Cost Optimization of Heat Exchangers

The objective function for the optimization problem is the total cost of the heat exchanger, consisting of the summation of the capital investment Ci and discounted operating cost CoD. This total cost is given Equation 21 [4, 7].

$$C_{total} = C_i + C_{oD} \tag{21}$$

here, the capital investment can be obtained from the Hall's correlation as a function of surface only as in Equation 22 [7].

$$C_i = a_1 + a_2 S^{a_3} \tag{22}$$

where, for heat exchangers made of stainless steel for both tubes and shells, a_1=8000, a_2=259.2 and a_3=0.91.

The operating cost is related to the pumping power to overcome the pressure drops resulted from friction losses. This can be computed as in Equation 23 [7].

$$C_o = PC_E H \tag{23}$$

where, P is the energy needed to overcome the total pressure drop (given in Equation 24), C_E is the cost of electricity and H is the yearly operation duration.

$$P = 1/\mu((\dot{m}_t/\rho_t)\Delta P_t + (\dot{m}_s/\rho_s)\Delta P_s) \tag{24}$$

Thus, the discounted operating cost at the beggining of the production of heat exchanger is obtained as in Equation 25.

$$C_{oD} = \sum_{k=1}^{n_y}(C_o/(1+i)^k) \tag{25}$$

6 Cost Optimization Problem

In this study, optimization of heat exchangers for three different processes is investigated. The design parameters for the processes are explained in the Table 1. The parameters can be considered as fixed optimization constraints or fixed parameters.

Table 1: Design parameters for different processes [11].

Processes	$\dot{m}$(kg/s)	T_i (°C)	T_o (°C)	ρ (kg/m^3)	c_p (kj/kg K)	μ (Pa s)	k (W/m K)	R_f (m^2 K/W)
Process 1								
Shellside: Methanol	27.8	95	40	750	2.84	0.00034	0.19	0.00033
Tubeside: Sea Water	68.9	25	40	995	4.2	0.00080	0.59	0.00020
Process 2								
Shellside: Kerosene	5.52	199	93.3	850	2.47	0.00040	0.13	0.00061
Tubeside: Crude Oil	18.8	37.8	76.7	995	2.05	0.00358	0.13	0.00061
Process 3								
Shellside: Distilled Water	22.07	33.9	29.4	995	4.18	0.00080	0.62	0.00017
Tubeside: Raw Water	35.31	23.9	26.7	999	4.18	0.00092	0.62	0.00017

There are four optimization variables to be computed by these parameters, which are number of tube side passages (2, triangular pitch or 4, square pitch), shell inside diameter (D_s), baffles spacing (B) and tube outside diameter (d_o). Here, to simplify the problem, the binary variable (number of tube side passages) can be reduced by conducting two different optimization at both values. Then, the minimum value obtained by these two optimizations can be compared, and the one with the lower value can be taken as the result. In example, Şahin A.Ş. et. al. [4] only considered the square pitch shell and tube heat exchangers in their work. However, Hadidi A. [9] considered 5 different tube pass scenarios for both triangular pitch and square pitch heat exchangers.

7 Results and Discussion

The optimization studies are conducted by using GA, PSO, ABC and BBO algorithms in the literature. Their results for crucial parameters of each process is given in the following Tables 2-4.

Table 2: Optimum values of the heat exchanger parameters evaluated for the process 1.

	Original Design[10]	GA[11]	PSO[12]	ABC[4]	BBO[9]
S (m^2)	278.6	262.8	243.2	230.109	229.95
U (W/m^2K)	615	660	713.9	832	755
L (m)	4.830	3.379	3.115	3.963	2.040
D_s (m)	0.894	0.830	0.81	1.3905	0.801
d_o (m)	0.020	0.016	0.015	0.0104	0.010
B (m)	0.356	0.500	0.424	0.4669	0.500
N_t	918	1567	1658	1528	3587
v_t (m/s)	0.75	0.69	0.67	0.36	0.77
v_s (m/s)	0.58	0.44	0.53	0.118	0.46
ΔP_t (Pa)	6251	4298	4171	3043	6156
ΔP_s (Pa)	35789	13267	20551	8390	13799
C_i (€)	51507	49259	46453	44559	44536
C_{oD} (€)	12973	5818	6778.2	6233.8	6046
C_{tot} (€)	64480	55077	53231.1	50793	50582

Table 3: Optimum values of the heat exchanger parameters evaluated for the process 2.

	Original Design[10]	GA[11]	PSO[12]	ABC[4]	BBO[9]
S (m^2)	61.5	52.9	47.5	61.566	60.35
U (W/m^2K)	317	376	409.3	323	317.75
L (m)	4.88	2.153	1.56	3.6468	1.199
D_s (m)	0.539	0.63	0.59	0.3293	0.74
d_o (m)	0.025	0.02	0.015	0.0105	0.015
B (m)	0.127	0.12	0.1112	0.0924	0.1066
N_t	158	391	646	511	1061
v_t (m/s)	1.44	0.87	0.93	0.43	0.69
v_s (m/s)	0.47	0.43	0.495	0.37	0.432
ΔP_t (Pa)	49245	14009	16926	1696	5109
ΔP_s (Pa)	24909	15717	21745	10667	15275
C_i (€)	19007	17599	16707	19014	18799
C_{oD} (€)	8012	2704	3215.6	1211.3	1010.25
C_{tot} (€)	27020	20303	19922.6	20225	19810

For process 1, the surface area has gradually decreased to reach 229.95 m^2 with each optimization method. This suggest that an absolute minimum may have been obtained for the capital investment amount. For process 2, the surface area firstly decreased for GA and PSO methods, but then increased to the original design value for the other algorithms. Thus, we conclude that two different local minimums have been reached, one with a higher capital investment and smaller operating cost, and one with a higher operating cost and smaller capital investment. These results are both candidates for being the absolute minimum value. For process 3, we see that the surface area is firstly increased to 62.5 m^2 by GA method, which resulted massive decrease in the operating costs. Then, the following studies decreased the surface area while managing to keep the operating costs close to each other.

Table 4: Optimum values of the heat exchanger parameters evaluated for the process 3.

	Original Design[10]	GA[11]	PSO[12]	ABC[4]	BBO[9]
S (m^2)	46.6	62.5	59.2	54.72	55.73
U (W/m^2K)	1471	1121	1177	1187	1384
L (m)	4.880	1.548	1.45	2.4	1.133
D_s (m)	0.387	0.62	0.0181	1.0024	0.55798
d_o (m)	0.019	0.016	0.0145	0.0103	0.01
B (m)	0.305	0.440	0.423	0.354	0.5
N_t	160	803	894	704	1565
v_t (m/s)	1.76	0.68	0.74	0.36	0.898
v_s (m/s)	0.94	0.41	0.375	0.12	0.398
ΔP_t (Pa)	62812	3673	4474	2046	4176
ΔP_s (Pa)	67684	4365	4271	2716	5917
C_i (€)	16549	19163	18614	17893	18059
C_{oD} (€)	27440	1671	1696	1584.2	1251.5
C_{tot} (€)	43989	20834	20310	19478	19310

The cost comparison of the optimization algorithms are illustrated in Figures 2-4 for each process. It is seen that; for all cases, the best results have been obtained by using BBO algorithm. However, when the pressure drop values and discounted operating cost are compared for each study (Figures 5-7), we see that BBO algorithm has resulted higher pressure drop values than ABC algorithm for each process, but nevertheless obtained lower discounted operating cost. This is a contradiction, since the discounted operating cost is obtained as in Equations 23 - 25, higher pressure drop values should result in higher discounted operating cost.

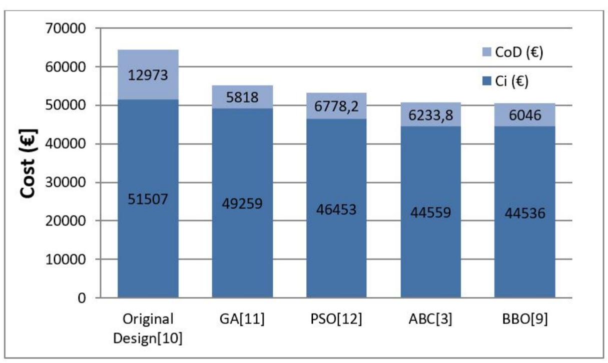

Figure 2: Optimum cost comparison for process 1.

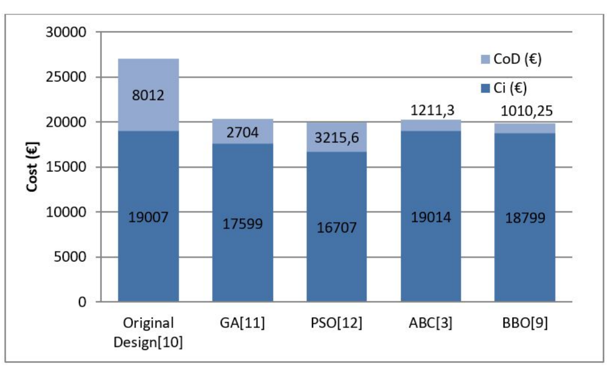

Figure 3: Optimum cost comparison for process 2.

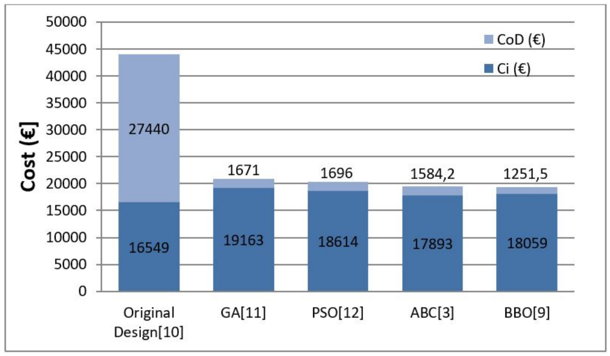

Figure 4: Optimum cost comparison for process 3.

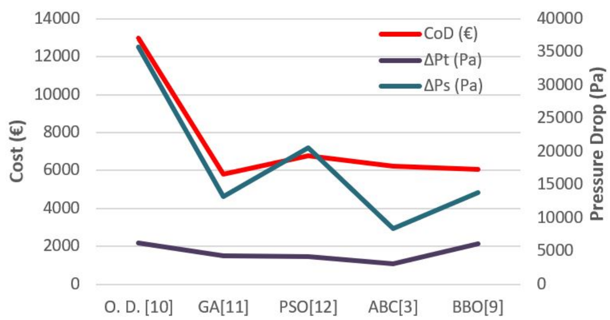

Figure 5: Resulted pressure drop and discounted operating cost values of optimization methods for process 1.

8 Conclusion

In this study, non-traditional optimization algorithms GA, PSO, ABC and BBO are considered for the optimization of heat exchanger. The results of each optimization method is compared by considering three different processes. While the best results are obtained by using BBO in each process, it is observed that the operating costs obtained by the BBO algorithm is contradicting with the operating costs obtained by other algorithms. Also, while ABC algorithm has resulted lower or similar values for surface area and pressure drop values than BBO algorithm, which are the fundamental optimization parameters directly linked to the objective function, the total cost of the ABC algorithm is reported higher than the cost of BBO algorithm. Thus, further investigation regarding to the validity and comparability of the algorithms is needed.

In addition, we have observed that the results of these non-traditional optimization algorithms seem to converge to a singular local minimum for process 1 and process 3. For process 2, it is observed that two different local minimum exists, one with a higher capital investment and one with a higher operating cost. While these extremums are all candidates for being the absolute minimum, it is not clear that the absolute minimum has been reached by any of these algorithms. The optimization may be conducted by utilizing different algorithms to obtain even better results.

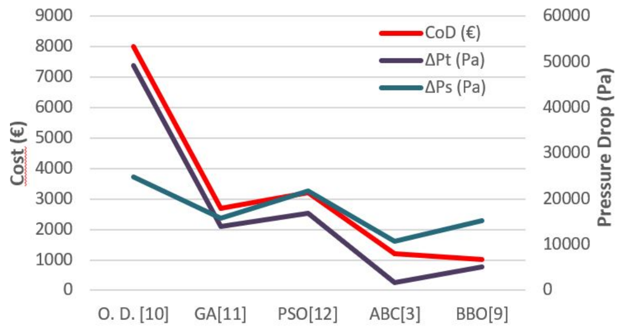

Figure 6: Resulted pressure drop and discounted operating cost values of optimization methods for process 2.

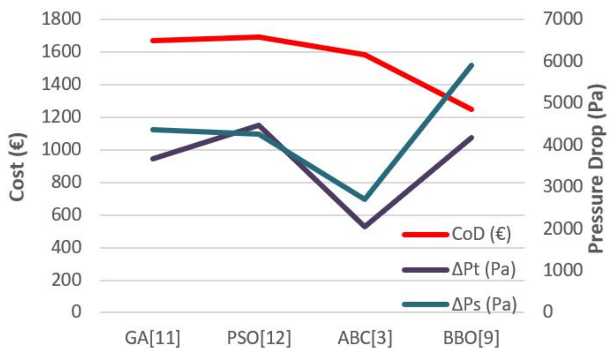

Figure 7: Resulted pressure drop and discounted operating cost values of optimization methods for process 3.

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NUMERICAL ANALYSIS OF AN ADAPTIVE NON-LINEAR FILTER BASED TIME REGULARIZATION MODEL FOR THE INCOMPRESSIBLE NON-ISOTHERMAL FLUID FLOWS

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ABSTRACT

This report proposes a modular filter based regularization model for the incompressible non-isothermal fluid flows, and a numerical method for solving that. The proposed model plugs a time relaxation term into Leray model which uses a deconvolution based indicator function. The task of this relaxation term is to drive the small scales to zero exponentially. The numerical algorithm for solving the model is stable over short time interval without any time step restriction, and has optimal convergence rates both in time and space. Numerical experiments verify theoretical convergence rates and compare the approximate solutions of the model with the non-regularized finite element spatial and backward-Euler temporal discretization of the flow problem.

Keywords Time relaxation · Finite element method · Non-linear filtering

1 Introduction

Finite element method is a powerful technique which is used to obtain approximate solutions for incompressible non-isothermal fluid flows (or the Boussinesq equations), which is governed by the Navier-Stokes and heat equations. Due to couple interaction of the evolution equations, getting full resolution of the system is impossible in many situations. One practical technique to overcome this issue is to use numerical regularization. In a recent paper [1], Monica and co-workers introduced a time regularization model via non-linear filtering of [9] to the incompressible Navier-Stokes equations (NSE). The proposed method adds a time relaxation term into NSE which uses a nonlinear filtering. This filtering step is applied as a modular step which solves a Stokes-Darcy type system, see [8]. Hence, the resulting algorithm can advantages computational point of view. We note that this type regularization has been studied extensively in [5, 7, 6].
The motivation behind this report is to apply this regularization model via non-linear filtering to

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incompressible non-isothermal fluid flows as in [9, 1]. The evolution equations which govern incompressible, non-isothermal flows are given by

$$\begin{aligned} u_t + u \cdot \nabla u - Re^{-1}\Delta u + \nabla p - RiT\hat{k} &= f, \\ \nabla \cdot u &= 0, \\ T_t + u \cdot \nabla T - (RePr)^{-1}\Delta T &= \gamma, \end{aligned} \tag{1}$$

where u is the velocity, p pressure, T temperature, f and γ are given forces, $\hat{k} :=< 0, \ldots, 0, 1 >$ is the unit vector, and Re, Ri, and Pr are the Reynolds, Richardson, and Prantdl numbers, respectively. Adding time relaxation term for velocity leads to the following model:

$$\begin{aligned} u_t + u \cdot \nabla u - Re^{-1}\Delta u + \nabla p + \chi(u - \bar{u}) - RiT\hat{k} &= f, \\ \nabla \cdot u &= 0, \\ T_t + u \cdot \nabla T - (RePr)^{-1}\Delta T &= \gamma, \\ -\alpha^2 \nabla \cdot (a(u)\nabla \overline{u}) + \overline{u} + \nabla \lambda &= u, \\ \nabla \cdot \overline{u} &= 0, \end{aligned} \tag{2}$$

where $\bar{u} := u - u'$ is unresolved (small) scales, u' is called the fluctuation. The term $\chi(u - \bar{u})$ a linear, lower order term which aims to to drive unresolved velocity scales to zero exponentially fast. To realize this aim, $\chi > 0$, and has unit $1/time$. In addition, $\alpha > 0$ is the spatial filtering radius, and $a(\cdot)(x)$ is a function satisfying $0 < a(\cdot)(x) \leqslant 1$ and

$$\begin{aligned} a(\cdot)(x) \approx 0 &\text{ in regions where } \phi \text{ does not need regularization,} \\ a(\cdot)(x) \approx 1 &\text{ in regions where } \phi \text{ does need regularization.} \end{aligned}$$

For further knowledge about non-linear filtering, one can see the paper [9].

2 Numerical scheme

We consider a domain $\Omega \subset \mathbb{R}^d$ (d=2 or 3) to be a convex polygon or polyhedra. The $L^2(\Omega)$ norm and inner product will be denoted as $\|\cdot\|$ and $(\cdot,\cdot)$, the $H^k(\Omega)$ norm by $\|\cdot\|_k$, and the $L^\infty(\Omega)$ norm by $\|\cdot\|_\infty$. All other norms will be clearly labeled.
For the finite element method, we assume a regular, conforming mesh τ_h, with maximum element diameter h, and associated velocity-pressure-temperature finite element (FE) spaces $X_h \subset H_0^1(\Omega)^d$, $Q_h \subset L_0^2(\Omega)$, and $Y_h \subset H_0^1(\Omega)$ satisfying approximation properties of piecewise polynomials of local degree $k, k-1$ and k, respectively, see [10]. We use skew symmetrized trilinear forms for the non-linear terms to ensure stability of the numerical method

$$\begin{aligned} b(u, v, w) &:= \frac{1}{2}\left[(u \cdot \nabla v, w) - (u \cdot \nabla w, v)\right], \ \forall u, v, w \in X, \\ c(u, \theta, \psi) &:= \frac{1}{2}\left[(u \cdot \nabla \theta, \psi) - (u \cdot \nabla \psi, \theta)\right], \quad \forall u \in X \text{ and } \theta, \psi \in Y. \end{aligned}$$

Remark 2.1. *The proposed algorithm is well-posed, and unconditionally stable with respect to time step $\Delta t > 0$. In addition, straightforward finite element error analysis, see [1], yield that the solutions of the proposed algorithm converges to the solutions of* (2)*, and this convergence is optimal both in time and space.*

$f \in L^\infty(0,t^*;H^{-1}(\Omega)^d)$, $\gamma \in L^\infty(0,t^*;H^{-1}(\Omega))$ be given. u_h^0, T_h^0 the interpolant of u^0 and T^0. Choose a finite end time t^* and time step Δt such that $t^*/\Delta t = M$. Denote $u_h^{n+1} := u_h(t^{n+1})$, $p_h^{n+1} := p_h(t^{n+1})$, $T_h^{n+1} := T_h(t^{n+1})$, $n = 0,1,2,...,M-1$ Find $(u_h^{n+1}, p_h^{n+1}, T_h^{n+1}) \in (X_h, Q_h, W_h)$ such that $\forall(v_h, q_h, \chi_h) \in (X_h, Q_h, W_h)$ it holds

Step 1: Find $(\overline{u_h^n}, \lambda_h^{n+1})$ such that it holds: $\forall(\chi_h, q_h) \in X_h \times Q_h$

$$\alpha^2(\nabla(a(u_h^n)\overline{u_h^n}, \nabla\chi_h) + (\overline{u_h^n}, \chi_h) - (\lambda_h^{n+1}, \nabla\cdot\chi_h) = (u_h^n, \chi_h),$$
$$(\nabla\cdot\overline{u_h^n}, q_h) = 0.$$

Step 2: Compute $(u_h^{n+1}, p_h^{n+1}, T_h^{n+1}) \in (X_h, Q_h, W_h)$

$$\left(\frac{u_h^{n+1}-u_h^n}{\Delta t}, v_h\right) + \nu(\nabla u_h^{n+1}, \nabla v_h) + b(u_h^n, u_h^{n+1}, v_h) - (p_h^{n+1}, \nabla\cdot v_h)$$
$$+\chi(u_h^{n+1} - \overline{u_h^n}, v_h) = Ri(T_h^n\hat{k}, v_h) + (f^{n+1}, v_h),$$
$$\left(\frac{T_h^{n+1}-T_h^n}{\Delta t}, S_h\right) + \kappa(\nabla T_h^{n+1}, \nabla S_h) + c(u_h^n, T_h^{n+1}, S_h) = (\gamma^{n+1}, S_h).$$

3 Numerical Experiments

The first numerical experiment confirms spatial and temporal convergence rates of Algorithm 2. With the Taylor-Hood finite element P_2, P_1, P_2-the velocity-pressure-temperature choice, one concludes that the velocity and temperature spatial convergence are of second-order, and and temporal errors are first order, i.e.

$$\||u-u_h\||_{2,1} + \||T-T_h\||_{2,1} = \mathcal{O}(h^2+\Delta t),$$

where $\||\cdot\||_{2,1} := \left(\Delta t \sum_{n=0}^{N-1} \|\nabla\cdot\|^2\right)^{1/2}$. To verify both in spatial and temporal convergence rates, we pick true velocity, pressure and temperature solutions as

$$u(x,t) = \begin{bmatrix} \cos(\pi(y-t)) \\ \sin(\pi(x+t)) \end{bmatrix} \exp(t), \;\; p(x,t) = \sin(x+y)(1+t^2), \;\; T(x,t) = \sin(\pi x) + y\exp(t),$$

on $\Omega = (0,1)\times(0,1)$. We picked the dimensionless flow parameters $\nu = \kappa = Ri = 1$, and filtering radius, relaxation parameter $\alpha = \mathcal{O}(h)$, $\chi = \mathcal{O}(\Delta t)$. We imposed the boundary conditions

$$u_h(x,t) = u(x,t), \;\; T_h(x,t) = T(x,t).$$

For spatial convergence rate verification, we fix the end time $t^* = 0.001$ and time step $\Delta t = 0.0001$. Then, we calculate the approximate solutions using smaller mesh size h, see Table 1. For

h	$\||u-u_h\||_{2,1}$	Rate	$\|\||T-T_h\||_{2,1}$	Rate
1/4	2.2701e-3	- - -	1.6005e-3	- - -
1/8	5.7243e-4	1.9876	4.02747e-4	1.9905
1/16	1.4299e-4	2.0012	1.0087e-4	1.9974
1/32	3.5717e-5	2.0012	2.5229e-5	1.9993
1/64	8.9283e-6	2.0002	6.3080e-6	2.0015

Table 1: Velocity and temperature spatial errors and convergence rates.

temporal convergence,we fixed the mesh size $h = 1/64$, calculate the approximate solutions using smaller time steps with end time $t^* = 1$. The results can be seen in Table 2.

Δt	$\|\|u-u_h\|\|_{2,1}$	Rate	$\|\|T-T_h\|\|_{2,1}$	Rate
$1/4$	2.3992e-1	- - -	2.2850e-2	- - -
$1/8$	1.3877e-1	0.7900	1.14507e-2	0.9967
$1/16$	7.2666e-2	0.9333	5.7067e-3	1.0046
$1/32$	3.6915e-2	0.9771	2.84988e-3	1.0018
$1/64$	1.8579e-2	0.9905	1.4329e-3	0.9920

Table 2: Velocity/Temperature Temporal Errors and Convergence Rates.

3.1 Marsigli's experiment

In this section, we test the proposed algorithm on a benchmark problem, named Marsigli's experiment. In the problem set-up, we follow the paper [4]. Flow region is an insulated box $[0, 8] \times [0, 1]$ divided at $x = 4$. The initial velocity is taken to be zero since the flow is at rest, and the initial temperature on the left hand side of the box is $T_0 = 1.0$, and on the right hand side $T_0 = 1.5$. The dimensionless flow parameters are set to be $Re = 1,000$, $Ri = 4$, $Pr = 1$, and the flow starts from rest.
We make two simulations related to this experiment. In both two simulations, we impose homogeneous Dirichlet boundary conditions for the velocity and the adiabatic boundary condition for the temperature, and compute solutions at $t^* = 2, 4, 8$, by picking (P_2, P_1, P_2)-velocity-pressure-temperature finite elements. We first perform the direct numerical simulations (DNS) of (1), i.e., no relaxation term for the velocity. We take time step $\Delta t = 0.025$ on a fine mesh, which provides $135,642$ velocity *degrees of freedom* (dof), $17,111$ pressure dof and $67,821$ the temperature dof. The temperature contours of this DNS are presented in the first plot of Figure 1, Figure 2 and Figure 3.
Next, we run Algorithm 2 with indicator functions a_{D_0} and a_{D_1}, and DNS on the same coarse mesh providing $36,814$ *DOF's* in total by taking time step $\Delta t = 0.005$. The results from these computations are shown in Figure 1- Figure 3. It can be clearly seen that the Algorithm 2 catches very well the flow pattern and temperature distribution of the DNS at each time level. However, DNS at this coarse mesh creates very poor solutions, and builds significant oscillations in temperature and velocity as time progresses.

4 Conclusions

In this work, we proposed, analyzed and tested a modular non-linear filter based stabilization method with time relaxation term. The proposed fully discrete finite element method for solving the model is unconditionally stable with respect to time step, has optimal convergences rates both in time and space, and gives much more accurate solutions on Marsigli's experiment when compared with the unstabilized method.

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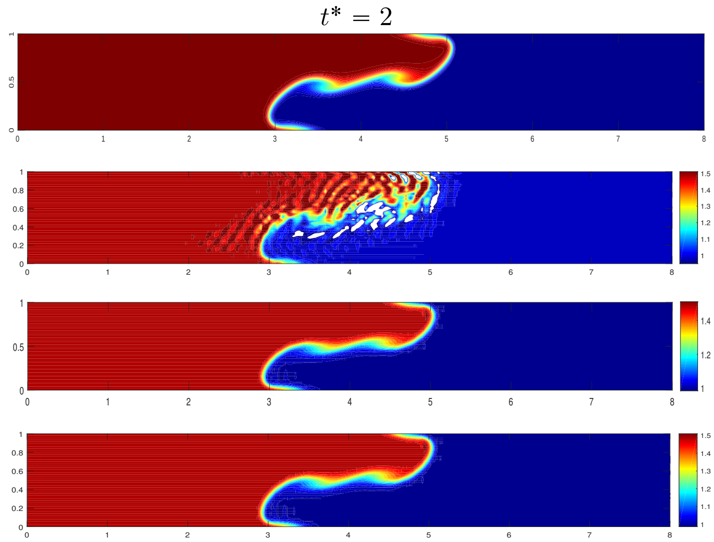

Figure 1: The temperature contours of DNS (fine mesh), DNS and our model's simulations with nonlinear filter that used indicator functions a_{D_0} and a_{D_1} (on the same coarse mesh) for 2D Marsigli's flow at $t^* = 2$.

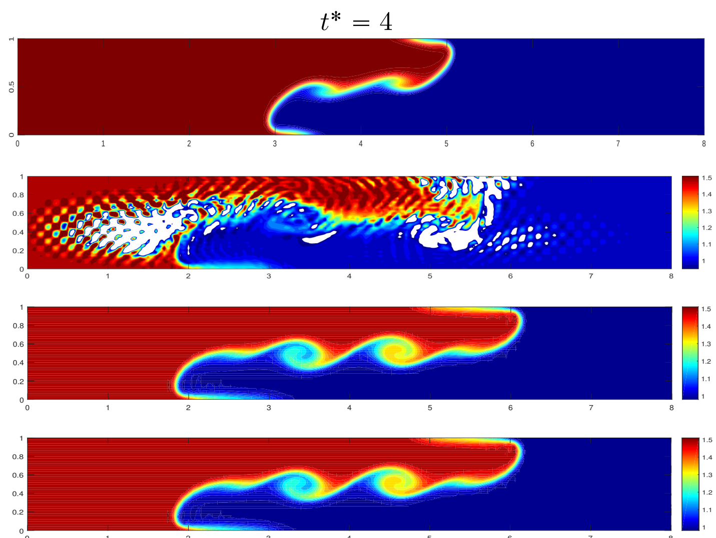

Figure 2: The temperature contours of DNS (fine mesh), DNS and our model's simulations with nonlinear filter that used indicator functions a_{D_0} and a_{D_1} (on the same coarse mesh) for 2D Marsigli's flow at $t^* = 4$.

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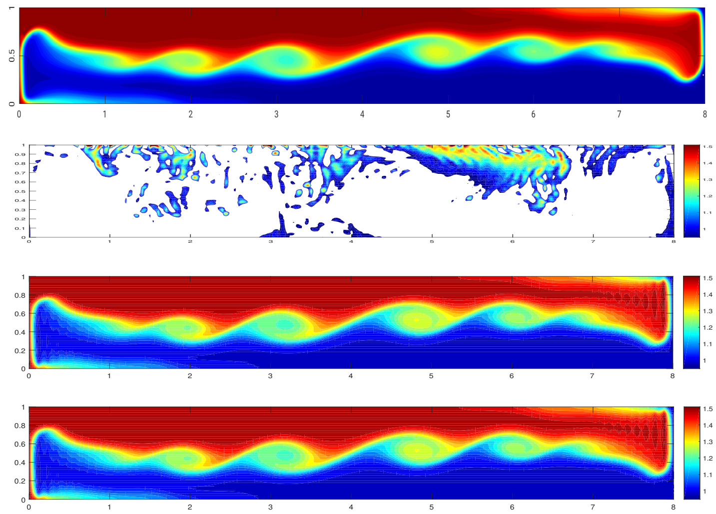

Figure 3: The temperature contours of DNS (fine mesh), DNS and our model's simulations with nonlinear filter that used indicator functions a_{D_0} and a_{D_1} (on the same coarse mesh) for 2D Marsigli's flow at $t^* = 8$.

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THE ROADS WE CHOOSE: CRITICAL THINKING EXPERIENCES IN MATHEMATICAL COURSES

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ABSTRACT

The constant technological development has changed profoundly the way we interact with each other, think and work, so the skills required by employers for new professionals graduated from universities also changed. This new paradigm requires educators, especially higher education teachers, to improve teaching and learning methodologies in order to promote critical thinking for solutions to complex and decision-making problems. Education should also take into account communication and collaboration, as well as the ability to master new technologies, avoiding the risks of their misuse. Along with them, skills such as critical thinking, creativity, originality, initiative, persuasion and negotiation should also be considered in training and they are in line with European Union (2018) documents along with the need to support the development of key skills in the acquisition of science, technology, engineering and mathematics (STEM) skills, taking into account their links with the arts, creativity and innovation and motivating young people to engage in STEM(/STEAM) careers. According to the report of the World Economic Forum (2018), politicians, educators and workers' representatives will gain if there is a deeper understanding of the labour market, as well as preparation for the ongoing changes. As teachers of higher education in science and technology and in particular in engineering, and as participants in the Erasmus+ CRITHINKEDU project (based on the knowledge and experience of European Higher Education Institutions, Enterprises and Non-Governmental Organisations), we believe that there should exist a constant concern to improve the quality of learning in higher education institutions in order to address the need for well-trained and well-educated citizens with skills tailored to labour market requirements. The present work arises from this concern in the Mathematics field and from the changes that the authors experienced fostered by the teacher-training course in Education for Critical Thinking of the Erasmus+ CRITHINKEDU project. This is a qualitative work, and we present a description of three different courses that were altered after the training course. The changes are presented at the level of planning, teaching and learning strategies used, as well as the learning assessment. Despite the introduced alterations, we know that we still have to continue the path started with training, as there are still aspects to improve to achieve all the aspects of

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the CRITHINKEDU project framework, in the teaching practices described, namely those that cover the critical thinking assessment.

***K*eywords** Mathematics · Critical thinking · Teaching methodology

1 Introduction

Nowadays higher education institutions (HEI) face several challenges, since the world is changing at an enormous velocity and "research on learning and education continues to influence engineering education. Examples include learning outcomes and teaching approaches, such as cooperative learning and inquiry that increase student engagement" [1]. This was important in order to improve the quality of learning in HEI in order to address the need for well-trained and well-educated citizens with skills tailored to labor market requirements. As teachers of higher education in science and technology and in particular in engineering, we participated in the Erasmus+ CRITHINKEDU project (based on the knowledge and experience of European Higher Education Institutions, Enterprises and Non-Governmental Organizations) and in this work we also adopted the Facione definition and framework [2].

> "We understand critical thinking to be purposeful, self-regulatory judgment which results in interpretation, analysis, evaluation, and inference, as well as explanation of the evidential, conceptual, methodological, criteriological, or contextual considerations upon which that judgment is based. CT is essential as a tool of inquiry. As such, CT is a liberating force in education and a powerful resource in one's personal and civic life. (...) The ideal critical thinker is habitually inquisitive, well-informed, trustful of reason, open-minded, flexible, fair minded in evaluation, honest in facing personal biases, prudent in making judgments, willing to reconsider, clear about issues, orderly in complex matters, diligent in seeking relevant information, reasonable in the selection of criteria, focused in inquiry, and persistent in seeking results which are as precise as the subject and the circumstances of inquiry permit." [3]

The present work arises from this concern in the Mathematics field and from the changes that the authors experienced fostered by the Critical Thinking (CT) of the Erasmus+ CRITHINKEDU project. Accompanying and participating in its outputs provided the experience and gave us the motivation to improve our own course design, mainly the teachers training week and the educational protocol to support the development of CT. This is a qualitative work, and we present a summary description of three different mathematical and statistical courses that have changed since the academic year 2017/2018. The changes enhance the connections with the educational protocol.

2 The Educational Protocol

In the report of output 4 of the Erasmus+ CRITHINKEDU project we may read.

> "The educational protocol rests on two major claims: 1) students will develop their critical thinking (CT) by explicitly engaging in appropriate learning activities, and 2) becoming stronger in critical thinking requires repeated engagement in critical thinking processes. The educational protocol has three parts: goals, conditions and supportive interventions." [2]

Figure 1 gives an idea of the protocol that was edited in a card form.

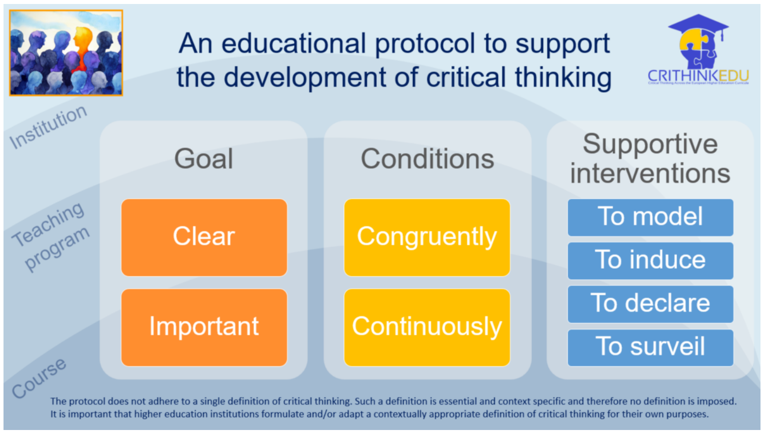

Figure 1: Card of the educational protocol to support the development of CT.

The first thing that we have to notice is that as an educational protocol we should develop it at all levels of HEI, that is CT has to be a goal at all the level of HEI, being a mission at the institutional level, and being a well described goal at the teaching program level and a well described learning outcome at the course level. Another important thing is that each HEI should use their one CT definition. Erasmus+ CRITHINKEDU project used Facione framework, but it was a partner's choice [2]. All the developed work took us to the protocol to support the development of CT and we summarize it before showing our own Mathematics course changes. In Figure 1, if you look at the left column of the card you have: Goal, Clear and Important. In order to support the development of CT, CT has to be a goal of education. Clear' means that an explicit clarification (by referring to the relevant literature) of the meaning of CT is provided. 'Important' means that not reaching the goal would be considered a failure [4]. If you look at the middle column of the card you have: Conditions, Continuously and Congruently. At the three levels of a HEI, CT is continuously and congruently allowed and made possible. 'Continuously' implies that the development of CT is not a one-shot operation. CT does not occur automatically or effortlessly. It needs continuous practice, reinforcement, and support. 'Congruently' implies that all actions with respect to CT are aligned to the goals [4]. If you look at the right column of the card you have: Supportive interventions, To model, To induce, To declare, and To surveil. Along the way for all supportive interventions the rule is that the support gradually withdraws. To model: CT development is supported when the institute (through its management structures), the teaching program (through its representatives) and the course (through its teachers) shows what it is to think critically [3]. To induce: Inducing CT implies that open questions are raised, ill-structured tasks are provided, complex problems are discussed and/or authentic, real-world issues remain at the core. What 'inducing' entails and how it can be done may vary for different fields and disciplines and may be done in different ways [4]. To declare: CT development is supported by declaring or making explicit what is at stake, what strategies can be used and what criteria are to be met. Declaring can be either spoken or written, but in all cases, it is both explicit and specific. What 'declaring' entails and how it can be done may also vary in different fields and disciplines [4]. To surveil: Surveillance monitors the ongoing efforts and activities, provides feedback on those efforts and activities, and helps to keep the efforts and activities oriented towards the (development of) CT. While differing in its concrete content and form among fields and disciplines, surveillance will always entail monitoring, feedback, and orientation [4].

3 Case 1. Linear Algebra

The first and the last authors were the teachers of the Linear Algebra course of the Communication and Multimedia Degree and decide to adopt a more active approach to motivate students in their classes [5]. Due to less years of Mathematics along the secondary education, these students struggle with a lot of difficulties in mathematics. Therefore, the learning outcomes are different but the design model 4C/ID of van Merriënboer et al. complex learning [6] was also used including the learning outcomes [7] (Figure 2). This course has as final goal "to discover the use/application of linear algebra in your area of study" and for that student had to accomplished different and more complex stages until the final of the course. Either the theoretical classes or the practical classes were taught 2h per week for 15 weeks.

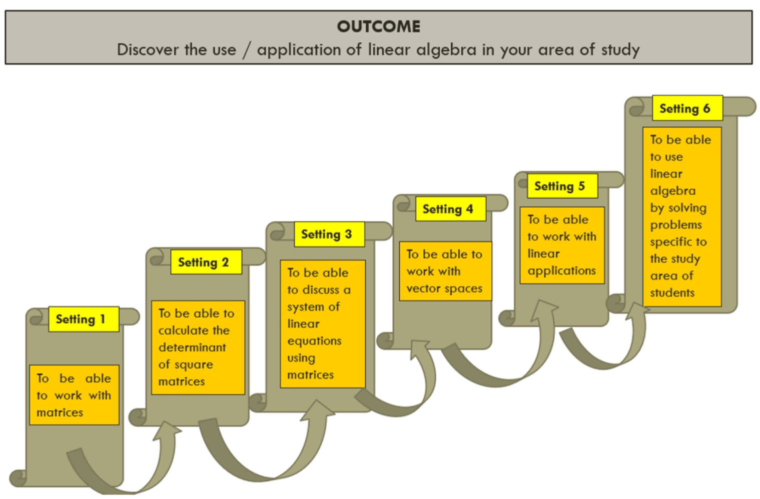

Figure 2: 4C/ID design for Linear Algebra and learning outcomes.

As an example of the tasks proposed to the students in the supportive intervention "To model" at the level course, teachers decide to ask the students: "Find out why you need to study matrices in this course". And that is an ill-structured problem and has no clear procedural or predetermined path to solve it and may have many different solutions. It was included in the first setting of the design "To be able to work with matrices". To solve this task, the teacher should address multiple internet sites about this topic and after, in group or in pairs, students should engage in critical discourse and group conversations about this topic. In a second stage, the teacher should address multiple images related with this topic (Figure 3).

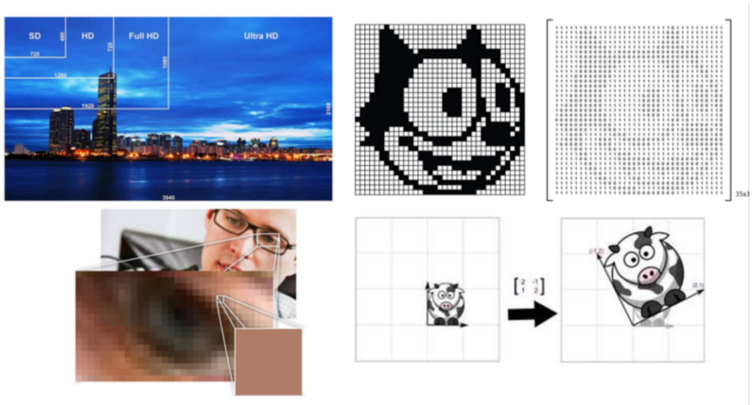

Figure 3: Images for Task 1 in the Linear Algebra.

And after, in group or in pairs, students should engage in critical discourse and group conversations about this topic. And the elements of this task "to induce" are presented in Figure 4.

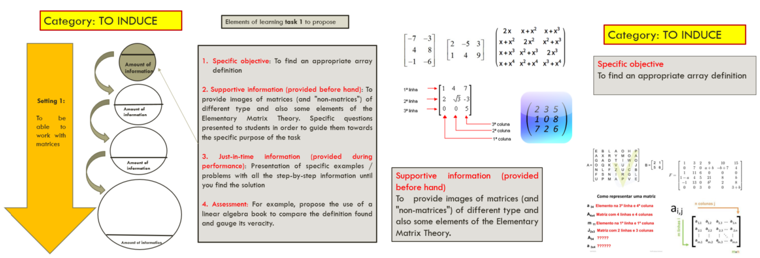

Figure 4: Task 1 "to induce" use in the Linear Algebra course level.

And for this first task the plan was: During the class, first the students work alone, and then work in pairs in order to compare their definitions of matrices; When each pair of students has a common definition, they share for all class their definition; There will be a discussion about it and in the end there will be a new definition - the definition of the class; Using a book of Linear Algebra the students can compare the class definition with the book's definition.

4 Case 2. Biostatistics

The third author was the teacher of the Biostatistics course of the Biology Degree. Also, in line with design model 4C/ID of van Merriënboer et al. complex learning [6] was also used (Figure 5) with the main outcome "Use the knowledge of probability theory to solve applied problems." As before, either the theoretical classes or the practical classes were taught 2h per week for 15 weeks.

All the supportive interventions were done through this Biostatistical course (Figure 6). Socrative questioning during the theoretical classes and on-line quizzes (Google Drive) at the end gave immediate feedback to students allowed them identify the mistakes made and to improve their performance and for the teacher allow her to improve the design and preparation of the learning strategies in the following classes.

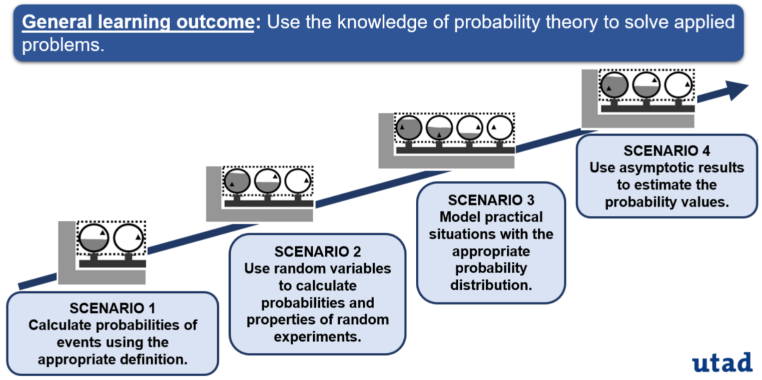

Figure 5: 4C/ID design for Biostatistics, including the learning outcomes.

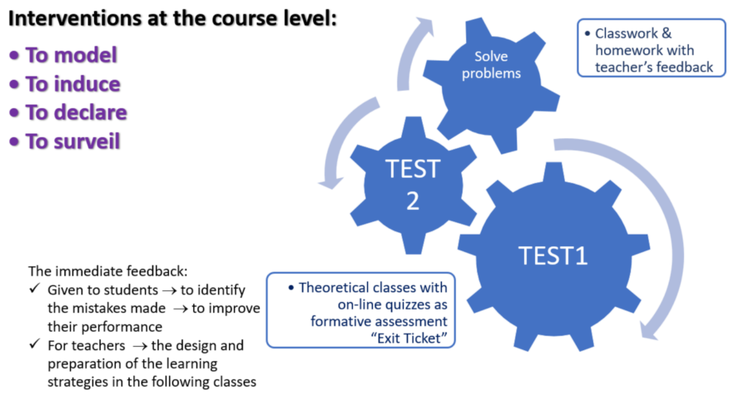

Figure 6: Interventions at Biostatistics course level.

5 Case 3. Statistical Methods

The second and third authors were the teacher of the Statistical Methods course of the Informatics Engineering Degree in the school year of 2018/2019. Also, in line with design model 4C/ID of van Merriënboer et al. complex learning [6] was also used (Figure 7) with the main outcome "To be able to select statistical concepts and data analysis methods in engineering" As before, either the theoretical classes or the practical classes were taught 2h per week during 15 weeks.

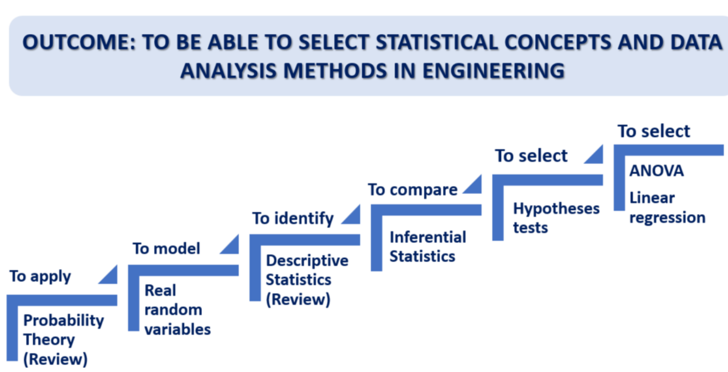

Figure 7: 4C/ID design for Statistical Methods, including the learning outcomes.

All the supportive interventions were done through this Statistical Methods course (Figure 8). During most of the theoretical classes and online quizzes (Google Drive) in the end of it gave immediate feedback to students and for the teachers as in Case 2. The students had a newspaper news to blind peer review (we planned) in small groups (pairs to three students) but instead of a full round we were only able to do the first round. That is students-authors did the comments (in a document word in Google Drive) but since that school year there were a lot of students, so we could not manage everything in order to the second and third rounds the small groups acting as student-reviewers of their work's colleagues. Nevertheless, we were able to manage accomplish the project work using the tutorial hours and some practical laboratory classes. The students fulfilled the ecological footprint survey, a database was built, and the students analyzed the data and presented a digital poster to the teachers during the last class and some questions were made. The supportive intervention "to declare" was the least accomplished of all since we fail to alert the students to the importance of the skills developed in all the classes and when they completed this course.

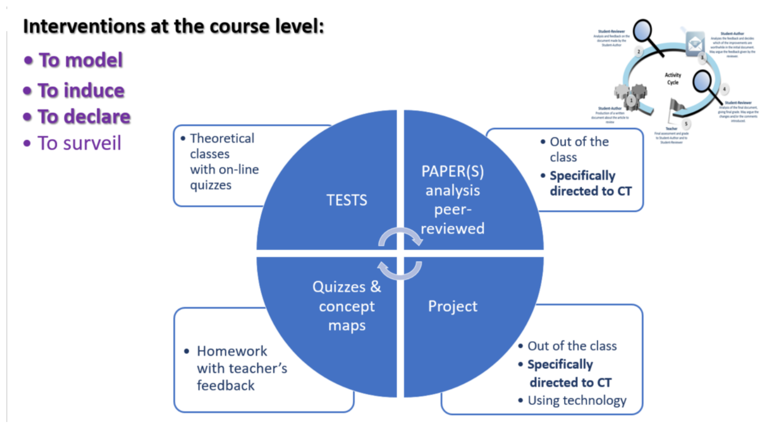

Figure 8: Interventions at Statistical Methods course level.

6 Concluding Remarks

We used the 4C/ID model [6] to rethink their curricular units – to model. The use of the taxonomy of learning outcomes is also clear Thus, we accomplish "to model".

In Case 1 in the written evaluations of 2017/2018, the questioning adopted aimed at students to mobilize their skills of interpretation, analysis and explanation. The objectives of the UC were aligned with those of the CT, but were not declared to the students, that is "to declare" must be improved.

In what concerns "to declare" in cases 2 and 3, references to general and specific objectives are still very brief. This aspect must be worked on and improved in line with the importance of "clear identification and clear definition of the CT skills to be developed" [7]. In Case 3 there is an explanation on the tasks of competences and dispositions of the CT they ended up not being explained to the students by the teacher and that must be declared [7]. Furthermore, "to declare" in Case 2 was slightly viewed in the quizzes that identified students' involvement with the CT in their texts. In other words, although incipient in the program, the students' texts recognize their presence at the Biostatistics of their Biology degree.

In the teachers' opinion of the three cases the course learning outcomes in CT were aligned with the curricular (course) learning goals of the cases reported here, accomplishing "to model".

In Case 3, although most students liked the group work and its dynamics, the fact that an assessment of the CT's skills and dispositions was not carried out – not surveiled - did not allow the teacher to assess progress, or the students reflect – to surveil.

Contact with the educational protocol to support the development of CT [4] has once again alerted teachers involved in the three cases that reflection and change have only just begun and that so that teaching practices cease to be adopted implicitly, unintentionally and to move to longer periods of time and, who knows, to involve other teachers of undergraduate degrees (from the same academic years or others), there is still a long way to go, there is still a lot of work to do.

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MMR Encryption Algorithm As An Alternative Encryption Algorithm to RSA Encryption Algorithm

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Abstract

In this study, a public key encryption algorithm is tried to be developed. Unlike other public key encryption algorithms, it is desired to create a monitoring key next to the open and closed keys. While creating the algorithm, the equivalence of $a^{10^{n-1}} - b^{10^{n-1}} \equiv 0(mod10^n)$ is used. Based on this equivalence, algorithms and keys have been created using modular arithmetic rules, Euclidean Algorithm, Euler Theorem, Euler Function, Factoring rules. The first difference of this algorithm from other algorithms is the observation key. In the event that private keys are stolen or cracked, hidden text or data cannot be accessed without a observation key. The second difference is that the receiver's private key, public key and observation key can take infinite values. For now, it is not a problem for the keys to take limited values in other algorithms. However, with the development and speed of the quantum computers, this will be a problem in the future. There are studies that have been successful in this regard. The third difference is that this algorithm has its own character code table. In addition, this algorithm is safer against side channel attacks. This algorithm can be used for secure e-mail, e-commerce, mobile banking and secure data transfer.

Keywords Encryption · closed key · Observation key · Euclid Algorithm · Euler's Theorem · Side Channel Attacks · Open Key encryption

1 Introduction

Encryption algorithms can be listed as Transposition method, Substitution method, Symmetric Key Cipher method, Open Key Cipher method. These encryption methods were developed on the basis of security, respectively [1]. In symmetric key encryption, there is only one key, and the key of the message sender and the message receiver is the same and unique. This method is fast but insufficient in terms of security. There is a problem if the key is cracked or stolen. In the public key encryption method, the message receiver and sender have their own private key. Also, encrypted text acts as a public key. The most commonly used is the RSA Asymmetric Encryption Algorithm and the Elliptic Curve Encryption Algorithm [2]. The RSA Encryption Algorithm is an algorithm

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based on the product of two very large prime numbers, and it is very difficult to decipher. However, the limited value of the private keys may be a problem in the future due to the increased computer speed. It is also open to timing analysis attacks, power analysis attacks, differential power analysis attacks, electromagnetic attacks, quantum computing power attack, acoustic crypto analysis attack and error analysis attacks, which are side channel attacks. Great progress has been made in this regard. Shamir and his team achieved successful results in 2013, especially in the Shor Algorithm, Acoustic crypto analysis attack on quantum computing power. Bellcore attack is known in 2010 for error analysis attack [3]. In the algorithm developed in this study, first of all, private keys are prevented from taking limited values. The encrypted text cannot be accessed without using the observation key with the private key.

2 Materials and Methods

Let a and b be two natural numbers whose last digits are the same. $a - b \equiv 0(mod10)$ x is the natural number greater than one. Let's find the value of x that provides $a^x - b^x \equiv 0(mod10^2)$

$$a^x - b^x = (a-b).(a^{x-1}b^0 + a^{x-2}b^1 + \ldots + a^0b^{x-1}) \equiv 0(mod10^2) \tag{1}$$

$$a \equiv b(mod10) => a^s \equiv b^s(mod10), \tag{2}$$

where "s" is a natural number. If (2) is written in (1);

$$\begin{aligned} a^x - b^x &= (a-b).(a^{x-1}a^0 + a^{x-2}a^1 + \ldots + a^0a^{x-1}) \equiv 0(mod10^2) \\ a^x - b^x &= (a-b).(a^{x-1} + a^{x-1} + \ldots + a^{x-1}) \equiv 0(mod10^2) \\ a^x - b^x &= (a-b).(x.a^{x-1}) \equiv 0(mod10^2) \end{aligned} \tag{3}$$

The expression must always be at least $x = 10$ to be provided. Let $a^{10} = m$ and $b^{10} = n$. y is a natural number greater than one. Let's find the value y that provides $m^y - n^y \equiv 0(mod10^3)$.

$$m^y - n^y = (m-n).(m^{y-1}n^0 + m^{y-2}n^1 + \ldots\ldots + m^0n^{y-1}) \equiv 0(mod10^3) \tag{4}$$

If (2) is written in (4);

$$\begin{aligned} m^y - n^y &= (m-n).(m^{y-1}m^0 + m^{y-2}m^1 + \ldots + m^0m^{y-1}) \equiv 0(mod10^3) \\ m^y - n^y &= (m-n).(m^{y-1} + m^{y-1} + \ldots + m^{y-1}) \equiv 0(mod10^3) \\ m^y - n^y &= (m-n).(y.m^{y-1}) \equiv 0(mod10^3) \end{aligned} \tag{5}$$

The expression must always be at least $y = 10$ to be provided.

$$(a^{10})^{10} = m, \quad (b^{10})^{10} = n. \tag{6}$$

Let $a^{10^2} = p$, $b^{10^2} = q$, z is a natural number greater than one. Let's find the value z that provides $p^z - q^z \equiv 0(mod10^4)$.

$$p^z - q^z = (p-q).(p^{z-1}q^0 + p^{z-2}q^1 + \ldots + p^0q^{z-1}) \equiv 0(mod10^4) \tag{7}$$

If (2) is written in (7);

$$\begin{aligned} p^z - q^z &= (p-q).(p^{z-1}p^0 + p^{z-2}p^1 + \ldots + p^0p^{z-1}) \equiv 0(mod10^4) \\ p^z - q^z &= (p-q).(p^{z-1} + p^{z-1} + \ldots + p^{z-1}) \equiv 0(mod10^4) \\ p^z - q^z &= (p-q).(z.p^{z-1}) \equiv 0(mod10^4) \end{aligned} \tag{8}$$

The expression must always be at least $z = 10$ to be provided. It becomes

$$a^{10^3} - b^{10^3} \equiv 0(mod 10^4). \tag{9}$$

If the same operations are done infinite times; "n" is a natural number. The equivalence

$$a^{10^{n-1}} - b^{10^{n-1}} \equiv 0(mod 10^n) \tag{10}$$

is obtained. In the equation in (10), if 2 is written in the modular part, a and b numbers can be taken as twin prime numbers. This algorithm is created according to the equivalence established according to the twin prime numbers in (11).

$$a^{2^{n-1}} - b^{2^{n-1}} \equiv 0(mod 2^n) \tag{11}$$

a and b twin prime numbers.

3 Results

3.1 Generating Algorithm Keys

Based on the equivalence found in (11), the numbers "e" and "d" required for the private keys were created first. Euclidean algorithm and modular arithmetic rules were used in creating these numbers.
Let be an "e" number such that $1 < e < 2^n$. Such that;
Let

$$e.a \equiv 1(mod 2^n). \tag{12}$$

Let's multiply each side of the equivalence in (11) by the number e^{2n-1}.

$$e^{2^{n-1}}.a^{2^{n-1}}. - e^{2^{n-1}}.b^{2^{n-1}} \equiv 0(mod 2^n)$$

If $(e.a)^{2n-1} - (b.e)^{2n-1} \equiv 0(mod 2^n)$ is applied in (12);

$$(b.e)^{2n-1} \equiv 1(mod 2^n). \tag{13}$$

Let be an "d" number such that $1 < d < 2^n$.Such that;
Let

$$d.b \equiv 1(mod 2^n). \tag{14}$$

Let's multiply each side of the equivalence in (11) by the number d^{2n-1}.

$$d^{2^{n-1}}.a^{2^{n-1}}. - d^{2^{n-1}}.b^{2^{n-1}} \equiv 0(mod 2^n).$$

If $(d.a)^{2n-1} - (b.d)^{2n-1} \equiv 0(mod 2^n)$ is applied in (14);

$$(a.d)^{2n-1} \equiv 1(mod 2^n) \tag{15}$$

If (13) and (15) are combined;

$$(a.b.e.d)^{2n-1} \equiv 1(mod 2^n). \tag{16}$$

If $(a.b.e.d)^{2^{n-1}} \equiv 1(mod2^n)$ is

$$(a.b.e.d)^{2^{n-1}} = 2^n.k + 1, \tag{17}$$

k is an integer. Accordingly, the keys are formed as follows;
1-) The sender's private key is "$a.b$", in other words the twin prime numbers selected in the equivalence.
2-) The private key of the person receiving the message is "$e.d$", the numbers coming from (12), (14) and found by Euclidean Algorithm.
3-) Observation key: 2^{n-1}
4-) The character of the character to be sent the public key is "m"; $m^{a.b}$

3.2 MMR Character Code Table

KOD	CHAR	KOD	CHAR	KOD	CHAR	KOD	CHAR	KOD	CHAR	KOD	CHAR
1	(nul)	89	+	177	W	279	Ë	373	╣	445	▌
3	(soh)	91	,	179	X	281	Ô	375	║	447	▐
5	(stx)	93	-	181	Y	283	Ó	377	╗	449	▀
7	(etx)	95	.	183	Z	285	ı	379	╝	451	α
9	(eot)	97	/	185	[	287	ƒ	381	┘	453	β
11	(enq)	99	0	187	\	289	≈	383	╛	455	γ
13	(ack)	101	1	189	]	291	...	385	┐	457	π
15	(bel)	103	2	191	^	293	Ê	387	└	459	Σ
17	(bs)	105	3	193	_	295	Δ	389	┴	461	σ
19	(tab)	107	4	195	,	297	Ù	391	┬	463	μ
21	(lf)	109	5	197	a	299	ˆ	393	├	465	τ
23	(vt)	111	6	199	b	301	Ú	395	─	467	Φ
25	(np)	113	7	201	c	303	º	397	┼	469	θ
27	(cr)	115	8	203	d	305	ˇ	399	╞	471	Ω
29	(so)	117	9	205	e	307	0	401	╟	473	δ
31	(si)	119	:	207	f	309	Ö	403	╚	475	∞
33	(dle)	121	;	209	g	311	Ü	405	╔	477	Ø
35	(dc1)	123	<	211	h	313	-	407	╩	479	ε
37	(dc2)	125	=	213	i	315	£	409	╦	481	∩
39	(dc3)	127	>	215	j	317		411	╠	483	≡
41	(dc4)	129	?	217	k	319	Ş	413	═	485	±
43	(nak)	131	@	219	l	321	Ş	415	╬	487	≥
45	(syn)	133	A	221	m	323	.	417	╧	489	≤
47	(etb)	135	B	223	n	325	Ì	419	╨	491	⌠
49	(can)	137	C	225	o	327	Û	421	╤	493	⌡
51	(em)	139	D	227	p	329	·	423	╥	495	÷
53	(eof)	141	E	229	q	331	Ò	425	╙	497	≈
55	(esc)	143	F	231	r	333	_	427	╘	499	o
59	(fs)	145	G	233	s	335		429	╒	501	“
61	(gs)	147	H	235	t	337	ğ	431	∏		-
63	(rs)	149	I	237	u	339	TL	433	╧	503	√
65	(us)	151	J	239	v	341		435	╪	505	∩
67	sp	153	K	241	w	343	“	437	┘	507	²
69	!	155	L	243	x	345	Ω	439	┌	509	■
71	ì	157	M	245	y	347	º	441	█	511	
73	#	159	N	247	z	349	0	443	▄		
75	$	161	O	249	{	351	´				
77	%	163	P	251		353	ª				
79	&	165	Q	253	}	355	░				
81	ë	167	R	255	~	357	▒				
83	(	169	S	257		359	▓				
85	)	171	T	259	Ç	361	│				
87	*	173	U	261	ü	363	┤				
		175	V	263	È	365	╡				
				265	,	367	╢				
				267	‰	369	∏				
				269	‡	371	╕				
				271	Â						
				273	ç						
				275	Í						
				277	Î						

Figure 1: MMR Character Code Table

3.3 Working Principle of Algorithm

The algorithm works as follows.
First, each character of a text to be sent will be converted to MMR Character Codes. Let the code of a character in the text to be sent be the number "m".
1-) The person sending the message will encrypt this code with their own private key "$a.b$ ". The number "$m^{a.b}$ " is an encrypted number and will now serve as a public key.
2-) The recipient of the message will apply their private key to the incoming public key. The number "$(m^{a.b})^{e.d} = m^{a.b.e.d}$" is still encrypted.
3-) The last observation key will be applied to the encrypted number.
It will be "$m^{(a.b.e.d)^{2n-1}} \equiv x(mod2^n)$" is written in place of (17);
It becomes

$$m^{(a.b.e.d)^{2n-1}} = \left(m^{2^n}\right)^k.m \equiv x(mod2^n) \tag{18}$$

Euler Function has

$$\phi(2^n) = 2^{n-1} \tag{19}$$

equation. "m" and "2" are coprime numbers
If

$$m^x \equiv 1(mod2^n), x = \phi(2^n) \tag{20}$$

value provides this equivalence. (18) in (19) and (20) are written in place;

$$\begin{aligned} m^{(a.b.e.d)^{2n-1}} &= \left(m^{2^n}\right)^k.m \equiv x(mod2^n) \\ m^{(a.b.e.d)^{2n-1}} &= \left(m^{2^{n-1}}\right)^{2k}.m \equiv x(mod2^n) \end{aligned}$$

From the expression $m^{(a.b.e.d)^{2n-1}} = (1)^{2.k}.m \equiv x(mod2^n)$ comes the result of

$$"x = m". \tag{21}$$

As can be seen, the MRR Character Code of each character comes out of the algorithm as its own value. So the algorithm works.

3.3.1 Flow Chart

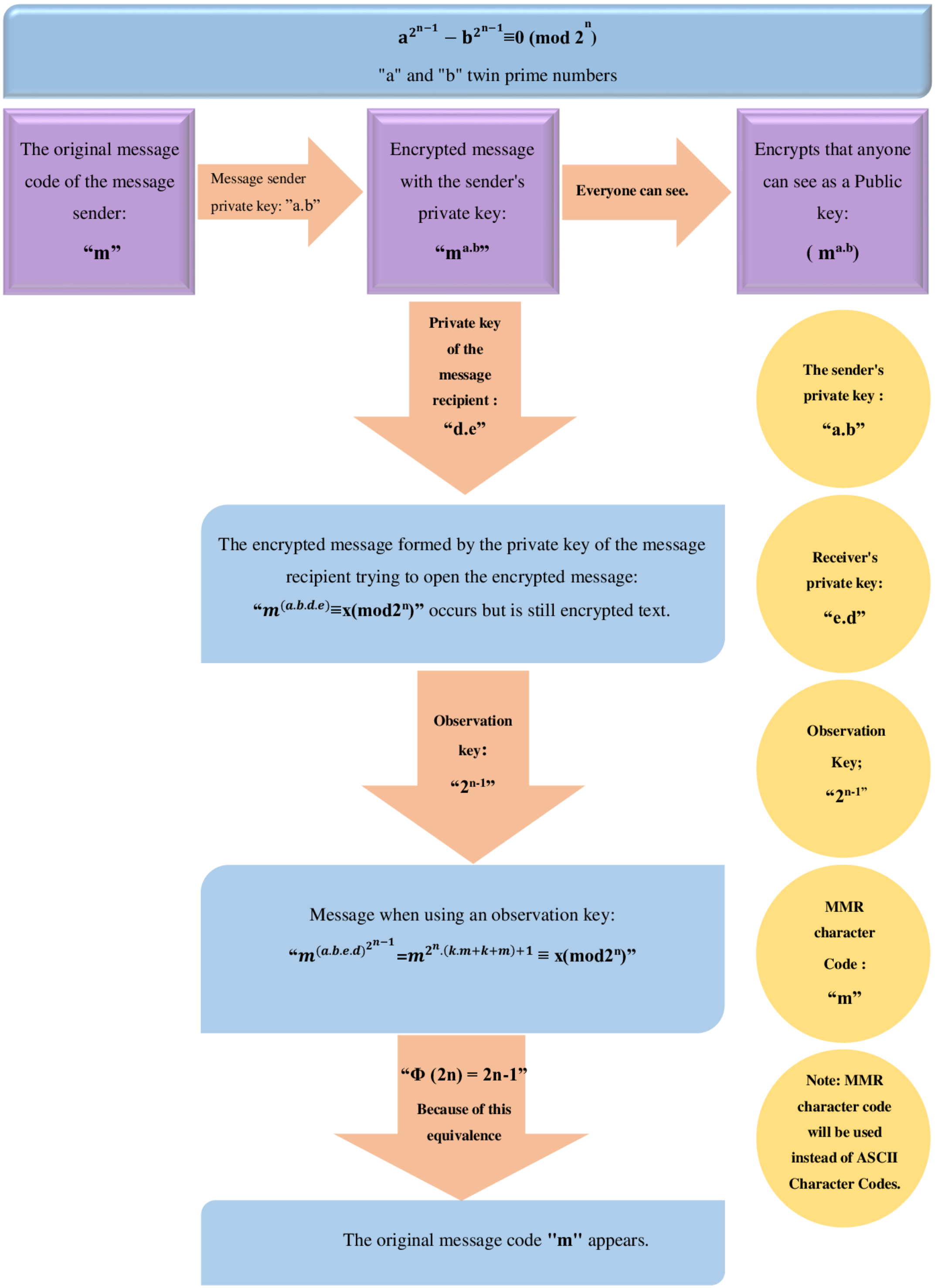

Figure 2: Summary template

4 Conclusion and Discussion

When the equivalence in (11) is generalized, in any number base the force that makes the desired digit of the two digits on the same base the same as the last digit can always be found.
There are two reasons for using twin prime numbers in this algorithm. First, twin prime numbers can take infinite values just like prime numbers. Just as it is difficult to find prime numbers, it is very difficult to find twin prime numbers. The second reason is the number in the modular part is 2. The provision of the Euler Theorem in the algorithm depends on this. Because the Euler Function value of the 2^n number is 2^{n-1} , it must provide the Euler Theorem.
The reason for taking the equivalents used in selecting the numbers "e" and "d" as in (12) and (14); both to make it complex and to use the numbers "a" and "b" in the equivalence found at (11).
The private keys of the sender and the message recipient are taken as "$a.b$" and "$e.d$" because it gives the character code immediately in other crosses. For example; in case of receiving the closed key of the message "$a.e$", "$a.d$", "$b.d$", "$b.e$"; The public key actually becomes the desired character code because m^1 is the remainder. Taking the keys in this way occurred after many attempts.
In fact, it may seem like a disadvantage that the sender's private key is the product of the twin prime numbers and is constant. However, even if private keys are found, it does not make sense without an observation key. Also, the large numerical value of the key is a disadvantage for cracking it.
The reason for performing the operations and creating equivalencies in (13), (15), (17); is for the algorithm to provide. It makes it compulsory to use the observation key. In MMR Encryption Algorithm, the observation key and depending on the observation key, the fact that the sender's private key can get infinite value may cause it to work slowly for now. However, it is thought that when working with quantum computers in the future, it will be very important for the keys to get infinite value.
One of the attacks on the RSA Algorithm is the N Number Attack. As a result, N is the product of two prime numbers. This number can be easily reached with the development of computer technology. For this reason, it is easy to reach private keys easily. So MMR keys must take infinite value.
Acoustic Crypto Analysis Attack is an attack developed based on the sounds of the computer while it is operating. In private keys with limited value, studies were made to reach the values of these keys from the sounds in the study. Positive results have also been obtained. The infinite value of the keys ensures that this attack can also be prevented.
In short, positive results were obtained in the side channel attacks on the RSA Algorithm. MMR Encryption Algorithm is more advantageous in this regard. The fact that the algorithm depends on the twin prime number and the long keys will provide many advantages in the future.
The reason for not using ASCII Character Codes in this algorithm is that Euler Theorem does not provide. For thisreason, MMR Character Code table was created.
In this algorithm, keys may not be made by just changing the number "n". Keys can also be created by changing the twin prime numbers "a" and "b". However, the generated twin prime numbers may not always be easy. It would make more sense to change the software during the update.

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RESULTS ON SET-VALUED PREŠIĆ TYPE MAPPINGS

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ABSTRACT

Fixed point theory is one of the most powerful and fruitful tools in nonlinear analysis. The Banach contraction principle [1] is the simplest and one of the most versatile elementary results in fixed point theory. Over the years, various extensions and generalizations of this principle have appeared in the literature. Moreover, among the different generalization of Banach contraction principle, Prešić [2] in 1965 gave a contractive condition on finite product of metric spaces and proved a fixed point theorem. Also Ćirić and Prešić [3] and Abbas et al. [4] extended and generalized these results. In the present study, we prove a fixed point theorem for set-valued Prešić type almost contractive mapping. After we give a fixed point theorem Prešić type almost F-contractive mapping in metric space.

***K*eywords** Fixed point · metric spaces · Prešic type contraction

1 Introduction and preliminaries

Considering the convergence of certain sequences S. B. Prešić [2] generalized Banach contraction principle as follows:

Theorem 1.1. *Let (X,d) be a complete metric space, k a positive integer and $T : X^k \to X$ a mapping satisfying the following contractive type condition*

$$d(T(x_1,x_2,...,x_k),T(x_2,x_3,...,x_{k+1})) \leqslant q_1 d(x_1,x_2) + q_2 d(x_2,x_3) + ... + q_k d(x_k,x_{k+1}), \quad (1)$$

for every $x_1,x_2,...,x_{k+1}$ in X, where $q_1,q_2,...,q_k$ are non negative constants such that $q_1+q_2+...+q_k < 1$. Then there exist a unique point x in X such that $T(x,x,...,x) = x$. Moreover, if $x_1,x_2,...,x_k$, are arbitrary points in X and for $n \in N$,

$$x_{n+k} = T(x_n,x_{n+1},...,x_{n+k-1}), \quad (n = 1,2,...)$$

then the sequence $\{x_n\}_{n=1}^{\infty}$ is convergent and

$$\lim x_n = T(\lim x_n, \lim x_n, ..., \lim x_n).$$

Remark that condition (1) in the case $k = 1$ reduces to the well-known Banach contraction mapping principle. So, Theorem 1.1 is a generalization of the Banach fixed point theorem.

Ćirić and Prešić [3] generalized the above result as follows:

Theorem 1.2. *Let (X, d) be a complete metric space, k a positive integer and $T : X^k \to X$ a mapping satisfying the following contractive type condition*

$$d(T(x_1, x_2, ..., x_k), T(x_2, x_3, ..., x_{k+1})) \leqslant \lambda \max_{1 \leqslant i \leqslant k} \{d(x_i, x_{i+1})\}, \tag{2}$$

where $\lambda \in (0, 1)$ is constant and $x_1, x_2, ..., x_{k+1}$ are arbitrary elements in X. Then there exist a point x in X such that $T(x, x, ..., x) = x$. Moreover, if $x_1, x_2, ..., x_k$, are arbitrary points in X and for $n \in N$,

$$x_{n+k} = T(x_n, x_{n+1}, ..., x_{n+k-1}), \quad (n = 1, 2, ...)$$

then the sequence $\{x_n\}_{n=1}^{\infty}$ is convergent and

$$\lim x_n = T(\lim x_n, \lim x_n, ..., \lim x_n).$$

If in addition we suppose that on a diagonal $\triangle \subset X^k$

$$d(T(u, u, ..., u), T(v, v, ..., v)) < d(u, v) \tag{3}$$

holds for all $u, v \in X$, with $u \neq v$, then x is the unique point in X with $T(x, x, ..., x) = x$.

Recently, Nadler [6], introduced the notion of multi-valued contraction mapping and proved well known Banach contraction principle. Our results is related to mappings $E : X \to CB(X)$. Then, head the pertaining lemma:

Lemma 1.1. *Let K and L be nonempty closed and bounded subsets of a metric space. Therefore, For any $k \in K$,*

$$D(k, L) \leqslant H(K, L).$$

Lemma 1.2. *[6] Let K and L be nonempty closed and bounded subsets of a partial metric space and $h > 1$. Then, for all $k \in K$, there exists $l \in L$ such that*

$$p(k, l) \leqslant hH_p(K, L).$$

Berinde [7, 8, 9] defined almost contraction (or (δ, L)-weak contraction) mappings in a metric space. In the same paper, Berinde [10] introduced the concepts of set-valued almost contraction (the original name was set-valued (δ, L)-weak contraction) and proved the following nice fixed point theorem:

Theorem 1.3. *Let (X, d) be a complete metric spaces, $M : X \to CB(X)$ be a set-valued almost contraction, that is, there exist two constants $\delta \in (0, 1)$ and $\lambda \geqslant 0$ such that*

$$H(M \nvdash, M \mathcal{A}) \leqslant \delta d(\nvdash, \mathcal{A}) + LD(\mathcal{A}, M \nvdash)$$

for all $\nvdash, \mathcal{A} \in X$. Then M is an set-valued almost contraction operator.

Wardowski [11] introduced concept of F-contractive mapping on metric space and proved a fixed point theorem for such a map on complete metric space. Let $\mathcal{F}$ be the set of all functions $F : \mathbb{R}_+ \to \mathbb{R}$ satisfying the following conditions:

(F1) F is strictly increasing. That is, $\beta < \gamma \Rightarrow F(\beta) < F(\gamma)$ for all $\beta, \gamma \in \mathbb{R}_+$

(F2) For every sequence $\{\beta_n\}_{n\in\mathbb{N}}$ in $\mathbb{R}_+$ we have $\lim_{n\to\infty} \beta_n = 0$ if and only if $\lim_{n\to\infty} F(\beta_n) = -\infty$

(F3) There exists a number $z \in (0,1)$ such that $\lim_{\beta\to 0^+} \beta^z F(\beta) = 0$.

2 Main Results

In this section, we introduce a fixed point theorem for set-valued Prešić type almost contractive mapping. Then we introduce fixed point theorems Prešić type almost F-contractive mapping.

Definition 2.1. *Let (X,d) be a metric space. We say that $M : X^r \to CB(X)$ is a set-valued Prešić type almost contraction mapping, where r is a positive integer, there exist $\delta \in (0,1)$ and $\lambda \geqslant 0$ such that*

$$H(M(\nvdash_1, \nvdash_2, ..., \nvdash_r), M(\nvdash_2, \nvdash_3, ..., \nvdash_{r+1})) \leqslant \delta \max_{1\leqslant t\leqslant r}\{d(\nvdash_t, \nvdash_{t+1})\} + \lambda \min_{1\leqslant t\leqslant r}\{D(\nvdash_{t+1}, M(\nvdash_1, \nvdash_2, ..., \nvdash_r))\}, \quad (4)$$

for all $(\nvdash_1, \nvdash_2, \cdots, \nvdash_{r+1}) \in X^{r+1}$.

Theorem 2.1. *Let (X,d) be a complete metric spaces, $M : X^r \to CB(X)$ a set-valued Prešić type almost contraction mapping, where r is a positive integer. There exists the sequence $(\nvdash_{n+r})$ defined by*

$$\nvdash_{n+r} \in M(\nvdash_n, \nvdash_{n+1}, ..., \nvdash_{n+r-1}), \quad (n = 1, 2, ...) \quad (5)$$

such that $\nvdash_{n+r} \in M(\nvdash_{n+r}, \nvdash_{n+r}, ..., \nvdash_{n+r})$, for any arbitrary points $\nvdash_1, \nvdash_2, ..., \nvdash_r \in X$. Then M has a fixed point.

Proof:

Let $\beta > 1$, $\nvdash_r \in X$ and $\nvdash_{r+1} \in M(\nvdash_1, \nvdash_2, ..., \nvdash_r)$. If there exists $r \in \mathbb{N}$ for which

$H(M(\nvdash_1, \nvdash_2, ..., \nvdash_r), M(\nvdash_2, \nvdash_3, ..., \nvdash_{r+1})) = 0$ then $\nvdash_{r+1} \in M(\nvdash_{r+1}, \nvdash_{r+1}, \ldots, \nvdash_{r+1})$ that is, $\nvdash_{r+1}$ is a fixed point of M. Let $H(M(\nvdash_1, \nvdash_2, ..., \nvdash_r), M(\nvdash_2, \nvdash_3, ..., \nvdash_{r+1})) \neq 0$. Using Lemma 1.2 there exists $\nvdash_{r+2} \in M(\nvdash_2, \nvdash_3, ..., \nvdash_{r+1})$ such that

$$d(\nvdash_{r+1}, \nvdash_{r+2}) \leqslant \beta H(M(\nvdash_1, \nvdash_2, ..., \nvdash_r), M(\nvdash_2, \nvdash_3, ..., \nvdash_{r+1})).$$

By (4), we obtain

$$d(\nvdash_{r+1}, \nvdash_{r+2}) \leqslant \beta[\delta \max_{1\leqslant t\leqslant r}\{d(\nvdash_t, \nvdash_{t+1})\} + \lambda \min_{1\leqslant t\leqslant r}\{D(\nvdash_{t+1}, M(\nvdash_1, \nvdash_2, ..., \nvdash_r))\}].$$

Using the fact that $\nvdash_{r+1} \in M(\nvdash_1, \nvdash_2, ..., \nvdash_r)$ implies

$$\min_{1\leqslant t\leqslant r}\{D(\nvdash_{t+1}, M(\nvdash_1, \nvdash_2, ..., \nvdash_r))\} = 0$$

Then we obtain

$$\begin{aligned} d(\nvdash_{r+1}, \nvdash_{r+2}) &\leqslant \beta[\delta \max_{1\leqslant t\leqslant r}\{d(\nvdash_t, \nvdash_{t+1})\} + \lambda \min_{1\leqslant t\leqslant r}\{D(\nvdash_{t+1}, M(\nvdash_1, \nvdash_2, ..., \nvdash_r))\}] \\ &= \beta\delta \max_{1\leqslant t\leqslant r}\{d(\nvdash_t, \nvdash_{t+1})\}. \end{aligned}$$

We take $\beta > 1$ such that $\gamma = \beta\delta < 1$ and so,

$$d(\nvdash_{r+1}, \nvdash_{r+2}) \leqslant \gamma \max_{1\leqslant t\leqslant r}\{d(\nvdash_t, \nvdash_{t+1})\}. \tag{6}$$

Similarly, there exists $\nvdash_{r+2}\in M(\nvdash_2, \nvdash_3, ..., \nvdash_{r+1})$ such that

$$d(\nvdash_{r+2}, \nvdash_{r+3}) \leqslant \beta[\delta \max_{2\leqslant t\leqslant r+1}\{d(\nvdash_t, \nvdash_{t+1})\} + \lambda \min_{2\leqslant t\leqslant r+1}\{D(\nvdash_{t+1}, M(\nvdash_2, \nvdash_3, ..., \nvdash_{r+1}))\}].$$

Using the fact that $\nvdash_{r+2}\in M(\nvdash_2, \nvdash_3, ..., \nvdash_{r+1})$ implies

$$\min_{2\leqslant t\leqslant r+1}\{D(\nvdash_{t+1}, M(\nvdash_2, \nvdash_3, ..., \nvdash_{r+1}))\} = 0$$

Then we obtain

$$\begin{aligned} d(\nvdash_{r+2}, \nvdash_{r+3}) &\leqslant \beta[\delta \max_{2\leqslant t\leqslant r+1}\{d(\nvdash_t, \nvdash_{t+1})\} + \lambda \min_{2\leqslant t\leqslant r+1}\{D(\nvdash_{t+1}, M(\nvdash_2, \nvdash_3, ..., \nvdash_{r+1}))\}] \\ &= \gamma \max_{2\leqslant t\leqslant r+1}\{d(\nvdash_t, \nvdash_{t+1})\}. \end{aligned}$$

Continuous this condition, we obtain $\nvdash_{n+r}\in M(\nvdash_n, \nvdash_{n+1}, ..., \nvdash_{n+r-1})$ such that

$$d(\nvdash_{n+r}, \nvdash_{n+r+1}) \leqslant \beta[\delta \max_{n\leqslant t\leqslant n+r-1}\{d(\nvdash_t, \nvdash_{t+1})\} + \lambda \min_{n\leqslant t\leqslant n+r-1}\{D(\nvdash_{t+1}, M(\nvdash_n, \nvdash_{n+1}, ..., \nvdash_{n+r-1}))\}].$$

Using the fact that $\nvdash_{n+r}\in M(\nvdash_n, \nvdash_{n+1}, ..., \nvdash_{n+r-1})$ implies

$$\min_{n\leqslant t\leqslant n+r-1}\{D(\nvdash_{t+1}, M(\nvdash_n, \nvdash_{n+1}, ..., \nvdash_{n+r-1}))\} = 0$$

Then we obtain

$$\begin{aligned} d(\nvdash_{n+r}, \nvdash_{n+r+1}) &\leqslant \beta[\delta \max_{n\leqslant t\leqslant n+r-1}\{d(\nvdash_t, \nvdash_{t+1})\} + \lambda \min_{n\leqslant t\leqslant n+r-1}\{D(\nvdash_{t+1}, M(\nvdash_n, \nvdash_{n+1}, ..., \nvdash_{n+r-1}))\}] \\ &= \gamma \max_{n\leqslant t\leqslant n+r-1}\{d(\nvdash_t, \nvdash_{t+1})\}. \end{aligned}$$

Let $w_n = d(\nvdash_n, \nvdash_{n+1})$ for all $n \in \mathbb{N}$ and $A = \max\{\frac{w_1}{s}, \frac{w_2}{s^2}, ..., \frac{w_r}{s^r}\}$ where $s = \gamma^{\frac{1}{r}}$. We shall prove by induction that for each $n \in \mathbb{N}$:

$$w_n \leqslant As^n. \tag{7}$$

By the definition of A it is clear that (7) is true for $n = 1, 2, ..., r$. Now let the following r inequalities:

$$w_n \leqslant As^n, \quad w_{n+1} \leqslant As^{n+1}, ..., w_{n+r-1} \leqslant As^{n+r-1}$$

be the induction hypothesis. Then we obtain:

$$\begin{aligned} w_{n+r} =& d(\nvdash_{n+r}, \nvdash_{n+r+1}) \\ \leqslant& \gamma \max_{n \leqslant t \leqslant n+r-1} \{d(\nvdash_t, \nvdash_{t+1})\} \\ =& \gamma \max\{w_n, w_{n+1}, ..., w_{n+r-1}\} \\ \leqslant& \gamma \max\{As^n, As^{n+1}, ..., As^{n+r-1}\} \\ =& \gamma As^n \\ =& As^{n+r} \end{aligned}$$

Then, inductive proof of (7) is complete. Using (7) for any $n, m \in \mathbb{N}$ we have the following:

$$\begin{aligned} d(\nvdash_n, \nvdash_m) \leqslant& d(\nvdash_n, \nvdash_{n+1}) + d(\nvdash_{n+1}, \nvdash_{n+2}) + \cdots + d(\nvdash_{m-1}, \nvdash_m) \\ \leqslant& w_n + w_{n+1} + \cdots + w_{m-1} \\ \leqslant& As^n + As^{n+1} + ... + As^{m-1} \\ \leqslant& As^n(1 + s + s^2 + ...) = \frac{As^n}{1-s}. \end{aligned}$$

As $s = \gamma^{\frac{1}{r}} < 1$, thus, $\frac{As^n}{1-s} \to 0$ as $n \to \infty$. Hence, $(\nvdash_n)$ is a Cauchy sequence. Since X is a complete space, there exists $a \in X$ such that $\lim_{n\to\infty} \nvdash_n = a$.

Now let's show a is a fixed point of M.

$$\begin{aligned} D(a, M(a, a, ..., a)) \leqslant& d(a, \nvdash_{n+k}) + D(\nvdash_{n+k}, M(a, a, ..., a)) \\ \leqslant& d(a, \nvdash_{n+k}) + H(M(\nvdash_n, \nvdash_{n+1}, ..., \nvdash_{n+r-1}), M(a, a, ..., a)) \\ \leqslant& d(a, \nvdash_{n+k}) + H(M(\nvdash_n, \nvdash_{n+1}, ..., \nvdash_{n+r-1}), M(\nvdash_{n+1}, \nvdash_{n+2}, ..., \nvdash_{n+r-1}, a)) \\ &+ H(M(\nvdash_{n+1}, \nvdash_{n+2}, ..., \nvdash_{n+r-1}, a), M(\nvdash_{n+2}, \nvdash_{n+3}, ..., \nvdash_{n+r-1}, a, a)) \\ &+ ... + H(M(\nvdash_{n+r-1}, a, ..., a), M(a, a, ..., a)) \end{aligned}$$

and by (4), we obtain

$$\begin{aligned} D(a, M(a, a, ..., a)) \leqslant& d(a, \nvdash_{n+k}) + \max\{d(\nvdash_n, \nvdash_{n+1}), d(\nvdash_{n+1}, \nvdash_{n+2}), ..., d(\nvdash_{n+r-1}, a)\} + \\ & \lambda \min\{D(\nvdash_{n+1}, M(\nvdash_n, \nvdash_{n+1}, ..., \nvdash_{n+r-1})), D(\nvdash_{n+2}, M(\nvdash_n, \nvdash_{n+1}, ..., \nvdash_{n+r-1})), \\ & ..., D(a, M(\nvdash_n, \nvdash_{n+1}, ..., \nvdash_{n+r-1}))\} \\ & + \max\{d(\nvdash_{n+1}, \nvdash_{n+2}), d(\nvdash_{n+2}, \nvdash_{n+3}), ..., d(\nvdash_{n+r-1}, a), d(a, a)\} + \\ & \lambda \min\{D(\nvdash_{n+2}, M(\nvdash_{n+1}, \nvdash_{n+2}, ..., \nvdash_{n+r-1}, a)), \\ & D(\nvdash_{n+3}, M(\nvdash_{n+1}, \nvdash_{n+2}, ..., \nvdash_{n+r-1}, a)), ..., D(a, M(\nvdash_{n+1}, \nvdash_{n+2}, ..., \nvdash_{n+r-1}, a))\} \\ & + ... + \max\{d(\nvdash_{n+r-1}, a), d(a, a)\} + \lambda \min\{D(a, M(\nvdash_{n+1}, a, ..., a, a))\}. \end{aligned}$$

Letting $n \to \infty$ we get $D(a, M(a, a, ..., a)) = 0$, that is, a is a fixed point of M. Therefore this completes the proof.

Definition 2.2. *Let (X, d) be a metric space. We say that $M : X^r \to X$ is a Prešić type almost F-contraction mapping, where r is a positive integer, if $F \in \mathcal{F}$, and there exist $\tau > 0$ and $\lambda \geqslant 0$*

such that

$$\tau + F(d(M(\nvdash_1, \nvdash_2, ..., \nvdash_r), M(\nvdash_2, \nvdash_3, ..., \nvdash_{r+1}))) \\ \leqslant F(\max_{1\leqslant t\leqslant r}\{d(\nvdash_t, \nvdash_{t+1})\} + \lambda \min_{1\leqslant t\leqslant r}\{d(\nvdash_{t+1}, M(\nvdash_1, \nvdash_2, ..., \nvdash_r))\}), \tag{8}$$

for all $(\nvdash_1, \nvdash_2, \cdots, \nvdash_{r+1}) \in X^{r+1}$.

Theorem 2.2. *Let* (X, d) *be a complete metric spaces,* $M : X^r \to X$ *a Prešić type almost* F*-contraction mapping, where* r *is a positive integer. There exists the sequence* $(\nvdash_{n+r})$ *defined by*

$$\nvdash_{n+r}= M(\nvdash_n, \nvdash_{n+1}, ..., \nvdash_{n+r-1}), \quad (n = 1, 2, ...) \tag{9}$$

such that $\nvdash_{n+r}= M(\nvdash_{n+r}, \nvdash_{n+r}, ..., \nvdash_{n+r})$*, for any arbitrary points* $\nvdash_1, \nvdash_2, ..., \nvdash_r\in X$*. Then* M *has a fixed point.*

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NEW RESULTS ON EXPANSIVE MAPPINGS

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ABSTRACT

Recently, Jleli and Samet [1] introduced a new concept of θ-contraction and proved a fixed point theorem which generalizes the Banach contraction principle. Following this direction of research, in this study, we present some new fixed point results for θ-expansive mappings, especially on a complete metric space.

***K*eywords** Fixed point · Metric spaces · Expansion mappings

1 Introduction

In 1984, Wang et al.[2] introduced the concept of expanding mappings and proved some fixed point theorems in complete metric spaces. Rhoades [3] and Taniguchi [4] generalized the results of Wang for pair of mappings. Thereafter, several authors obtained many fixed point theorems for expansive mappings. For more details see [5, 6, 7].

Theorem 1.1. *[2] Let (Y, d) be a complete metric space and A a self mapping on Y. If A is surjective and satisfies*

$$d(A\mp, A\not{a}) \geqslant qd(\mp, \not{a}) \tag{1}$$

for all $\mp, \not{a} \in Y$, with $q > 1$ then A has a unique fixed point in Y.

Jleli and Samet [1] introduced the family of all functions, $\theta : (0, \infty) \to (1, \infty)$ supplying the following particulars by Θ:

- (Θ_1) θ is nondecreasing;
- (Θ_2) For each sequence $\{s_n\} \subset (0, \infty)$, $\lim_{n\to\infty} \theta(s_n) = 1$ if and only if $\lim_{n\to\infty} s_n = 0^+$;
- (Θ_3) There exists $m \in (0, 1)$ and $z \in (0, \infty]$ such that $\lim_{s\to 0^+} \frac{\theta(s)-1}{s^m} = z$.

Lemma 1.1. *If $A : Y \to Y$ is surjective, then exists a mapping $A^* : Y \to Y$, (Y, d) such that $A \circ A^*$ is the identity mapping on Y.*

2 Main Results

In this section, we introduce the concept of θ-expansive mappings in metric spaces and prove fixed point theorems for such mappings.

Theorem 2.1. *Let (Y, d) be a complete metric space. We say that the $A : Y \to Y$ is surjective θ-expanding mapping if there exists a constant $\eta > 1$ and $\theta \in \Theta$ such that*

$$\theta(d(A\digamma, A\mathscr{A})) \geqslant [\theta(d(\digamma, \mathscr{A}))]^{\eta} \tag{2}$$

for all $\digamma, \mathscr{A} \in Y$, then A has a unique fixed point in Y.

Proof:

Let $\digamma_0$ be an arbitrary point of Y and since A is surjective there exists $\digamma_1 \in Y$ such that $\digamma_0 = A\digamma_1$. In general, having chosen $\digamma_n \in Y$ we choose $\digamma_{n+1} \in Y$ so that $\digamma_n = A\digamma_{n+1}$ for all $n = 0, 1, 2, \ldots$. Given that there exists $n \in \mathbb{N}$ for which $\digamma_n = \digamma_{n+1}$, we obtain $\digamma_{n+1} = \digamma_n = A\digamma_{n+1}$. Then $\digamma_{n+1}$ is a fixed point of A. Now assume that $n \in \mathbb{N}$ for which $\digamma_n \neq \digamma_{n+1}$. Then from (2) for $\digamma = \digamma_n$ and $\mathscr{A} = \digamma_{n+1}$ we obtain

$$\theta(d(\digamma_{n-1}, \digamma_n)) = \theta(d(A\digamma_n, A\digamma_{n+1})) \geqslant [\theta(d(\digamma_n, \digamma_{n+1}))]^{\eta} \tag{3}$$

for all $n \geqslant 1$. Letting $n \to \infty$, and so

$$\theta(d(\digamma_{n-1}, \digamma_n)) \geqslant [\theta(d(\digamma_n, \digamma_{n+1}))]^{\eta}. \tag{4}$$

We given that $\eta > 1$ and let $s = \frac{1}{\eta}$. Using (4) the following takes for every $n \geqslant 1$:

$$\begin{aligned}
\theta(d(\digamma_n, \digamma_{n+1})) &\leqslant [\theta(d(\digamma_{n-1}, \digamma_n))]^{s} \\
&\leqslant [\theta(d(\digamma_{n-2}, \digamma_{n-1}))]^{s^2} \\
&\;\;\vdots \\
&\leqslant [\theta(d(\digamma_0, \digamma_1))]^{s^n}.
\end{aligned} \tag{5}$$

Letting $n \to \infty$, we have

$$\lim_{n\to\infty} \theta(d(\digamma_n, \digamma_{n+1})) = 1, \tag{6}$$

which implies from (Θ_2) that

$$\lim_{n\to\infty} d(\digamma_n, \digamma_{n+1}) = 0.$$

Let

$$k_n = d(\digamma_n, \digamma_{n+1}),$$

for all $n \in \mathbb{N}$.

From condition (Θ_3), there exists $c \in (0, 1)$ and $G \in (0, \infty]$ such that

$$\lim_{n\to\infty} \frac{\theta(k_n) - 1}{(k_n)^c} = G. \tag{7}$$

Assume that $G < \infty$. Then, let $J = \frac{G}{2} > 0$. Than the description of the limit, there exists $n_0 \in \mathbb{N}$ such that

$$\left| \frac{\theta(k_n) - 1}{(k_n)^c} - G \right| \leqslant J, \text{ for all } n \geqslant n_0.$$

This refers that

$$\frac{\theta(k_n) - 1}{(k_n)^c} \geqslant G - J = J, \text{ for all } n \geqslant n_0.$$

Hence

$$n(k_n)^c \leqslant Hn[\theta(k_n) - 1],$$

for all $n \geqslant n_0$ here $H = \frac{1}{J}$. Now assume that $G = \infty$ and $J > 0$ be an arbitrary number. Than the description of the limit, there exists $n_0 \in \mathbb{N}$ such that

$$\frac{\theta(k_n) - 1}{(k_n)^c} \geqslant J,$$

for all $n \geqslant n_0$. This refers that

$$n(k_n)^c \leqslant Hn[\theta(k_n) - 1],$$

for all $n \geqslant n_0$, where $H = \frac{1}{J}$. Therefore, in two conditions, there exist $H > 0$ and $n_0 \in \mathbb{N}$ such that

$$n(k_n)^c \leqslant Hn[\theta(k_n) - 1],$$

for all $n \geqslant n_0$. Using (5), we get

$$n(k_n)^c \leqslant Hn([\theta(k_0)]^{s^n} - 1), \tag{8}$$

for all $n \geqslant n_0$. Letting $n \to \infty$ in (8) we get

$$\lim_{n \to \infty} n(k_n)^c = 0.$$

Therefore, there exists $n_1 \in \mathbb{N}$ such that

$$k_n \leqslant \frac{1}{n^{\frac{1}{c}}}, \text{ for all } n \geqslant n_1. \tag{9}$$

For any $n, m \in \mathbb{N}$ with $m > n \geqslant n_0$we have the following argument:

$$\begin{aligned} d(\not\top_n, \not\top_m) \leqslant & d(\not\top_n, \not\top_{n+1}) + d(\not\top_{n+1}, \not\top_{n+2}) + \cdots + d(\not\top_{m-1}, \not\top_m) \\ \leqslant & k_n + k_{n+1} + \cdots + k_{m-1} \\ \leqslant & \sum_{i=n}^{m-1} \frac{1}{i^{\frac{1}{c}}} \end{aligned}$$

where we take the limit for $n \to \infty$, this show that $d(\not\top_n, \not\top_m) \to 0$. Hence the sequence $\{\not\top_n\}$ is Cauchy. Therefore (Y, d) is complete, these show that $\{\not\top_n\}$ is a Cauchy sequence converging to some point $a \in Y$. That is

$$\lim_{n\to\infty} \text{Ŧ}_n = a. \tag{10}$$

Now let's show a is a fixed point of A. Since A is surjective, there exists $b \in Y$ such that $a = Ab$. On account of (2) for $\text{Ŧ} = \text{Ŧ}_n$, $\text{ѧ} = b$ with $n \geqslant 1$ we obtain

$$\theta(d(\text{Ŧ}_n, a)) = \theta(d(A\text{Ŧ}_{n+1}, Ab)) \geqslant [\theta(d(\text{Ŧ}_{n+1}, b))]^\eta \geqslant \theta(d(\text{Ŧ}_{n+1}, b)) \tag{11}$$

which implies from (Θ_1) that

$$d(\text{Ŧ}_n, a) \geqslant d(\text{Ŧ}_{n+1}, b) \tag{12}$$

where we take the limit for $n \to \infty$, this show that $a = Ab = Aa$. Therefore $a = Aa$ which deduces the proof.

Also if A is continuous we have

$$a = \lim_{n\to\infty} \text{Ŧ}_n = A(\lim_{n\to\infty} \text{Ŧ}_{n+1}) = Aa.$$

Now let's show that A is the only fixed point. Given that there exists $a, p \in Y$ such that $Aa = a$, $Ap = p$. Then we obtain

$$\begin{aligned}\theta(d(a,p)) =& \theta(d(Aa, Ap))\\ \geqslant& [\theta(d(a,p)]^\eta\end{aligned}$$

which is a contraction. Therefore a is the unique fixed point of A.

Let $\theta : (0,\infty) \to (1,\infty)$ be given by the formulae

$$\theta(w) = 2 - \frac{2}{\pi}\arctan\left(\frac{1}{w^\xi}\right), \quad \xi \in (0,1), \quad w > 0.$$

Corollary 2.1. *Let (Y, d) be a complete metric space. We say that the $A : Y \to Y$ is surjective θ-expanding mapping if there exists $\xi \in (0,1)$, a constant $\eta > 1$ and $\theta \in \Theta$ such that*

$$2 - \frac{2}{\pi}\arctan\left(\frac{1}{d(A\text{Ŧ}, A\text{ѧ})^\xi}\right) \geqslant \left[2 - \frac{2}{\pi}\arctan\left(\frac{1}{d(\text{Ŧ}, \text{ѧ})^\xi}\right)\right]^\eta \tag{13}$$

for all $\text{Ŧ}, \text{ѧ} \in Y$, then A has a fixed point.

Corollary 2.2. *Let (Y, d) be a complete metric space and $A : Y \to Y$ be a surjection. Assume that $\theta \in \Theta$ and a constant $\eta > 1$ such that*

$$\theta(d(A\text{Ŧ}, A\text{ѧ})) \geqslant [\theta(d(\text{Ŧ}, \text{ѧ}))]^\eta,$$

for all $\text{Ŧ}, \text{ѧ} \in Y$. Now let we show that expensive mapping of Corollary 2.2. If A is expensive mapping there exists $\eta > 1$ such that

$$d(A\text{Ŧ}, A\text{ѧ}) \geqslant \eta d(\text{Ŧ}, \text{ѧ}), \quad \forall \text{Ŧ}, \text{ѧ} \in Y$$

then we have

$$e^{d(A\text{Ŧ}, A\text{ѧ})} \geqslant [e^{d(\text{Ŧ}, \text{ѧ})}]^\eta.$$

Therefore the function $\theta : (0,\infty) \to (1,\infty)$ defined by $\theta(k) = e^k$ belong to Θ.

3 Conclusion

The study of expansive mappings is a very interesting research area in fixed point theory. Wang, et al.[2] proved some fixed point theorems for expansion mappings, which correspond to some contractive mappings in metric spaces. Rhoades [3] and Taniguchi [4] generalized the results of Wang for pair of mappings. In the following, we introduce a new approach to expansion mappings in fixed point theory. We introduce the concept of θ-expansive mappings in metric spaces and prove fixed point theorem and results for such mappings.

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Some Constant Angle Spacelike Surfaces Construct on a Curve in De-Sitter $3-$Space

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Abstract

In this study, the surfaces constructed on a curve have been examined in the de-Sitter 3- space. Spacelike ones of these types of surfaces are discussed and their properties are given under the condition of being a fixed angle surface.

Keywords Normal Surface · Binormal Surface · Darboux Surface · Ruled Surface · Spacelike Surface · de-Sitter Space

1 Introduction

Constant angle surfaces were studied in three dimensional Euclidean space E^3 by Munteanu and Nistor and entire class of constant angle surfaces in E^3 were obtained [1] and in E^n were studied by Scala and Hernandez [2],[3]. Germelli and Scala applied constant angle surfaces to liquid layers and liquid crystal theory [4].

S^2 and H^2 are spherical and hyperbolic planes respectively, constant angle surfaces were studied in multiplication spaces such as $S^2 \times R, H^2 \times R$ and Nil_3 [5],[6],[7].

Lopez and Munteanu studied and classified such surfaces in Minkowski space E_1^3. In addition in these studies, they delivered required and sufficient condition that a extensile tangential surface to be a constant angle surface [8].

Constant angle surfaces similar to those in Lorentz space of helicoid surfaces, which are well known in the Euclidean and Lorentz spaces, were also studied in hyperbolic $3-$space and de Sitter $3-$space [9],[10],[11],[12],[13]. The constant angle conditions of a surface in hyperbolic and de Sitter spaces were determined and the invariants of these surfaces were investigated.

Constant angle tanget surfaces were given as typical examples of constant angle surfaces in H^3 and S_1^3 [9]. Also, a ruled surface was formed by moving a line along a curve in hyperbolic $3-$space [9].

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Such constant angle surfaces construct on the curves were studied by Nistor, in three dimensional Euclidean space [10]. Again in Minkowski space E_1^3, Karakus studied under constant angle normal, binormal, rectifiying developable, darboux developable and conic surfaces [11].

In this study, some special spacelike surfaces in de-Sitter 3- space are discussed. Spacelike ones of normal, binormal and Darboux surfaces were investigated in De-Sitter 3-space. Due to the variation in the causal character of a vector, a curve, and a surface in this space, this space has a rich structure. This rich structure gives the opportunity to diversify the surfaces we consider. The situation of normal, binormal and Darboux surfaces that are spacelike are constant angle surfaces were investigated. Since these surfaces are built on a curve, the causal character of the curve is also very important for these types of surfaces. The concept of constant angle surface, similar to the Helisoid surfaces in Lorentz space, which has many applications in the technique, occupies a huge place in the establishment of these spacelike surfaces in the de-Sitter space. Also, since these special surfaces will be constructed as constant angle surfaces, attention should be paid to the causal character of the constant vector area of the de-Sitter space and the causal character of the fixed angle when creating these surfaces. In this study, with all these important situations, using the rich structure of de-Sitter space, the spacelike ones of normal, binormal and Darboux surfaces will be introduced. In recent years, similar types of surface types well known in Euclidean space are also being investigated in Sitter space. De-Sitter space is a model for physical events and many physical events can be interpreted in this space. Since the surface types in these spaces will guide the areas related to our daily life, this type of surface is of great importance. It is possible to see this from the structures used in the history of architecture. The importance of such surfaces in this space is clear and this study will contribute to geometry.

Let us now consider the differential geometry of curves and surfaces in this space before investigating these types of surfaces in de-Sitter $3-$space.

2 Preliminaries

In this section, Differential geometry of curves and surfaces are summarized in de Sitter space S_1^3. Let $\mathbb{R}^4$ be 4-dimensional vector space equipped with the scalar product $\langle , \rangle$ which is defined by

$$\langle x, y\rangle = -x_1y_1 + x_2y_2 + x_3y_3 + x_4y_4 \, .$$

$\mathbb{R}_1^4$ is 4-dimensional vector space equipped with the scalar product $\langle , \rangle$, than $\mathbb{R}_1^4$ is called Lorentzian 4- space or 4-dimensional Minkowski space. The Lorentzian norm (length) of x is defined to be

$$\|x\| = |\langle x, x\rangle|^{\frac{1}{2}} \, .$$

If $(x_0^i, x_1^i, x_2^i, x_3^i)$ is the coordinate of x_i with respect to canonical basis $\{e_0, e_1, e_2, e_3\}$ of $\mathbb{R}_1^4$, then the lorentzian cross product $x_1 \wedge x_2 \wedge x_3$ is defined by the symbolic determinant

$$x_1 \wedge x_2 \wedge x_3 = \begin{vmatrix} -e_0 & e_1 & e_2 & e_3 \\ x_0^1 & x_1^1 & x_2^1 & x_3^1 \\ x_0^2 & x_1^2 & x_2^2 & x_3^2 \\ x_0^3 & x_1^3 & x_2^3 & x_3^3 \end{vmatrix} .$$

One can easly see that

$$\langle x_1 \wedge x_2 \wedge x_3, x_4 \rangle = \det(x_1, x_2, x_3, x_4).$$

If $\langle x, x\rangle > 0$, $\langle x, x\rangle = 0$ or $\langle x, x\rangle < 0$ for any non-zero $x \in \mathbb{R}^4_1$, then we call that x is spacelike, ligtlike or timelike ,respectively. In the rest of this section, we give background of context in [12].

Let $U \subset \mathbb{R}^2$ be open subset, and let $x : U \to S^3_1$ be an embedding. If the vector subspace $\tilde{U}$ whih generated by $\{x_{u_1}, x_{u_2}\}$ is spacelike, then x is called spacelike surface, if $\tilde{U}$ contain at least a timelike vector field, then x is called timelike surface in S^3_1.

In point of view Kasedou [12], we construct the extrinsic differential geometry on curves in S^3_1. Since S^3_1 is a Riemannian manifold, the regular curve $\gamma : I \to S^3_1$ is given by arclength parameter.

Let $\gamma : I \to S^3_1$ is a spacelike and regular curve with unit speed. So we have the tangent vector $t(s) = \gamma'(s)$ so that

$$\|t(s)\| = 1.$$

The unit normal vector of the spacelike curve-γ is defined as

$$n(s) = \frac{t'(s) + \gamma(s)}{\|t'(s) + \gamma(s)\|},$$

where $\langle t'(s), t'(s)\rangle \neq 1$. Also, the binormal vector of the spacelike curve-γ is defined as

$$b(s) = \gamma(s) \wedge t(s) \wedge n(s).$$

The $\{\gamma(s), t(s), n(s), b(s)\}$ frame obtained from here is called the pseudo orthogonal frame of R^4_1 along the spacelike curve-γ.

Similarly, let $\gamma : I \to S^3_1$ is a timelike and regular curve with unit speed. So we have the tangent vector $t(s) = \gamma'(s)$ so that

$$\langle t(s), t(s)\rangle = -1.$$

The unit normal vector of the timelike curve-γ is defined as

$$n(s) = \frac{t'(s) - \gamma(s)}{\|t'(s) - \gamma(s)\|},$$

where $\langle t'(s), t'(s)\rangle \neq 1$. Also, the binormal vector of the timelike curve-γ is defined as

$$b(s) = \gamma(s) \wedge t(s) \wedge n(s).$$

The $\{\gamma(s), t(s), n(s), b(s)\}$ frame obtained from here is called the pseudo orthogonal frame of R^4_1 along the timelike curve-γ.

If the curve-γ is spacelike, the value of

$$\kappa_d(s) = \|t'(s) + \gamma(s)\|$$

is called the de-Sitter curvature of the spacelike curve-γ and the value of

$$\tau_d(s) = -\frac{\det(\gamma, \gamma', \gamma'', \gamma''')}{[\kappa_d(s)]^2}$$

is called the de-Sitter torsion of the spacelike curve-γ.

Similarly, if the curve-γ is timelike, the value of

$$\kappa_d(s) = \|t'(s) - \gamma(s)\|$$

is called the de-Sitter curvature of the timelike curve-γ and the value of

$$\tau_d(s) = -\frac{\det(\gamma, \gamma', \gamma'', \gamma''')}{[\kappa_d(s)]^2}$$

is called the de-Sitter torsion of the timelike curve-γ.

Theorem 2.1. *If $\gamma : I \to S_1^3$ is curve with unit speed, then Frenet-Serre type formulae is obtained*

$$\begin{cases} \gamma'(s) &= t(s) \\ t'(s) &= -\delta_0\delta_1\gamma(s) + \kappa_d(s)\, n(s) \\ n'(s) &= -\delta_1\delta_2\kappa_d(s)\, t(s) - \delta_1\delta_3\tau_d(s)\, b(s) \\ b'(s) &= \delta_1\delta_2\tau_d(s)\, n(s) \end{cases} \tag{1}$$

where

$$\delta_0 = sgn(\gamma(s)), \delta_1 = sgn(t(s)), \delta_2 = sgn(n(s)), \delta_3 = sgn(b(s)). \tag{2}$$

Due to the diversity of the causal charecter of a vector field in Minkowski space R_1^4, there are multiple angle concepts between the arbitrary two vectors.

Definition 2.1. *Let x and y be spacelike vectors in R_1^4.*
i) If x and y span a timelike vector subspace, then we have

$$|\langle x, y\rangle| > \|x\|\,\|y\|$$

and hence, there is a uniqe positive real number θ such that

$$|\langle x, y\rangle| = \|x\|\,\|y\| \cosh\theta.$$

The real number θ is called Lorentz timelike angle between x and y [13].
ii) If x and y span a spacelike vector subspace, then we have

$$|\langle x, y\rangle| \leqslant \|x\|\,\|y\|$$

and hence,there is a unique real number $\theta \in \left[0, \frac{\pi}{2}\right]$ such that

$$|\langle x, y\rangle| = \|x\|\,\|y\| \cos\theta.$$

The real number θ is called the Lorentz spacelike angle between x and y [13].

Definition 2.2. *Let x and y be future pointing timelike vectors in R_1^4, then there is a unique non-negative rel number θ such that*

$$|\langle x, y\rangle| = \|x\|\,\|y\| \cosh\theta.$$

The real number θ is called the Lorentz timelike angle between x and y [13].

Definition 2.3. *Let x be a spacelike vector and y a future pointing timelike vector in R_1^4, then there is a uniue non-negative real number θ such that*

$$|\langle x, y\rangle| = \|x\| \, \|y\| \sinh\theta.$$

The real number θ is called the Lorentz timelike angle between x and y [13].

3 Constant Angle Spacelike Surfaces Construct on a Curve

In this section, normal, binormal and Darboux surfaces, a special class of constant angle spacelike surfaces, will be construct in de-Sitter 3- space. Since such surfaces will be construct on a curve, the causal character of the curve used is very important. For this reason, such constant angle surfaces should be examined in two parts: surfaces produced with a spacelike curve and surfaces produced with a timelike curve.

Let's examine these special types of surface types.

Definition 3.1. *Let $\alpha : I \to S_1^3 \subset R_1^4$ is a unit speed spacelike curve given by arclenght, $x : U \subset \mathbb{R}^2 \to S_1^3 \subset R_1^4$ and $M = x(U)$ is a embeding. The surface M produced by spacelike curve-α is called* ***normal surface*** *in de-Sitter $3-$space S_1^3 given by*

$$x(s,t) = (\cos t)\,\alpha(s) + (\sin t)\,n(s)\,, \;\; (s,t) \in I \times R, \tag{2}$$

here, $n(s)$ is unit normal vector of regular curve $\alpha(s)$.

Remark 3.1. *While constructing the normal surface given by parameterization (2), the vectors of $\{\alpha(s), t(s), n(s), b(s)\}$ frame obtained from the pseudo orthogonal frame of R_1^4 along the $\alpha(s)$ were used. Here $\alpha(s), t(s), n(s)$ are spacelike vectors and $b(s)$ is timelike vector. If vectors $\alpha(s), t(s), b(s)$ are spacelike and $n(s)$ vector is timelike, set $\{\alpha(s), t(s), n(s), b(s)\}$ again indicates an pseudo orthogonal frame of R_1^4. However, using such a frame, a spacelike normal surface cannot be produced in de-Sitter $3-$space. Because the normal surface set up with such a frame has the*

$$x(s,t) = (\cosh t)\,\alpha(s) + (\sinh t)\,n(s)\,, \;\; (s,t) \in I \times R,$$

parameterization. In this case,

$$\begin{cases} E &= \langle x_s, x_s\rangle = (\cosh t + \kappa_d(s)\sinh t)^2 + (\tau_d(s)\sinh t)^2 \\ F &= \langle x_s, x_t\rangle = 0 \\ G &= \langle x_t, x_t\rangle = \sinh^2 t - \cosh^2 t = -1 \end{cases}$$

is obtained and it is clear that

$$\langle \xi, \xi\rangle = F^2 - EG = E.$$

Since the surface we want to achieve here will be spacelike, the unit normal of this spacelike surface must be timelike. Since $E < 0$ cannot be used, the ξ vector cannot be selected to be timelike and therefore a spacelike surface cannot be established with such a frame.

Remark 3.2. *After that, for the normal surface produced with the α-spacelike curve, Frenet-Serret frame will be used along the α-curve, such that the vectors $\alpha(s), t(s), n(s)$ are spacelike, and the vector $b(s)$ is timelike.*

Lemma 3.1. *The normal surface M produced by the spacelike curve-α in the de-Sitter $3-$space must be*

$$(\cos t-\kappa_d(s)\sin t)^2-(\tau_d(s)\sin t)^2>0$$

to be a spacelike surface.

If the expression $\dfrac{x\wedge x_s\wedge x_s}{\|x\wedge x_s\wedge x\|}$ is calculated to find the timelike normal vector ξ of the spacelike normal surface M, we can express the following lemma.

Lemma 3.2. *In the de-Sitter $3-$space, the timelike unit normal vector of the normal surface M produced by the spacelike curve-α is*

$$\xi=\frac{(\tau_d(s)\sin t)}{\sqrt{\Delta}}t(s)+\frac{(\cos t-\kappa_d(s)\sin t)}{\sqrt{\Delta}}b(s) \tag{3}$$

where

$$\Delta=\left|(\tau_d(s)\sin t)^2-(\cos t-\kappa_d(s)\sin t)^2\right|.$$

Let us now examine the case where the surface M given by equation (3) is a constant angle surface. In order to the normal surface M to be a constant angle surface, the normal ξ of the surface and the constant direction of the de-Sitter $3-$space must make a constant angle. We can select this constant direction in the de-Sitter $3-$space to be spacelike or timelike. Furthermore, considering the causal character of the angle between the unit normal vector field of the surface and the constant direction of the de-Sitter $3-$space, the normal surfaces produced by a spacelike curve are divided into two parts as follows.

Definition 3.2. *Let $U\subset\mathbb{R}^2$ be an open set, $x:U\to S_1^3\subset R_1^4$ is the spacelike normal surface given by the (2) parameterization, and ξ is the timelike unit normal vector area of the surface M. If there is a constant timelike (spacelike) direction d_1 as $\theta(\xi,d_1)$ timelike angle on surface M is constant, then surface M is called a* ***spacelike normal surface****, which is produced with a spacelike curve, with* ***a constant timelike angle****,* ***constant timelike (spacelike) direction*** *in de-Sitter $3-$space S_1^3.*

Theorem 3.1. *The normal spacelike surfaces with a constant timelike angle, constant timelike (spacelike) direction, produced with a spacelike curve in S_1^3 are Lorentz plane parts.*

Now let's find one of the specially selected constant directions of the spacelike normal surface M in de-Sitter $3-$space S_1^3.

Lemma 3.3. *$x:M\to S_1^3\subset R_1^4$ is a embeding and M is a constant timelike angled spacelike normal surface in de-Sitter $3-$space S_1^3. In this case, constant timelike direction of surface M is as*

$$\begin{aligned} d_1 \;=\;& \frac{(\cos t-\kappa_d(s)\sin t)\sqrt{\left|\cosh^2\theta-\cosh^2\varphi\right|}}{\sqrt{\Delta}}t(s)+\frac{(\tau_d(s)\sin t)\sqrt{\left|\cosh^2\theta-\cosh^2\varphi\right|}}{\sqrt{\Delta}}b(s)\\ &+(\cosh\theta)\,\xi \end{aligned}$$

where φ is the angle between the constant timelike direction d_1 and the spacelike vector x.

Corollary 3.1. *The spacelike normal surfaces in de-Sitter $3-$space are ruled surface.*

Remark 3.3. *The spacelike normal surfaces produced with the spacelike curve-α are given above. By making similar calculations, it can be easily demonstrated that the spacelike normal surfaces produced with the timelike curve-α will produce similar results with the above lemma and theorems.*

Definition 3.3. *Let $\alpha : I \to S_1^3 \subset R_1^4$ is a unit speed spacelike curve given by arclenght, $x : U \subset \mathbb{R}^2 \to S_1^3 \subset R_1^4$ and $M = x(U)$ is a embeding. The surface M produced by spacelike curve α is called **binormal surface** in de-Sitter $3-$space S_1^3 given by*

$$x(s,t) = (\cos t)\,\alpha(s) + (\sin t)\,b(s)\,,\ \ (s,t) \in I \times R, \tag{4}$$

here, $b(s)$ is unit binormal vector of regular curve $\alpha(s)$.

Lemma 3.4. *The binormal surface M produced by the spacelike curve-α in the de-Sitter $3-$space must be*

$$(\cos t)^2 - (\tau_d(s)\sin t)^2 > 0$$

to be a spacelike surface.

Lemma 3.5. *In the de-Sitter $3-$space, the timelike unit normal vector of the binormal surface M produced by the spacelike curve-α is*

$$\xi = \frac{(\tau_d(s)\sin t)}{\sqrt{\Delta_1}}t(s) + \frac{(\cos t)}{\sqrt{\Delta_1}}n(s) \tag{5}$$

where

$$\Delta_1 = \left|(\tau_d(s)\sin t)^2 - (\cos t)^2\right|.$$

Definition 3.4. *Let $U \subset \mathbb{R}^2$ be an open set, $x : U \to S_1^3 \subset R_1^4$ is the spacelike binormal surface given by the (4) parameterization, and ξ is the timelike unit normal vector area of the surface M. If there is a constant timelike (spacelike) direction d_3 as $\theta(\xi, d_3)$ timelike angle on M surface is constant, then surface M is called a **spacelike binormal surface**, which is produced a spacelike curve, with **a constant timelike angle**, **constant timelike (spacelike) direction** in de-Sitter $3-$space S_1^3.*

Theorem 3.2. *The spacelike binormal surfaces with a constant timelike angle, constant timelike (spacelike) direction, produced with a spacelike curve in S_1^3 are Lorentz plane parts.*

Corollary 3.2. *The spacelike binormal surfaces in de-Sitter $3-$space are ruled surface.*

Definition 3.5. *Let $\alpha : I \to S_1^3 \subset R_1^4$ is a unit spacelike curve given by arclenght, $x : U \subset \mathbb{R}^2 \to S_1^3 \subset R_1^4$ and $M = x(U)$ is a embeding. The surface M is called **Darboux surface** in de-Sitter $3-$space S_1^3 given by*

$$x(s,t) = (\cos t)\,t(s) + (\sin t)\,b(s)\,,\ \ (s,t) \in I \times R, \tag{6}$$

here, $t(s)$ and $b(s)$ are unit tangent vector and unit binormal vectors of regular curve $\alpha(s)$, respectively.

Remark 3.4. *After that, for the Darboux surface produced with the α-spacelike curve, Frenet-Serret frame will be used along the α-curve, such that the vectors $\alpha(s), t(s), b(s)$ are spacelike, and the vector $n(s)$ is timelike. Otherwise, if the vectors $\alpha(s), t(s), n(s)$ are spacelike and the vector $b(s)$ is timelike, a Darboux surface cannot be created to be spacelike with such a frame.*

Lemma 3.6. *The Darboux surface M produced by the spacelike curve-α in the de-Sitter $3-$space must be*

$$(\cos t)^2 - (\kappa_d(s)\cos t - \tau_d(s)\sin t)^2 > 0$$

to be a spacelike surface.

Lemma 3.7. *In the de-Sitter $3-$space, the timelike unit normal vector of the Darboux surface M is*

$$\xi = \frac{(\kappa_d(s)\cos t - \tau_d(s)\sin t)}{\sqrt{\Delta_2}}\alpha(s) + \frac{(\cos t)}{\sqrt{\Delta_2}}n(s) \tag{7}$$

where

$$\Delta_2 = (\kappa_d(s)\cos t - \tau_d(s)\sin t)^2 - (\cos t)^2.$$

Let us now examine the case where the Darboux surface M given by equation (6) is a constant angle surface. In order to the Darboux surface M to be a constant angle surface, the normal ξ of the surface and the constant direction of the de-Sitter $3-$space must make a constant angle. We can select this constant direction in the de-Sitter $3-$space to be spacelike or timelike. Furthermore, considering the causal character of the angle between the unit normal vector field of the surface and the constant direction of the de-Sitter $3-$space, the Darboux surfaces produced by a spacelike curve are divided into two parts as follows.

Definition 3.6. *Let $U \subset \mathbb{R}^2$ be an open set, $x : U \to S_1^3 \subset R_1^4$ is the spacelike Darboux surface given by the (6) parameterization, and ξ is the timelike unit normal vector area of the surface M. If there is a timelike (spacelike) constant direction d_5 as $\theta(\xi, d_5)$ timelike angle on surface M is constant, then surface M is called a* ***spacelike Darboux surface,****which is produced a spacelike curve, with* ***a constant timelike angle****,* ***constant timelike direction*** *in de-Sitter $3-$space S_1^3.*

Theorem 3.3. *The spacelike Darboux surfaces with a constant timelike angle, constant timelike (spacelike) direction in S_1^3 are Lorentz plane parts.*

Corollary 3.3. *The spacelike Darboux surfaces in de-Sitter $3-$space are ruled surface.*

Corollary 3.4. *Any constant angle spacelike surface is isometric to a plane, tangent surface, normal surface, binormal surface or Darboux surface.*

Conclusion 3.1. *For many years, many studies have been done on the geometry of surfaces in the Minkowski space. This study has been prepared to contribute to making more detailed studies on constant angle regle surfaces. In the first two section, a summary of the literature, basic definitions and theorems are given for a better understanding of the subject. In the following sections, Constant Angle Spacelike Surfaces Construct on a Curve are defined and examined in detail. As a result, this study has been presented to the literature as a resource that will be used by every scientist who will study surfaces in the de-Sitter 3-space.*

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USING FREEWARE MATHEMATICAL SOFTWARE IN CALCULUS CLASSES

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ABSTRACT

Due to the Bologna Accord, the teaching of mathematics has undergone important changes. Some of the most visible modifications have been the need to complement the traditional teaching-learning process with practical, real-life cases and the possibility to reinforce the introduction and usage of key concepts through mathematical software. Nowadays, there exist many computational packages dealing with mathematics, some of the best-known being Mathematica and Matlab. However, although they are very complete and powerful, they demand the use of commercial licences, which can be a problem for some education institutions or in the cases where students desire to use the software in an unlimited number of devices or to access from several of them simultaneously.

In this article, we show how to apply GeoGebra and WolframAlpha to the process of teaching Calculus to first-year university students. While GeoGebra is an interactive geometry, algebra, statistics, and calculus application available both as an online resource and a native application in Windows, macOS, and Linux systems, WolframAlpha is a computational knowledge engine developed by a subsidiary of Wolfram Research, the company behind Mathematica. However, unlike that product, WolframAlpha can be accessed by any individual as a web service free of charge. One of the key aspects of WolframAlpha is the possibility to use natural language and Mathematica syntax for requesting computations, which allows users to benefit from a large amount of Mathematica resources.

Being able to use GeoGebra and WolframAlpha as web services without downloading and installing software is another important advantage, as it avoids the need to have adminstrator rights to use those computational engines, which typically represents a problem in education centres where lab computers are locked so students cannot inadvertently install malware that can compromise the university's network.

As the best way to show a topic in mathematics is to provide examples, this contribution focuses on the main topics associated to a first-year Calculus

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class (limits, continuity, derivatives, curve interpolation and integrals), providing examples with GeoGebra and WolframAlpha for the computations and concrete examples used in actual Calculus classes.

***K*eywords** Calculus · Freeware · Mathematics · Software

1 Introduction

The Bologna Accord is an agreement on a common model of higher education reached on 1999 that implies the creation of a common European area of university studies. It emphasizes the creation of a European Area of Higher Education (EAHE) as a key to promote students' mobility, aiming to simplify Europe's educational qualifications and ensuring that credentials granted by an institution in one country are comparable with those earned elsewhere [1].

Spain is one of the 48 countries currently involved in the Bologna Process. The cornerstones of such an open space are mutual recognition of degrees and other higher education qualifications, transparency (readable and comparable degrees organised in a three-cycle structure) and European cooperation in quality assurance.

Due to the Bologna Accord, the teaching of mathematics has undergone important changes, such as the need to reinforce the traditional teaching-learning process with practical, real-life cases and the possibility to introduce some key concepts by using mathematical software. Nowadays, there are many computational packages focused on mathematics, where Mathematica and Matlab are some of the best known. However, even though they are certainly very complete and powerful, they require to use commercial licences, which can be a problem for some education institution or in the cases where students desire to use the software in an unlimited number of devices or to obtain access from several of them simultaneously.

In this contribution, our goal is to show how to apply freeware mathematical software to the teaching of Calculus for first-year university students. In order to do that, GeoGebra and WolframAlpha will be used for providing actual examples used at class.

The rest of this contribution is organized as follows: Section 2 introduces U-tad, the university centre where the Calculus classes mentioned in the article are imparted. In addition to that, this section also includes details about the syllabus of the Calculus class. Section 3 presents the most relevant information about GeoGebra and WolframAlpha, while Section 4 provides several examples used at class. Finally, in Section 5 we offer some conclusions and ideas for future work.

2 U-tad

U-tad is the acronym for Centro Universitario de Tecnología y Arte Digital (Technology and Digital Art University Centre, Figure 1) [2], a private university centre founded in 2011 with a strong focus on the creation, programming, and management of digital content, products, and services. U-tad is based near Madrid, and its current academic offer includes three higher technical education courses, five undergraduate degrees and twelve postgraduate courses.

Figure 1: Centro Universitario de Tecnología y Arte Digital (U-tad).

At the Software Engineering Degree imparted at U-tad, Calculus is a first-year course focused on single variable calculus [3]. This introductory Calculus course covers the following topics:

- Concepts of function, limits and continuity.
- Differentiation rules, application to graphing, rates, approximations and extremum problems.
- Definite and indefinite integration.
- The fundamental theorem of calculus.
- Applications of integrals to geometry: area, volume and arc length.

The evaluation scheme used during last years for computing the final grade of Calculus students is the following one:

- 40% first partial exam.
- 40% second partial exam.
- 20% homework and assignments.

Homework and assignments can take two forms: traditional exercises that have to be completed using pen and paper or exercises that require the use of a computational engine. Even though in the latter case it is inevitable to use computational software, it is recommended that they also use it in traditional exercises in order to check their answers before submitting them for grading.

3 Computational engines

There are basically three possibilities regarding the usage of computational engines:

- Commercial software: Matlab, Mathematica, etc.
- Free software: GeoGebra, WolframAlpha, Maxima, SageMath, etc.
- Programming languages such as Python.

Each of these options has its benefits and disadvantages. Applications like Mathematica are very powerful, but obviously they require commercial licences and the installation of many software packages that in some applications have to be managed manually and that in any case could potentially need to allocate several gigabytes of hard drive space.

In comparison, the computational capabilities of free software are considerably lower, but for introductory subjects they may be more than enough.

Finally, programming languages like Python are very versatile and allow to perform symbolic and numeric calculations, but most first-year students are not familiar with them.

That is why, in this contribution, we are going to focus on how to apply free software to the teaching of Calculus for first-year university students.

Focusing on the freeware computational engines, even though at class students are given information about GeoGebra, WolframAlpha, Maxima and SageMath, most of them choose GeoGebra and WolframAlpha for solving the exercises, and that is the reason why in the rest of this contribution we will make reference exclusively to those two options.

3.1 GeoGebra

GeoGebra (`www.geogebra.org`) is an interactive geometry, algebra, statistics, and calculus application available both as an online resource and a native application in Windows, macOS, and Linux systems [4].

The website includes several services such as a calculator and a graphics plotter, but the most widely used option is what is called GeoGebra Classic, which puts together those individual tools.

Figure 2 shows the GeoGebra Classic interface, where it is possible to find modules for 2- and 3-dimension plotting, an input bar or the Computer Algebra System module, among others.

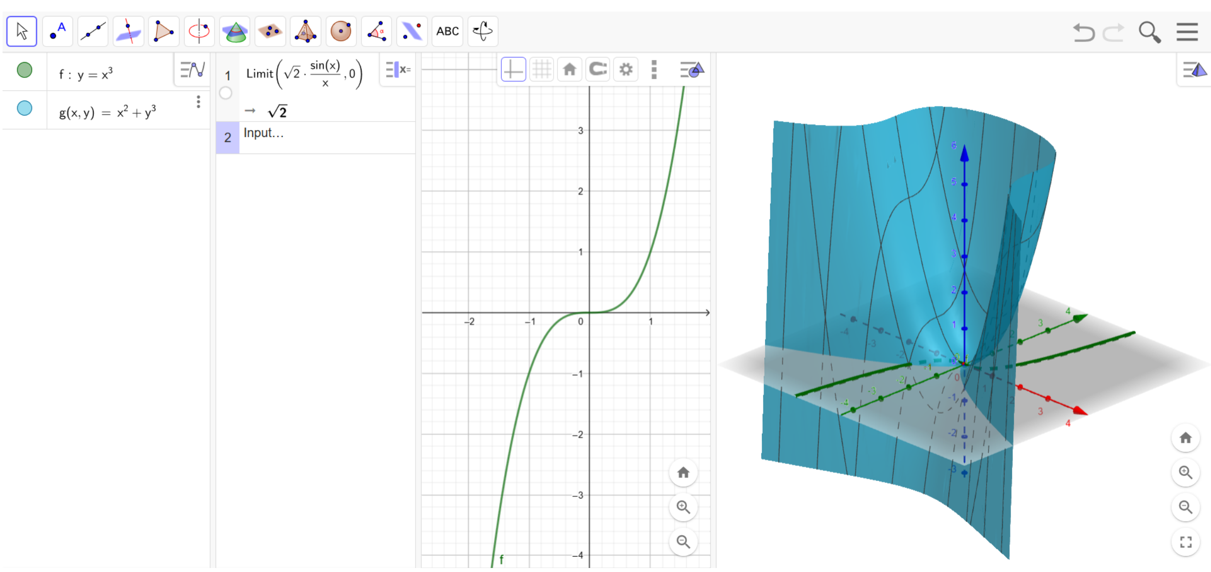

Figure 2: GeoGebra Classic screen.

GeoGebra's interface is easy to use and allows the configuration of several aspects associated to function representation, such as line width, color, and style. Those representations can be integrated into online books that can be shared with students so, for instance, they can navigate through all the examples and solutions associated to a certain topic.

3.2 WolframAlpha

WolframAlpha is a computational knowledge engine developed by a subsidiary of Wolfram Research, the company behind Mathematica [5]. Given that WolframAlpha is a reduced version of the Mathematica software, all options must be entered as text in the application's input box. However, the website provides access to many examples (see Figure 3), so students can find the right expression in a relatively short time. Obviously, the advantage of using WolframAlpha instead of Mathematica is that it can be accessed by anyone as a web service free of charge.

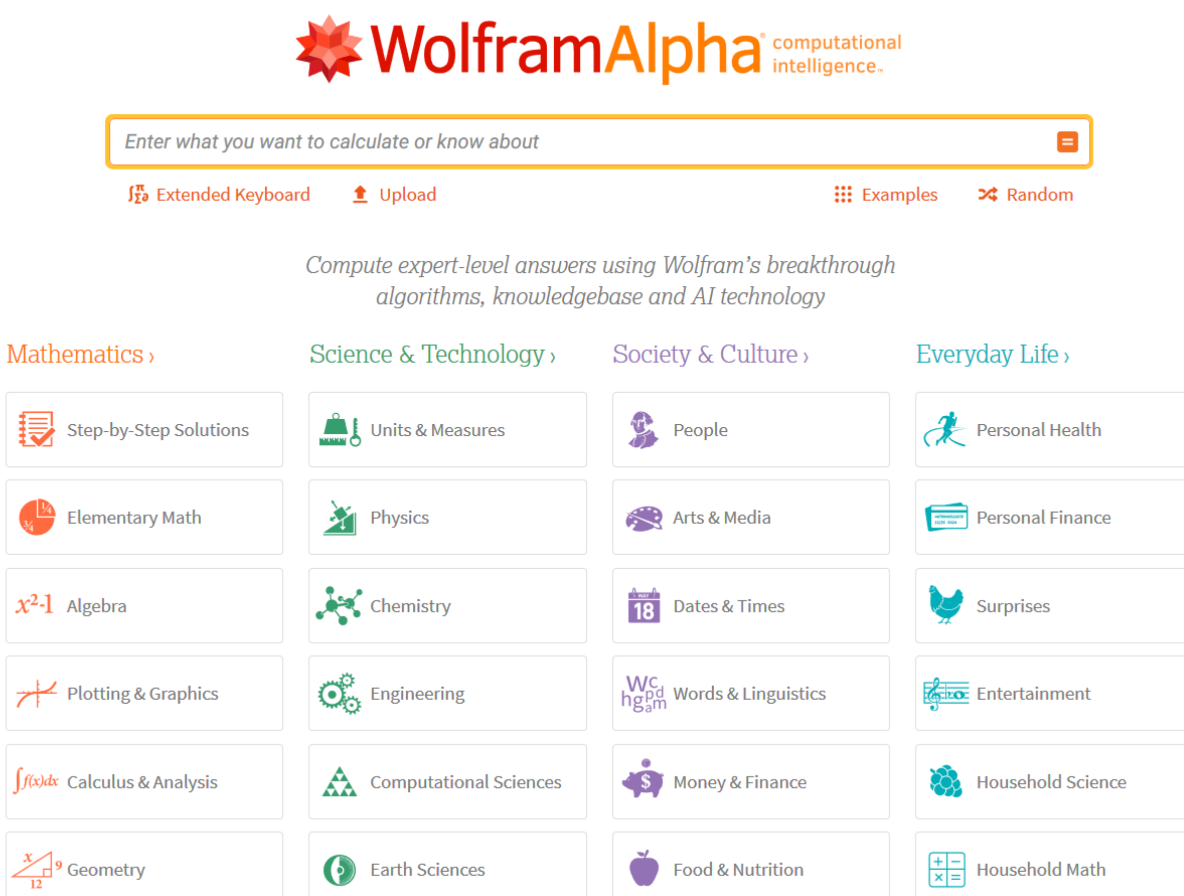

Figure 3: WolframAlpha website.

One of the key aspects of WolframAlpha is the possibility to use both natural language and Mathematica syntax for requesting computations. Figures 4 and 5 show how to generate the same calculation using the two options.

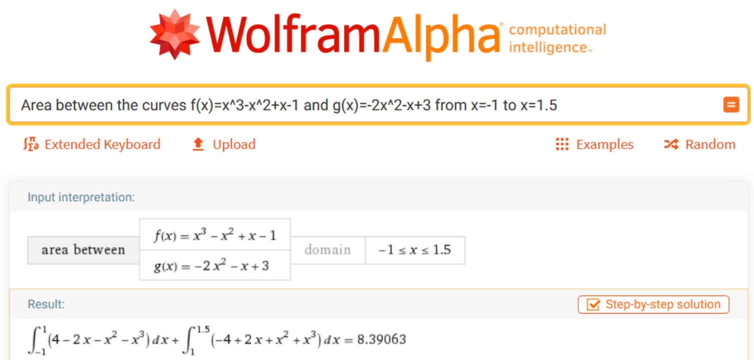

Figure 4: Example using natural language in WolframAlpha.

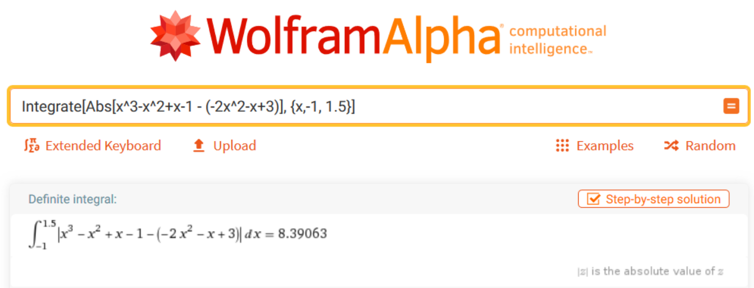

Figure 5: Example using Mathematica syntax in WolframAlpha.

4 Examples

As mentioned before, some Calculus key concepts can be reinforced or at least better understood by students when presented in a graphic way. Allowing students to replicate some model computations in other similar problems has the benefit to provide a durable link between what is taught at class and what they study at home.

Figure 6 shows an example associated to the graphic representation of a function and its asymptotes.

If, for instance, we need to show how the Taylor polynomials work, we can include in the same solution the initial function and Taylor polynomials of different degrees, so students can realize that a higher degree implies a better approximation for real functions (see Figure 7).

Even though calculus of several variables is not included in the contents of the first-year subject taught at U-tad, in other subjects it is necessary to correctly interpret and visualize that kind of functions. In that regard, GeoGebra is a suitable option given that it allows students to rotate the image in both directions. As an example, Figure 8 shows how to represent the intersection of two surfaces.

Both engines are supported by a large number of developers who make available their work, so in both cases it is possible to access many great online demonstrations and practical examples. This feature is particularly interesting when teaching theorems and their applications, as it is a topic

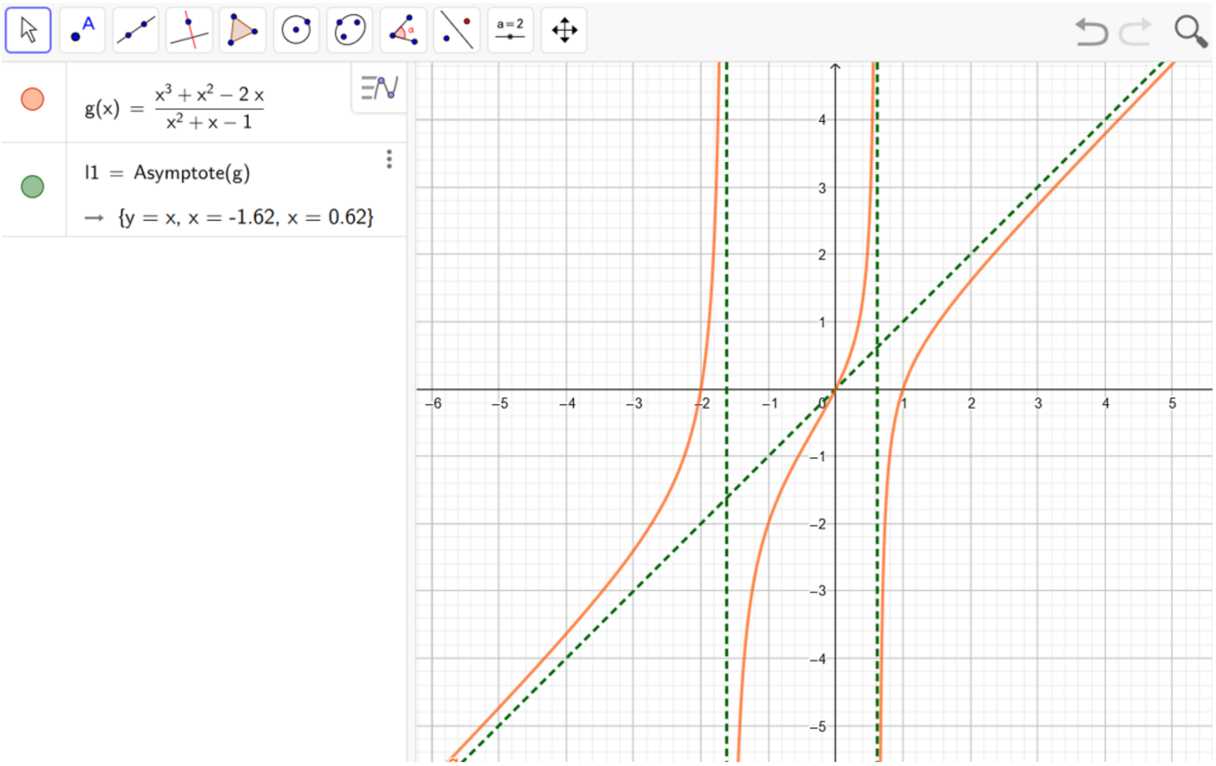

Figure 6: GeoGebra example about function representation.

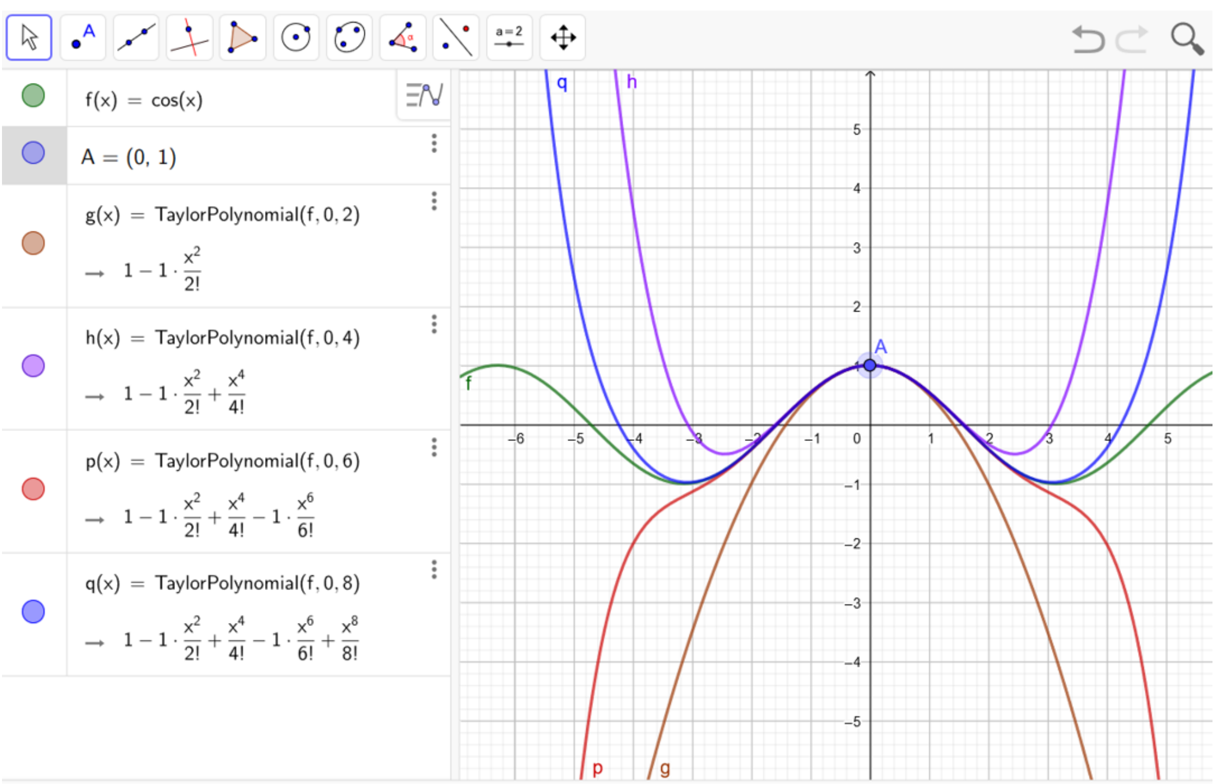

Figure 7: GeoGebra example about Taylor polynomials.

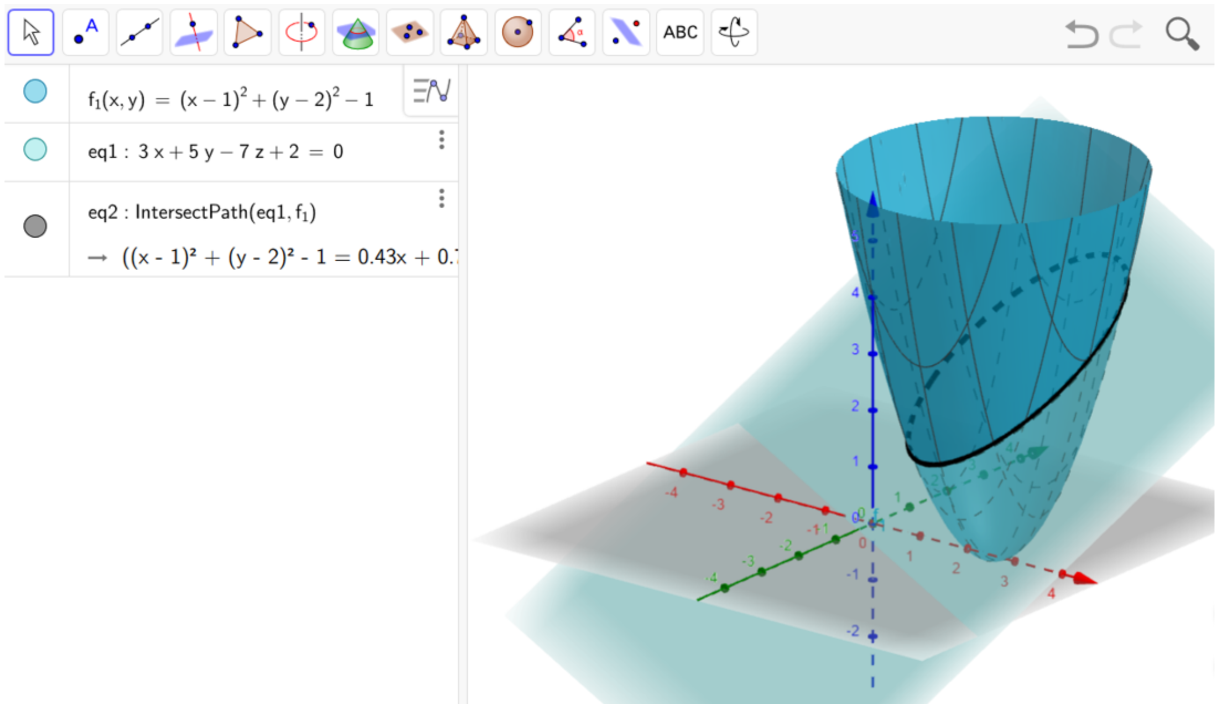

Figure 8: Intersection of two surfaces using GeoGebra.

where many students face some difficulties. Figures 9 and 10 show how to represent Lagrange's theorem [6] and the Integral Mean Value theorem [7].

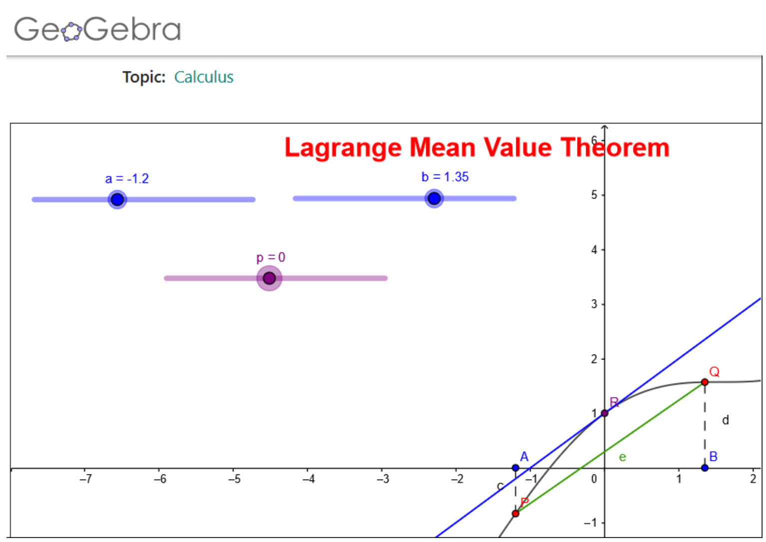

Figure 9: Lagrange Mean Value Theorem (author: Ravinder Kumar [6]).

5 Conclusions

In this contribution, we have shown how to use two of the best-known freeware mathematical software, GeoGebra and WolframAlpha, in order to enhance the comprehension of mathematical

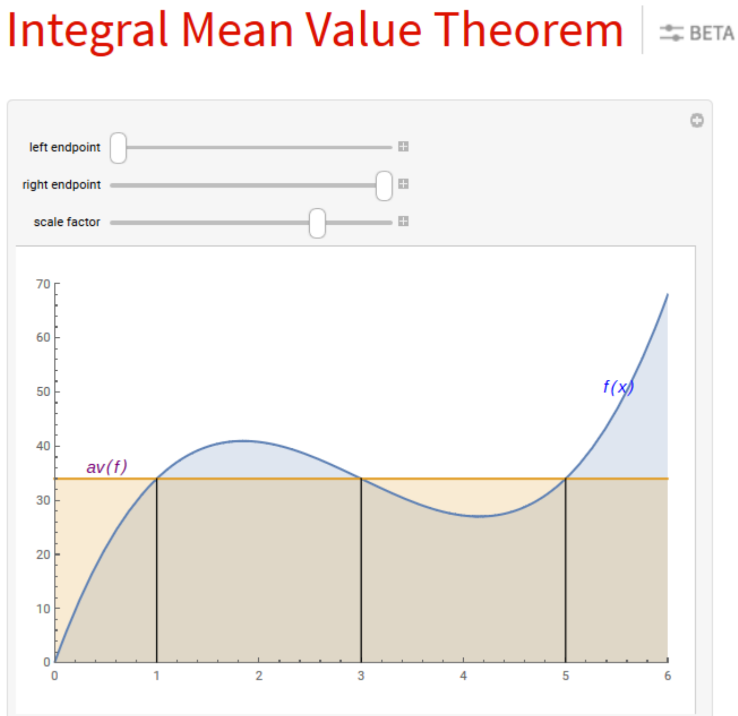

Figure 10: Integral Mean Value Theorem (author: Chris Boucher [7]).

concepts associated to a first-year single variable Calculus class. Both engines allow students to grasp the key concepts seen at class and to practice problems at their leisure, resulting in better learning outcomes and grades.

Regarding the future work, it would be interesting to centralize demonstrations about all the theorems included in the subject's syllabus as a public, online book, and to extend the usage of these engines to other mathematics courses such as Complex Numbers, Differential Geometry or Ordinary Differential Equations, which will be attempted in next years.

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On Reduction of a (2+1)-Dimensional Nonlinear Schrödinger Equation via Conservation Laws

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Abstract

In this work we consider a (2+1)-dimensional nonlinear Schrödinger equation which is one of the important models in branches of plasma physics, nonlinear optics, fluid dynamics, etc. Considering the Lie point symmetries and conservation laws obtained with the help of methods available in the literature, the association between symmetries and conservation laws leads to a reduction in the equation. This theory reduces both the order and the number of independent variables involved in underlying equation and quite useful to obtain new solutions.

Keywords Conservation laws · Lie symmetries · Schrödinger equation

1 Introduction

The $(2+1)$-dimensional the Heisenberg ferromagnetic spin chain (HFSC) equation which is governed by an integrable $(2+1)$-dimensional nonlinear Schrödinger type equation is given by [18, 1, 10]

$$iq_t + aq_{xx} + bq_{yy} + cq_{xy} - d|q|^2 q = 0 \tag{1}$$

where $a = \tau^4(\vartheta + \vartheta_2), b = \tau^4(\vartheta_1 + \vartheta_2), c = 2\tau^4\vartheta_2, d = 2\tau^4 A$. Here, ϑ,ϑ_1 are the coefficients of bilinear exchange interactions along the x- and y-directions, respectively, τ is lattice parameter, A is the uniaxial crystal field anisotropy parameter and ϑ_2 refers to the neighboring interaction along the diagonal. HFSC, different magnetic interactions have various integration features and soliton spin excitations in classical and semiclassical continuum limit. Therefore, the nonlinear magnetization dynamics of HFSC, which can be governed by the nonlinear Schrödinger (NLS) equations, have significantly increased researchers' interest in soliton theory and condensed matter physics [18, 1, 10, 9, 17].

Double reduction theory is a developed method from the field of Lie symmetries. To apply this method, the relation between the symmetries and conservation laws of the equation are needed. Various researches have been carried out by many researchers on this subject [3, 11, 12, 13, 15, 16]. In this paper, some preliminaries about double reduction are introduced. Next, double reduction will be implemented to HFSC equation and we get a reduction both the order of equation and the number of independent variables.

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2 Overview of double reduction method

We denoted an sth-order PDE of m independent variables $x = (x^1, x^2, ..., x^m)$ and n dependent variables $u = (u_1, u_2, ..., u_n)$

$$F^\alpha(x, u, u_{(1)}, ..., u_{(s)}) = 0, \quad \alpha = 1, ..., n, \tag{2}$$

where $u_{(1)}, u_{(2)}, ..., u_{(s)}$ symbolize the first, second, . . . , sth order partial derivatives, i.e., $u_i^\alpha = D_i(u^\alpha)$,$u_{ij}^\alpha = D_jD_i(u^\alpha)$,... respectively, with the total differentiation operator with respect to x^i given by

$$D_i = \frac{\partial}{\partial x^i} + u_i^\alpha \frac{\partial}{\partial u^\alpha} + u_{ij}^\alpha \frac{\partial}{u_j^\alpha} + ..., \;\; i = 1, ..., n. \tag{3}$$

The following definitions are acknowledged (see, e.g. [2, 5, 6]). The variational operator given by

$$\frac{\delta}{\delta u^\alpha} = \frac{\partial}{\partial u^\alpha} + \sum_{s\geqslant 1}(-1)^s D_{i_1 ...} D_{i_s} \frac{\partial}{\partial v^\alpha_{i_1 i_2 ... i_s}}, \;\; \alpha = 1, ..., m. \tag{4}$$

The Lie-Bäcklund operator is given as

$$\Gamma = \xi^i \frac{\partial}{\partial x^i} + \eta^\alpha \frac{\partial}{\partial u^\alpha}, \quad \xi^i, \eta^\alpha \in \mathcal{S}, \tag{5}$$

where $\mathcal{S}$ is the space of differential functions. The operator (5) is an abbreviated version of the infinite formal sum

$$\Gamma = \xi^i \frac{\partial}{\partial x^i} + \eta^\alpha \frac{\partial}{\partial u^\alpha} + \sum_{s\geqslant 1} \zeta^\alpha_{i_1 i_2 ... i_s} \frac{\partial}{\partial u^\alpha_{i_1 i_2 ... i_s}}, \tag{6}$$

where the extension coefficients are given by the extension formulae

$$\begin{aligned} \zeta_i^\alpha &= D_i(\mathcal{W}^\alpha) + \xi^j u_{ij}^\alpha \\ \zeta^\alpha_{i_1 ... i_s} &= D_{i_1 ... i_s}(\mathcal{W}^\alpha) + \xi^j u^\alpha_{j i_1 ... i_s} \;\; s > 1, \end{aligned} \tag{7}$$

where $\mathcal{W}^\alpha$ is the Lie characteristic function $\mathcal{W}^\alpha = \eta^\alpha - \xi^j u_j^\alpha$. The m-tuple vector $T = (T^1, T^2, ..., T^m)$, $T^j \in S$, $j = 1, ..., m$ is a conserved vector of (2) if T^i satisfies

$$D_i T^i|_{(2)} = 0. \tag{8}$$

We now give the relevant results used in this study below.
Assume that Γ is any Lie-Bäcklund operator of Eq. (2) and the components of conserved vector of (2) are given by T^i. Then [6, 8]

$$T^{*i} = [T^i, \Gamma] = \Gamma(T^i) + T^i D_j \xi^j - T^j D_j \xi^i, \quad i = 1, ..., m, \tag{9}$$

construct the components of a conserved vector of (2), i.e.,

$$D_i T^{*i}|_{(2)} = 0.$$

If formula T^{*i} in (9) is equal to zero, that means this two pair are associative, if they are not equal to zero, then the result will be a new conserved vector, and hence, we use the new conserved vector with the Lie symmetry until we get the result equal to zero.

Theorem [3] : Assume that $D_i T^i = 0$ is a conservation law of the PDE system (2). Then under a similarity transformation, there exists functions $\tilde{T}^i$ such that $JD_i T^i = \tilde{D}_i \tilde{T}^i$ where $\tilde{T}^i$ is given by

$$\begin{pmatrix} \tilde{T}^1 \\ \tilde{T}^2 \\ \vdots \\ \tilde{T}^m \end{pmatrix} = J \left(A^{-1}\right)^T \begin{pmatrix} T^1 \\ T^2 \\ \vdots \\ T^n \end{pmatrix}, \quad J \begin{pmatrix} T^1 \\ T^2 \\ \vdots \\ T^m \end{pmatrix} = A^T \begin{pmatrix} \tilde{T}^1 \\ \tilde{T}^2 \\ \vdots \\ \tilde{T}^m \end{pmatrix} \tag{10}$$

in which

$$A = \begin{pmatrix} \tilde{D}_1 x^1 & \dots & \tilde{D}_1 x^m \\ \vdots & \vdots & \vdots \\ \tilde{D}_m x^1 & \dots & \tilde{D}_m x^m \end{pmatrix}, \quad A^{-1} = \begin{pmatrix} D_1 \tilde{x}^1 & \dots & D_1 \tilde{x}^m \\ \vdots & \vdots & \vdots \\ D_m \tilde{x}^1 & \dots & D_m \tilde{x}^m \end{pmatrix} \tag{11}$$

and $J = det(A)$.

2.1 Governing equation

In this subsection, to create a system of equations, we substitute $q = u + iv$ into Eq. (1). Separating into real and imaginary parts, we yield

$$\begin{aligned} &-v_t + au_{xx} + bu_{yy} + cu_{xy} - duv^2 - du^3 = 0, \\ &u_t + av_{xx} + bv_{yy} + cv_{xy} - dvu^2 - dv^3 = 0. \end{aligned} \tag{12}$$

3 Double reduction of HFSC equation

The symmetry group of the HFSC equation (1) will be generated by the vector field of the form

$$\Gamma = \tau(x,t,u)\frac{\partial}{\partial t} + \zeta(x,t,u)\frac{\partial}{\partial x} + \eta(x,t,u)\frac{\partial}{\partial u}. \tag{13}$$

We obtain an overdetermined system of linear PDEs implementing the second prolongation $\Gamma^{[2]}$ to Eq. (1). Then, solving the obtained system, we get Lie point symmetries of (1) with the help of SADE (in Maple) [14]:

$$\begin{aligned}
\Gamma_1 &= \frac{\partial}{\partial y}, \\
\Gamma_2 &= \frac{\partial}{\partial t}, \\
\Gamma_3 &= \frac{\partial}{\partial x}, \\
\Gamma_4 &= u\frac{\partial}{\partial v} - v\frac{\partial}{\partial u} \\
\Gamma_5 &= -u\frac{\partial}{\partial u} - v\frac{\partial}{\partial v} + 2\,t\frac{\partial}{\partial t} + x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} \\
\Gamma_6 &= -\frac{1}{2}\,\frac{uc}{b}\frac{\partial}{\partial u} - \frac{1}{2}\,\frac{vc}{b}\frac{\partial}{\partial v} + \frac{ct}{b}\frac{\partial}{\partial t} - \frac{(ay-xc)}{b}\frac{\partial}{\partial x} + x\frac{\partial}{\partial y} \\
\Gamma_7 &= -\frac{v\,(-xc+2\,ay)\,\frac{\partial}{\partial u}}{4\,ab-c^2} + \frac{u\,(-xc+2\,ay)\,\frac{\partial}{\partial v}}{4\,ab-c^2} + t\frac{\partial}{\partial y} \\
\Gamma_8 &= -\frac{v\,(2\,bx-cy)\,\frac{\partial}{\partial u}}{4\,ab-c^2} + \frac{u\,(2\,bx-cy)\,\frac{\partial}{\partial v}}{4\,ab-c^2} + t\frac{\partial}{\partial x} \\
\Gamma_9 &= -\frac{(bx^2v+4\,baut-utc^2-xyvc+avy^2)\,\frac{\partial}{\partial u}}{4\,ab-c^2} - \frac{(4\,v_tab-v_tc^2-x^2ub+xucy-ay^2u)}{4\,ab-c^2}\frac{\partial}{\partial v} \\
&\quad +t^2\frac{\partial}{\partial t} + tx\frac{\partial}{\partial x} + yt\frac{\partial}{\partial y}
\end{aligned} \tag{14}$$

It is shown by Du et al.[4] that (1) accepts the conserved vectors given in Table 1.

With the help of the double reduction theory, we compute the exact solutions of Eq. (1). If the following expression is satisfied

$$T^* = \Gamma\begin{pmatrix} T^t \\ T^x \\ T^y \end{pmatrix} - \begin{pmatrix} D_t\xi^t & D_x\xi^t & D_y\xi^t \\ D_t\xi^x & D_x\xi^x & D_y\xi^x \\ D_t\xi^y & D_x\xi^y & D_y\xi^y \end{pmatrix}\begin{pmatrix} T^t \\ T^x \\ T^y \end{pmatrix} + \left(D_t\xi^t + D_x\xi^x + D_y\xi^y\right)\begin{pmatrix} T^t \\ T^x \\ T^y \end{pmatrix} = 0, \tag{16}$$

then the Lie-Bäcklund symmetry generator Γ is associated with a conserved vector T of Eq. (1).

$$
\begin{aligned}
T_1^t &= \frac{1}{2}\,avv_{xx} + \frac{1}{2}\,auu_{xx} + \frac{1}{2}\,buu_{yy} + \frac{1}{2}\,bvv_{yy} - \frac{1}{4}\,du^4 - \frac{1}{2}\,du^2v^2 + \frac{1}{2}\,ucu_{xy} + \frac{1}{2}\,vcv_{xy} - \frac{1}{4}\,dv^4 \qquad (15)\\
T_1^x &= -\frac{1}{2}\,avv_tx - \frac{1}{2}\,auu_{tx} + \frac{1}{2}\,av_x\,v_t + \frac{1}{2}\,au_x\,u_t + \frac{1}{4}\,cu_y\,ut + \frac{1}{4}\,cv_y\,v_t - \frac{1}{4}\,ucu_{ty} - \frac{1}{4}\,vcv_{ty}\\
T_1^y &= -\frac{1}{2}\,buuty + \frac{1}{2}\,bu_y\,ut - \frac{1}{2}\,bvv_ty + \frac{1}{2}\,bv_y\,v_t - \frac{1}{4}\,ucutx + \frac{1}{4}\,cu_x\,ut - \frac{1}{4}\,vcv_tx + \frac{1}{4}\,cv_x\,v_t\\
T_2^t &= \frac{1}{2}\,\frac{-uv_x\,tc^2 + vc^2u_x\,t + u^2cy - 2\,bu^2x - 2\,bxv^2 + cv^2y + 4\,uv_x\,tba - 4\,vbau_x\,t}{4\,ba - c^2}\\
T_2^x &= \frac{1}{4\,(4\,ba - c^2)}\left(\begin{array}{c}
-8\,utbav_t + 8\,utb^2au_{yy} - 2\,utc^2bu_{yy} - 4\,ucv_y\,bx - 8\,uv_x\,abx + 4\,uv_x\,acy + 4\,cu_y\,bxv + 4\,vcv_{xy}\,tba\\
+8\,au_x\,bxv - 4\,au_x\,cv_y + 8\,v_tut\,ba + 8\,v_tb^2v_{yy}\,a - 2\,v_tbv_{yy}\,c^2 + 2\,tc^2du^2v^2 - 4\,tbadu^4 - 4\,tdv^4ba\\
-c^3v_y\,v_x\,t - vc^3v_{xy}\,t - c^3u_y\,u_x\,t - uc^3u_{xy}\,t + tc^2du^4 + tdv^4c^2 + 2\,utc^2v_t - 2\,v_tut\,c^2 - 2\,c^2u_y\,v_y + 8\,a^2{u_x}^2bt\\
-2\,a{u_x}^2c^2t + 8\,{v_x}^2a^2tb - 2\,{v_x}^2atc^2 + 2\,uc^2v_y\,y - 8\,tbadu^2v^2 + 4\,cu_y\,bau_x\,t + 4\,ucu_{xy}\,tba + 4\,cv_y\,v_x\,tba
\end{array}\right)\\
T_2^y &= -\frac{1}{4\,(4\,ba - c^2)}\left(\begin{array}{c}
4\,ucu_{xx}\,tba + 4\,vcv_{xx}\,tba - 4\,ubv_y\,cy + 8\,ub^2u_{xy}\,ta - 2\,ubu_{xy}\,tc^2 + 4\,ucv_x\,bx - 8\,b^2u_y\,au_x\,t + 2\,bu_y\,c^2u_x\,t\\
-4\,c{u_x}^2bat - 4\,cu_x\,bxv + 8\,vb^2v_{xy}\,ta - 2\,vbv_{xy}\,tc^2 - 8\,b^2v_y\,v_x\,ta + 2\,bv_y\,v_x\,tc^2 - 4\,c{v_x}^2tba + 4\,bu_y\,cv_y\\
+8\,ub^2v_y\,x - 2\,uc^2v_x\,y - uc^3u_{xx}\,t - 8\,b^2u_y\,xv + 2\,c^2u_x\,v_y - vc^3v_{xx}\,t + c^3{u_x}^2t + c^3{v_x}^2t
\end{array}\right)\\
T_3^t &= \frac{1}{2}\,uv_x - \frac{1}{2}\,vu_x\\
T_3^x &= \frac{1}{2}\,a{u_x}^2 + \frac{1}{2}\,a{v_x}^2 + \frac{1}{2}\,buu_{yy} + \frac{1}{2}\,bvv_{yy} + \frac{1}{4}\,cv_y\,v_x + \frac{1}{4}\,ucu_{xy} + \frac{1}{4}\,vcv_{xy} - \frac{1}{2}\,uv_t + \frac{1}{4}\,cu_y\,u_x - \frac{1}{4}\,du^4 + \frac{1}{2}\,vut - \frac{1}{4}\,dv^4 - \frac{1}{2}\,du^2v^2\\
T_3^y &= -\frac{1}{2}\,ubu_{xy} + \frac{1}{2}\,bu_y\,u_x - \frac{1}{2}\,vbv_{xy} + \frac{1}{2}\,bv_y\,v_x - \frac{1}{4}\,ucu_{xx} - \frac{1}{4}\,vcv_{xx} + \frac{1}{4}\,c{v_x}^2 + \frac{1}{4}\,c{u_x}^2\\
T_4^t &= \frac{1}{2}\,u^2 + \frac{1}{2}\,v^2\\
T_4^x &= auv_x - avu_x + \frac{1}{2}\,ucv_y - \frac{1}{2}\,cu_y\,v\\
T_4^y &= ubv_y - vbu_y + \frac{1}{2}\,ucv_x - \frac{1}{2}\,vcu_x
\end{aligned}
$$

Table 1: Conserved vectors of (12)

3.1 A double reduction of (12) by Γ_5

We now show that Γ_5 is associated with T_4. We obtain

$$\begin{pmatrix} T_4^{*t} \\ T_4^{*x} \\ T_4^{*y} \end{pmatrix} = \Gamma_5^{[1]} \begin{pmatrix} T_4^{t} \\ T_4^{x} \\ T_4^{y} \end{pmatrix} - \begin{pmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} T_4^{t} \\ T_4^{x} \\ T_4^{y} \end{pmatrix} + (4) \begin{pmatrix} T_4^{t} \\ T_4^{x} \\ T_4^{y} \end{pmatrix} \tag{17}$$

from (16). Here $\Gamma_5^{[1]} = -u\frac{\partial}{\partial u} - v\frac{\partial}{\partial v} + 2\,t\frac{\partial}{\partial t} + x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} - 2u_x\partial\partial u_x - 2v_x\partial\partial v_x - 2u_y\partial\partial u_y - 2v_y\partial\partial v_y - 3u_t\partial\partial u_t - 3v_t\partial\partial v_t$.(17) shows that

$$\begin{aligned} T_4^{*t} &= \Gamma_5^{[1]}T_4^t + 2T_4^t, \\ T_4^{*x} &= \Gamma_5^{[1]}T_4^x + 3T_4^x \\ T_4^{*y} &= \Gamma_5^{[1]}T_4^y + 3T_4^y. \end{aligned}$$

After some calculations, one can see that

$$T_4^{*t} = 0, T_4^{*x} = 0, T_4^{*y} = 0..$$

Thus, Γ_5 is associated with T_4 [7].

Double reduction is continuing by finding the invariants. Γ_5 is then transformed into its canonical form $Y = \frac{\partial}{\partial q}$ where we suppose that Y has the following form

$$Y = 0\frac{\partial}{\partial r} + \frac{\partial}{\partial q} + 0\frac{\partial}{\partial s} + 0\frac{\partial}{\partial p} + 0\frac{\partial}{\partial \omega}. \tag{18}$$

Thus, we get

$$\frac{dx}{x} = \frac{dt}{2t} = \frac{dy}{y} = \frac{du}{-u} = \frac{dv}{-v} = \frac{ds}{0} = \frac{dr}{0} = \frac{dq}{1} = \frac{d\omega}{0} = \frac{dp}{0}. \tag{19}$$

The invariants of Γ_5 from (19) are given by

$$\left\{ \begin{array}{c} \dfrac{dt}{2t} = \dfrac{dx}{x}, \dfrac{dt}{2t} = \dfrac{dy}{y}, \\ \dfrac{dx}{x} = \dfrac{du}{-u}, \dfrac{du}{-u} = \dfrac{dv}{-v}, \frac{dt}{2t} = \dfrac{dq}{1} \\ \dfrac{d\omega}{0}, \dfrac{dr}{0}, \dfrac{dp}{0}, \dfrac{ds}{0} \end{array} \right. \tag{20}$$

and

$$\begin{gathered} b_1 = \tfrac{t}{x^2}, b_2 = \tfrac{t}{y^2}, b_3 = xu, \\ b_4 = \tfrac{u}{v}, b_5 = \ln(t) - 2q, b_6 = r, \\ b_7 = p, b_8 = s, b_9 = \omega. \end{gathered}$$

By choosing $b_1 = b_6, b_3 = b_7, b_2 = b_8, b_4 = b_9, b_5 = 0$, we obtain the canonical coordinates

$$r = \frac{t}{x^2}, s = \frac{t}{y^2}, q = \frac{1}{2}\ln(t), w = \frac{u}{v}, p = uv \tag{21}$$

where $w = w(r, s), p = p(r, s)$. The inverse canonical coordinates are presented below

$$x = \frac{e^q}{\sqrt{r}}, y = \frac{e^q}{\sqrt{s}}, t = e^{2q}, u = \frac{p\sqrt{r}}{e^q}, v = \frac{p\sqrt{r}}{\omega e^q}. \tag{22}$$

The matrices A and A^{-1} can be computed using the canonical coordinates above

$$A = \begin{pmatrix} 0 & -\frac{1}{2}\,\frac{\mathrm{e}^q}{r^{\frac{3}{2}}} & 0 \\ 0 & 0 & -\frac{1}{2}\,\frac{\mathrm{e}^q}{s^{\frac{3}{2}}} \\ 2\,\mathrm{e}^{2\,q} & \frac{\mathrm{e}^q}{\sqrt{r}} & \frac{\mathrm{e}^q}{\sqrt{s}} \end{pmatrix}$$

and

$$\left(A^{-1}\right)^T = \begin{pmatrix} \frac{r}{\mathrm{e}^{2\,q}} & -2\,\frac{r^{\frac{3}{2}}}{\mathrm{e}^q} & 0 \\ \frac{s}{\mathrm{e}^{2\,q}} & 0 & -2\,\frac{s^{\frac{3}{2}}}{\mathrm{e}^q} \\ \frac{1}{2}\,\left(\mathrm{e}^{2\,q}\right)^{-1} & 0 & 0 \end{pmatrix}$$

where $J = detA = \frac{1}{2}\,\frac{\mathrm{e}^{2\,q}(\mathrm{e}^q)^2}{r^{\frac{3}{2}}s^{\frac{3}{2}}}$. Using (21) and the partial derivatives of u, v, the reduced conserved form is given by

$$\begin{pmatrix} T_4^r \\ T_4^s \\ T_4^q \end{pmatrix} = J(A^{-1})^T \begin{pmatrix} T_4^t \\ T_4^x \\ T_4^y \end{pmatrix} = \begin{pmatrix} -\frac{1}{4}\,\frac{(p)^2\left(-\sqrt{r}(\omega)^2-\sqrt{r}+8\,r^{\frac{5}{2}}a\frac{\partial}{\partial r}\omega+4\,rcs^{\frac{3}{2}}\,\frac{\partial}{\partial s}w\right)}{s^{\frac{3}{2}}\,(\omega)^2} \\ -\frac{1}{4}\,\frac{(p)^2\left(-(\omega)^2-1+4\,\sqrt{s}cr^{\frac{3}{2}}\,\frac{\partial}{\partial r}\omega+8\,s^2b\frac{\partial}{\partial s}\omega\right)}{\sqrt{r}\sqrt{s}(\omega)^2} \\ \frac{1}{8}\,\frac{(p)^2\left((\omega)^2+1\right)}{\sqrt{r}s^{\frac{3}{2}}\,(\omega)^2} \end{pmatrix}. \tag{23}$$

where the reduced conserved form (23) satisfies

$$D_rT_4^r + D_sT_4^s + D_qT_4^q = 0. \tag{24}$$

The reduced form (24) also satisfies $D_rT_4^r + D_sT_4^s = 0$. This yields

$$\begin{gathered}\left(-8\,p_r\,\omega a\omega_r - 4\,\omega_{rr}\,pa\omega + 8\,{\omega_r}^2pa\right)r^{\frac{7}{2}} - 10\,p\omega a\omega_r\,r^{\frac{5}{2}} \\ + \left(-4\,p_r\,\omega c\omega_s - 4\,pc\omega_{rs}\,\omega + 8\,\omega_r\,pc\omega_s - 4\,p_s\,\omega c\omega_r\right)s^{\frac{3}{2}}r^2 \\ + \left(p_r\,\omega - \omega_r\,p + p_r\,\omega^3\right)r^{\frac{3}{2}} - 2\,p\omega cs^{\frac{3}{2}}\omega_s\,r \\ + \left(\left(-8\,p_s\,\omega b\omega_s - 4\,\omega_{ss}\,pb\omega + 8\,{\omega_s}^2pb\right)s^3 - 6\,p\omega s^2b\omega_s + \left(p_s\,\omega^3 + p_s\,\omega - \omega_s\,p\right)s\right)\sqrt{r} = k_1\end{gathered} \tag{25}$$

Equation (25) does not admit any symmetry and conservation laws, which mean no further integration can be done.

3.2 A double reduction of (12) by T_4

We now show that Γ_2 is associated with T_4. We obtain

$$\begin{pmatrix} T_4^{*t} \\ T_4^{*x} \\ T_4^{*y} \end{pmatrix} = \Gamma_2^{[1]} \begin{pmatrix} T_4^t \\ T_4^x \\ T_4^y \end{pmatrix} - \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} T_4^t \\ T_4^x \\ T_4^y \end{pmatrix} + (0) \begin{pmatrix} T_4^t \\ T_4^x \\ T_4^y \end{pmatrix} \tag{26}$$

from (16). Here $\Gamma_2^{[1]} = \frac{\partial}{\partial t}$. We obtain

$$T_4^{*t} = 0, T_4^{*x} = 0, T_4^{*y} = 0..$$

Thus, Γ_2 is associated with T_4 [7].

Similarly, since $\Gamma_1^{[1]} = \frac{\partial}{\partial y}$ and $\Gamma_3^{[1]} = \frac{\partial}{\partial x}$, Γ_1 and Γ_3 are associated with T_4.

Now, we consider Γ_4 and show that this symmetry generator is associated with T_4 [7].

$$\begin{pmatrix} T_4^{*t} \\ T_4^{*x} \\ T_4^{*y} \end{pmatrix} = \Gamma_4^{[1]} \begin{pmatrix} T_4^{t} \\ T_4^{x} \\ T_4^{y} \end{pmatrix} - \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} T_4^{t} \\ T_4^{x} \\ T_4^{y} \end{pmatrix} + (0) \begin{pmatrix} T_4^{t} \\ T_4^{x} \\ T_4^{y} \end{pmatrix} \tag{27}$$

Here $\Gamma_4^{[1]} = u\frac{\partial}{\partial v} - v\frac{\partial}{\partial u} - v_x\frac{\partial}{\partial u_x} + u_x\frac{\partial}{\partial v_x} - v_y\frac{\partial}{\partial u_y} - v_t\frac{\partial}{\partial u_t} + u_y\frac{\partial}{\partial v_y} + u_t\frac{\partial}{\partial v_t}$. We obtain

$$T_4^{*t} = 0, T_4^{*x} = 0, T_4^{*y} = 0.$$

Thus, Γ_4 is associated with T_4 [7].

So, we can get a reduced conserved form by the combination of Γ_1,Γ_2 and Γ_4 as $\mathbf{\Gamma} = \frac{\partial}{\partial t} + R\frac{\partial}{\partial y} - v\frac{\partial}{\partial u} + u\frac{\partial}{\partial v}$. $\mathbf{\Gamma}$ has a canonical form $Y = \frac{\partial}{\partial q} = \frac{\partial}{\partial q} + 0\frac{\partial}{\partial s} + o\frac{\partial}{\partial p} + \frac{\partial}{\partial w}$. We obtain

$$\frac{dx}{0} = \frac{dy}{R} = \frac{dt}{1} = \frac{du}{-v} = \frac{dv}{u} = \frac{ds}{0} = \frac{dr}{0} = \frac{dq}{1} = \frac{d\omega}{0} = \frac{dz}{0}. \tag{28}$$

The invariants are given by

$$\begin{cases} \dfrac{dt}{t} = \dfrac{dy}{R}, \dfrac{du}{-v} = \dfrac{dv}{u}, \\ \dfrac{dx}{0}, \frac{vdu-udv}{-v^2-u^2} = dt \\ \dfrac{d\omega}{0}, \dfrac{dr}{0}, \dfrac{dp}{0}, \dfrac{ds}{0} \end{cases} \tag{29}$$

and

$$b_1 = y - Rt, b_2 = r, b_3 = x,$$
$$b_4 = s, b_5 = q, b_6 = \omega,$$
$$b_7 = t - q, b_8 = u^2 + v^2, b_9 = \arctan(\tfrac{v}{u}) - t.$$

By choosing $b_1 = b_4$, $b_3 = b_2, b_6 = b_8, b_7 = 0, b_5 = b_9$, we obtain the canonical coordinates

$$r = x, s = y - Rt, q = t, \omega = \sqrt{u^2 + v^2}, p = \arctan(\frac{v}{u}) - t \tag{30}$$

where $w = w(r, s), z = z(r, s)$. The inverse canonical coordinates are presented below

$$x = r, y = s + Rq, t = q, u = w\cos(p + t), v = w\sin(p + t). \tag{31}$$

The matrices A and A^{-1} can be computed using the canonical coordinates above

$$A = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & R \end{pmatrix}$$

and

$$\left(A^{-1}\right)^T = \begin{pmatrix} 0 & 1 & 0 \\ -R & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}$$

where $J = detA = 1$. With the help olf canonical coordinates and partial derviatives we obtain

$$\begin{pmatrix} T_4^r \\ T_4^s \\ T_4^q \end{pmatrix} = J(A^{-1})^T \begin{pmatrix} T_4^t \\ T_4^x \\ T_4^y \end{pmatrix} = \begin{pmatrix} T_4^x \\ -RT_4^t + T_4^y \\ T_4^t \end{pmatrix} = \begin{pmatrix} \frac{1}{2}\omega^2(2ap_r + cp_s) \\ -R\frac{1}{2}\omega^2 + \frac{1}{2}\omega^2(2bp_s + cp_r) \\ \frac{1}{2}\omega^2 \end{pmatrix}. \quad (32)$$

Therefore, since $D_q T_4^q = 0$, the reduced conserved form (32) satisfies

$$D_r T_4^r + D_s T_4^s = 0. \quad (33)$$

This yields

$$w\left(2w_r ap_r + w_r cp_s + wap_{rr} + wcp_{rs} - w_s R + w_s cp_r + 2w_s bp_s + wbp_{ss}\right) = 0. \quad (34)$$

The reduced form admmits the inherited symmetries:

$$X_1 = \frac{\partial}{\partial s}, X_2 = \frac{\partial}{\partial r}.$$

One can see that both X_1 and X_2 are an associated symmetries. Let us take the linear combination of these symmetries

$$\tilde{X} = \frac{\partial}{\partial s} + \gamma\frac{\partial}{\partial r}.$$

We can get a reduced conserved form by $\tilde{X}$, where the generator Y has a canonical form $Y = \frac{\partial}{\partial m}$ when

$$\frac{ds}{1} = \frac{dr}{\gamma} = \frac{dp}{0} = \frac{d\omega}{0} = \frac{dn}{0} = \frac{dm}{1} = \frac{d\alpha}{0} = \frac{d\beta}{0}. \quad (35)$$

So the canonical variables are

$$n = \gamma s - r, m = s, \alpha = \omega, \beta = p.$$

Using partial derivatives according to canonical variables, we can get the reduced conserved form

$$\begin{pmatrix} T_4^n \\ T_4^m \end{pmatrix} = J(A^{-1})^T \begin{pmatrix} T_4^r \\ T_4^s \end{pmatrix} = \begin{pmatrix} T_4^r - \gamma T_4^s \\ -T_4^s \end{pmatrix}. \quad (36)$$

since $D_m T_4^m = 0$, the reduced conserved form (36) satisfies

$$D_n T_4^n = 0. \quad (37)$$

This yields

$$-1/2\ (\alpha)^2\left(2\,a\beta_n - 2\,c\beta_n\gamma - \gamma\,R + 2\,b\beta_n\gamma^2\right) = k_1. \quad (38)$$

4 Conclusion

Thus, with the help of the double reduction theory applied on the equation under consideration, we achieved a reduction in both the order of the equation and the number of variables. Solutions of the last equation can be calculated using numerical methods.

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